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SOME FORMULAS RELATED TO GAUSS'S SUM

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1. Chowla [1] has proved the formula

$$(1) \qquad \sum_{s=0}^{m-1} (-1)^s e^{\pi i n(2s+1)^2/(4m)} = \frac{e^{\pi i/4}}{\sqrt{mn}} \sum_{s=1}^{mn} e^{-\pi i s^2/mn} \sec \frac{\pi s}{m},$$

where m, n are arbitrary odd positive integers. By contour integration it is shown that

$$(2) \qquad \int_0^\infty \frac{e^{-\pi i m x^2/n}}{\cosh \pi x} dx = \frac{1}{2} \sum_{s=0}^{n-1} (-1)^s e^{-\pi i m(2s+1)^2/(4n)} - \frac{1}{2m} \sum_{s=1}^{mn} e^{\pi i s^2/mn} \sec \frac{\pi s}{m},$$

where m, n are odd and positive. Ramanujan [2] proved that

$$(3) \qquad \int_0^\infty \frac{e^{-\pi i m x^2/n}}{\cosh \pi x} dx = \frac{1}{2} \sum_{s=0}^{n-1} (-1)^s e^{\pi i m(2s+1)^2/(4n)} + \frac{1}{2} \sqrt{\frac{n}{m}}\, e^{-\pi i/4} \sum_{s=0}^{m-1} (-1)^s e^{-\pi i n(2s+1)^2/(4m)}.$$

Comparison of (2) and (3) gives (1).

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In the present note we shall give a simple elementary proof of (1). Indeed we shall prove the slightly more general formula

$$(4)\qquad e^{-\pi i a'/m} \sum_{s=1}^{mn} e^{\pi i a s^2/mn}$$

$$= \left(\frac{-1}{mn}\right)^{(a'-1)/2} \left(\frac{a}{mn}\right) \sqrt{mn} \sum_{k=0}^{m-1} (-1)^k \exp\{-\pi i (2k+1)^2 a' n/(4m)\},$$

where m, n are odd, $(a, 2mn) = 1$, $aa' \equiv 1 \pmod{2mn}$ and (a/mn) is the Legendre-Jacobi symbol. For $a = a' = -1$, (4) reduces to (1).

In addition we prove

$$(5)\qquad \sum_{s=1}^{mn} e^{2\pi i a s^2/mn} \sec\frac{2\pi s}{m} = \left(\frac{-a}{mn}\right) \sqrt{mn} \sum_{k=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-n)} (-1)^k e^{-2\pi i k^2 a' n/m},$$

where m is odd and $(a, mn) = 1$.

2. If m is odd and $\zeta = e^{\pi i m}$, it is clear that

$$\sec\frac{\pi s}{m} = (-1)^s \frac{\zeta^{ms} + \zeta^{-ms}}{\zeta^s + \zeta^{-s}} = (-1)^s \sum_{k=0}^{m-1} (-1)^k \zeta^{(m-2k-1)s}.$$

Let a be an integer prime to $2mn$. Then

$$S = \sum_{s=1}^{mn} e^{\pi i a s^2/mn} \sec\frac{\pi s}{m} = \frac{1}{2} \sum_{s=1}^{2mn} e^{\pi i a s^2/mn} \sec\frac{\pi s}{m}$$

$$= \frac{1}{2} \sum_{s=1}^{2mn} (-1)^s e^{\pi i a s^2/mn} \sum_{k=0}^{m-1} (-1)^k \zeta^{(m-2k-1)s}$$

$$= \frac{1}{2} \sum_{k=0}^{m-1} (-1)^k \sum_{s=1}^{2mn} (-1)^s \exp\{\pi i [as^2 + (m-2k-1)ns]/mn\}.$$

Let $aa' \equiv 1 \pmod{2mn}$; then

$$as^2 + (m-2k-1)ns \equiv a\left[s + \frac{1}{2}(m-2k-1)a'n\right]^2$$

$$- \frac{1}{4}(m-2k-1)^2 a' n^2 \pmod{2mn},$$

so that

$$(6)\quad \begin{aligned} S = \frac{1}{2} \sum_{k=0}^{m-1} (-1)^k \exp\{-\pi i (m-2k-1)^2 a' n/(4m)\} \\ \cdot \sum_{s=1}^{2mn} (-1)^s \exp\left\{\pi i a \left[s + \frac{1}{2}(m-2k-1) a' n\right]^2 / mn\right\}. \end{aligned}$$

Since

$$\sum_{s=1}^{2mn} (-1)^s \exp\left\{\pi i a \left[s + \frac{1}{2}(m-2k-1) a' n\right]^2 / mn\right\}$$

$$= (-1)^{\frac{1}{2}(m-1)+k} \sum_{t=1}^{2mn} (-1)^t e^{\pi i a t^2/mn}$$

and

$$\exp\{-\pi i (m-2k-1)^2 a' n/(4m)\}$$

$$= (-1)^k \exp\{-\pi i (m-2) a' n/4\} \exp\{-\pi i (2k+1)^2 a' n/(4m)\},$$

(6) becomes

$$(7)\quad \begin{aligned} S = \frac{1}{2} e^{\pi i a' mn/4} \sum_{k=0}^{m-1} (-1)^k \exp\{-\pi i (2k+1)^2 a' n[(4m)\} \\ \cdot \sum_{t=1}^{2mn} (-1)^t e^{\pi i a t^2/mn}. \end{aligned}$$

We shall now show that

$$(8)\quad \sum_{t=1}^{2n} (-1)^t e^{\pi i a t^2/n} = 2 \sum_{s=1}^{n} e^{4\pi i a s^2/n},$$

where a and n are any odd integers. Indeed

$$\sum_{t=1}^{2n} (-1)^t e^{\pi i a t^2/n} = \sum_{s=1}^{n} e^{4\pi i a s^2/n} - \sum_{s=1}^{n} e^{\pi i a (2s-1)^2/n}$$

and

$$\sum_{s=1}^{n} e^{\pi i a (2s-1)^2/n} = e^{\pi i a/n} \sum_{s=1}^{n} e^{4\pi i a (s^2-s)/n}$$

$$= e^{\pi i a/n} \sum_{t=1}^{n} e^{4\pi i a t^2/n} \cdot \exp\{-\pi i a (n-1)^2/n\} = -\sum_{t=1}^{n} e^{4\pi i a t^2/n},$$

which evidently yields (6).

Substituting from (3) in (7) we get

$$(9)\qquad \sum_{s=1}^{mn} e^{\pi i a s^2/mn} \sec \frac{\pi s}{m}$$

$$= e^{\pi i a' mn/4} \sum_{k=0}^{m-1} (-1)^k \exp\{-\pi i (2k+1)^2 a' n/(4m)\} \cdot \sum_{s=1}^{mn} e^{4\pi i a s^2/mn}.$$

But, by a familiar formula for Gaussian sums,

$$(10)\qquad \sum_{s=1}^{mn} e^{4\pi i a s^2/mn} = \left(\frac{2a}{mn}\right) i^{(mn-1)^2/4} \sqrt{mn},$$

where (a/mn) is the Legendre-Jacobi symbol. Thus (9) reduces after a little manipulation to

$$(11)\qquad e^{-\pi i a'/4} \left(\frac{-1}{mn}\right)^{(a'-1)/2} \left(\frac{a}{mn}\right) \sqrt{mn} \sum_{k=0}^{m-1} (-1)^k \exp\{-\pi i (2k+1)^2 a' n/(4m)\}.$$

3. We consider now the sum

$$(12)\qquad T = \sum_{s=0}^{mn} e^{2\pi i a s^2/mn} \sec \frac{2\pi s}{m},$$

where now m is odd and $(a, mn) = 1$.

Put $\zeta = e^{2\pi i/m}$. Then

$$\sec \frac{2\pi s}{m} = \frac{\zeta^{ms} = \zeta^{-ms}}{\zeta^s + \zeta^{-s}} = \sum_{k=0}^{m-1} (-1)^k \zeta^{(m-2k-1)s},$$

so that

$$T = \sum_{k=0}^{m-1} (-1)^k \sum_{s=1}^{mn} \exp\{2\pi i [as^2 + (m-2k-1) ns]/mn\}.$$

Let $aa' \equiv 1 \pmod{mn}$; then

$$as^2 + (m-2k-1)\, ns \equiv a \left[s + \frac{1}{2}(m-2k-1)\, a' n\right]^2$$

$$- \left(\frac{m-1}{2} - k\right)^2 a' n^2 \pmod{mn}.$$

It follows that

$$T = \sum_{k=0}^{m-1} (-1)^k \exp\left\{-2\pi i \left(\frac{m-1}{2} - k\right)^2 a'\, n/m\right\}$$

$$\cdot \sum_{s=1}^{mn} \exp\left\{2\pi i a \left[s + \frac{1}{2}(m-2k-1)\, a'\, n\right]^2 /mn\right\}$$

$$= (-1)^{\frac{1}{2}(m-1)} \sum_{k=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} (-1)^k \exp(-2\pi i k^2 a'\, n/m)$$

$$\cdot \sum_{s=1}^{mn} \exp(2\pi\, i a s^2/mn) = \left(\frac{-a}{mn}\right)\sqrt{mn} \sum_{k=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} (-1)^k \exp(-2\pi i k^2 a'\, n/m),$$

by (10). This evidently completes the proof of (5).

REFERENCES

[1] S. Chowla, *Some formalae of the Gauss sum type (II)*, Tohoku Mathematical Journal, vol. 32 (1929-30), pp. 352-353.

[2] S. Ramanujan, *Some definite integrals connected with Gauss's sums*, Messenger of Mathematics, vol. 44 (1915). pp. 75-85.

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