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L. CARLITZ

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# THE GENERATING FUNCTION FOR THE JACOBI POLYNOMIAL

L. CARLITZ \*)

1. Put

$$P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n \binom{n+\alpha}{n-k} \binom{n+\beta}{k} \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k}.$$

Then it is familiar that

$$(1) \quad \sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(x) z^n = 2^{\alpha+\beta} R^{-1} (1 - z + R)^{-\alpha} (1 + z + R)^{-\beta},$$

where

$$R = (1 - 2xz + z^2)^{\frac{1}{2}}.$$

For proofs of (1) see [2, p. 127], [3, p. 140], [4, pp. 69-70].

If we put

$$u = \frac{1}{2}(x-1)z, \quad v = \frac{1}{2}(x+1)z,$$

we have

$$(2) \quad R = ((1-u-v)^2 - 4uv)^{\frac{1}{2}}$$

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\*) Indirizzo dell'A.: Depart of Math., Duke University, Durham, North Carolina USA.

and (1) becomes

$$(3) \quad \sum_{j, k=0}^{\infty} \binom{\alpha + j + k}{j} \binom{\beta + j + k}{k} u^j v^k \\ = 2^{\alpha+\beta} R^{-1} (1 - u + v + R)^{-\alpha} (1 + u - v + R)^{-\beta},$$

where  $R$  is defined by (2).

The object of this note is to give a direct and elementary proof of (3).

## 2. Consider the sum

$$\frac{1}{(1-x)^{\alpha+1} (1-y)^{\beta+1}} \sum_{j, k=0}^{\infty} \frac{(\alpha+1)_{j+k}}{j! (\alpha+1)_k} \frac{(\beta+1)_{k+j}}{k! (\beta+1)_j} \frac{(-1)^{j+k} x^j y^k}{(1-x)^{j+k} (1-y)^{j+k}} \\ = \sum_{j, k=0}^{\infty} (-1)^{j+k} \frac{(\alpha+1)_{j+k} (\beta+1)_{k+j}}{j! (\alpha+1)_k k! (\beta+1)_j} x^j y^k \\ \cdot \sum_{r=0}^{\infty} \frac{(\alpha+j+k+1)_r}{r!} x^r \sum_{s=0}^{\infty} \frac{(\beta+j+k+1)_s}{s!} y^s = \\ \sum_{m, n=0}^{\infty} \frac{(\alpha+1)_m (\beta+1)_n}{m! n!} x^m y^n \sum_{j=0}^m \sum_{k=0}^n \frac{(-m)_j (-n)_k}{j! k!} \frac{(\alpha+m+1)_k (\beta+n+1)_j}{(\alpha+1)_k (\beta+1)_j}.$$

The inner sum is equal to

$$\sum_{j=0}^m \frac{(-m)_j (\beta+n+1)_j}{j! (\beta+1)_j} \sum_{k=0}^n \frac{(-n)_k (\alpha+m+1)_k}{k! (\alpha+1)_k} = \frac{(-m)_m}{(\beta+1)_m} \frac{(-m)_n}{(\alpha+1)_n},$$

which vanishes unless  $m = n$ . It follows at once that

$$(4) \quad \frac{1}{(1-x)^{\alpha+1} (1-y)^{\beta+1}} \sum_{j, k=0}^{\infty} \binom{\alpha + j + k}{j} \binom{\beta + j + k}{k} \\ \cdot \frac{(-1)^{j+k} x^j y^k}{(1-x)^{j+k} (1-y)^{j+k}} = \frac{1}{1-xy}.$$

If we put

$$(5) \quad u = -\frac{x}{(1-x)(1-y)}, \quad v = -\frac{y}{(1-x)(1-y)},$$

it is easily verified that

$$1-x = \frac{2}{1-u+v+R}, \quad 1-y = \frac{2}{1+u-v+R}$$

and

$$\frac{1-xy}{(1-x)(1-y)} = R.$$

Thus (4) becomes

$$\begin{aligned} \sum_{j,k=0}^{\infty} \binom{\alpha+j+k}{j} \binom{\beta+j+k}{k} u^j v^k &= R^{-1} (1-x)^{\alpha} (1-y)^{\beta} \\ &= 2^{\alpha+\beta} R^{-1} (1-u+v+R)^{-\alpha} (1+u-v+R)^{-\beta}. \end{aligned}$$

This evidently completes the proof of (3).

The above proof may be compared with [1, p. 82].

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