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SOME GENERATING FUNCTIONS FOR ASSOCIATED LEGENDRE FUNCTIONS

G. K. DHAWAN *)

1. INTRODUCTION. This paper will present a couple of generating functions for associated Legendre Functions defined [1, p. 122] by

$$(1.1) \quad P_n^m(x) = \frac{1}{\Gamma(1-m)} \left(\frac{x+1}{x-1} \right)^{\frac{1}{2}m} {}_2F_1 \left[\begin{matrix} -n, n+1 \\ 1-m \end{matrix}; \frac{1}{2} - \frac{1}{2}x \right]$$

(|1 - x| < 2)

and

$$(1.2) \quad Q_n^m(x) = e^{mi\pi} 2^{-1-n} \pi^{1/2} \frac{p(n+m+1)}{p\left(n+\frac{3}{2}\right)} x^{-n-m-1} (x^2-1)^{\frac{1}{2}m} \times$$

$$\times {}_2F_1 \left[\begin{matrix} \frac{1}{2}n + \frac{1}{2}m + 1, \frac{1}{2}n + \frac{1}{2}m + \frac{1}{2} \\ n + \frac{3}{2} \end{matrix}; x^{-2} \right] \quad (|x| > 1)$$

where the functions $P_n^m(x)$ and $Q_n^m(x)$ are known as the Legendre functions of the first and second kind, respectively.

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The generating functions developed are :

$$(1.3) \quad \sum_{n=0}^{\infty} e^{im\pi} Q_n^m(x) t^n = \frac{\Gamma(-m) \Gamma(1+m)}{2} \left[\left(\frac{x-1}{x+1} \right)^m - 1 \right] \varrho^{-1-m} \times \\ \times \left[\frac{1}{2} (x+1) t \right]^{\frac{1}{2}m} \cdot P_m^m \left(\frac{1-t}{\varrho} \right)$$

where

$$\varrho = (1 - 2xt + t^2)^{\frac{1}{2}}$$

and with the restriction that m is not zero or positive or negative integer

$$(1.4) \quad \sum_{n=0}^{\infty} \frac{(1-m)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 1 \\ 1-m \end{matrix}; y \right] P_n^m(x) t^n = \\ = \mu^{-1-m} \left[-\frac{1}{2} ty (x-t+\varrho) \right]^{\frac{1}{2}m} P_m^m \left(\frac{\varrho+ty}{\mu} \right)$$

where

$$\mu = [1 - 2xt(1-y) + t^2(1-y)^2]^{\frac{1}{2}}$$

and

$$\varrho = (1 - 2xt + t^2)^{\frac{1}{2}}$$

$$(1.5) \quad \sum_{n=0}^{\infty} \frac{(1-m)_n}{n!} {}_1F_2 \left[\begin{matrix} -n \\ 1+m, 1-m \end{matrix}; y \right] P_n^m(x) t^n = \\ = \frac{\varrho^{-1}}{\Gamma(1+m)} e^{\frac{1}{2}mi\pi} J_{-m} \left\{ \left[\frac{2ty(x+t-\varrho)}{\varrho^2} \right]^{\frac{1}{2}} \right\} \times I_m \left\{ \left[\frac{-2ty(x-t+\varrho)}{\varrho^2} \right]^{\frac{1}{2}} \right\}$$

where

$$\varrho = (1 - 2xt + t^2)^{\frac{1}{2}}$$

and with the restriction that m is not an integer.

2. In this section we enlist some results for ready reference :

$$(2.1) \quad F_4[a, b; c, d; x, y] = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a)_{n+k} (b)_{n+k}}{k! n! (c)_k (d)_n} x^k y^n$$

which is formula (9) in [1; p. 224]

$$(2.2) \quad F_4 \left[\begin{matrix} \alpha, \beta; \\ 1 + \alpha - \beta, \beta; \end{matrix} \frac{-u}{(1-u)(1-v)}, \frac{-v}{(1-u)(1-v)} \right] = \\ = (1-v)^{\alpha} {}_2F_1 \left[\begin{matrix} \alpha, \\ 1 + \alpha - \beta; \end{matrix} \frac{\beta; -u(1-v)}{1-u} \right]$$

which is formula (8) in [1; p. 238]

$$(2.3) \quad {}_2F_1 \left[\begin{matrix} a, b; x \\ c; \end{matrix} \right] = (1-x)^{-b} {}_2F_1 \left[\begin{matrix} c-a, b; \\ .c; \end{matrix} \frac{x}{x-1} \right]$$

which is formula (22) in [1; p. 64]

$$(2.4) \quad \varrho^{-1-n} P_n^m \left(\frac{x-t}{\varrho} \right) = \sum_{k=0}^{\infty} \binom{n-m+k}{k} P_{n+k}^m(x) t^k$$

which is formula (2) in [2; p. 264]

$$(2.5) \quad P_n^{(\alpha, -\alpha)}(x) = \Gamma(1+\alpha) \left(\frac{x+1}{x-1} \right)^{\frac{1}{2}\alpha} \frac{(1+\alpha)_n}{n!} P_n^{-\alpha}(x)$$

which is result (8) in [4].

3. PROOF OF (1.3). In the proof, I shall use the expansion of $Q_n^m(x)$ in terms of $\frac{x-1}{x+1}$ as [1; p. 132]

$$e^{-im\pi} Q_n^m(x) = 2^{-1-n} \Gamma(m) (x+1)^{n+\frac{1}{2}} (x-1)^{-\frac{1}{2}} {}_2F_1 \left(-n, -n-m; 1-m; \frac{x-1}{x+1} \right) +$$

$$(3.1) \quad + 2^{-1-n} \frac{\Gamma(1+m+n) \Gamma(-m)}{\Gamma(1+n-m)} (x+1)^{-\frac{1}{2}m+n} (x+1)^{\frac{1}{2}m} \times \\ \times {}_2F_1\left(-n, -n+m; 1+m; \frac{x-1}{x+1}\right).$$

$$= A + B \text{ (say)}$$

Now

$$A = 2^{-1-n} \Gamma(m) (x+1)^{n+\frac{1}{2}m} (x-1)^{-\frac{1}{2}m} {}_2F_1\left(-n, -n-m; 1-m; \frac{x-1}{x+1}\right) \\ = \frac{\Gamma(m)}{2} \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} \sum_{k=0}^n \frac{(1)_n (1+m)_n}{k! (n-k)! (1-m)_k (1+m)_{n-k}} \times \\ \times \left[\frac{1}{2}(x-1)\right]^k \left[\frac{1}{2}(x+1)\right]^{n-k}$$

Multiplying both sides by t^n and summing as indicated, we have

$$\sum_{n=0}^{\infty} A t^n = \frac{\Gamma(m)}{2} \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(1)_n (1+m)_n}{k! (n-k)! (1-m)_k (1+m)_{n-k}} \times \\ \times \left[\frac{1}{2}(x-1)t\right]^k \left[\frac{1}{2}(x+1)t\right]^{n-k} \\ = \frac{\Gamma(m)}{2} \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1)_{n+k} (1+m)_{n+k}}{k! n! (1-m)_k (1+m)_n} \times \\ \times \left[\frac{1}{2}(x+1)t\right]^k \left[\frac{1}{2}(x+1)t\right]^n$$

and using the result (2.1)

$$(3.2) \quad = \frac{\Gamma(m)}{2} \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} F_4\left[1, 1+m; 1-m, 1+m; \frac{1}{2}(x-1)t, \frac{1}{2}(x+1)t\right].$$

The hypergeometric function of two arguments in (3.2) can be represented as the product of two functions of unit argument.

Taking

$$\frac{-u}{(1-u)(1-v)} = \frac{1}{2}(x-1)t; \quad \frac{-v}{(1-u)(1-v)} = \frac{1}{2}(x+1)t$$

$$\text{i.e.} \quad u = 1 - \frac{2}{1+t+\varrho}; \quad v = 1 - \frac{2}{1-t+\varrho}$$

where

$$\varrho = (1 - 2xt + t^2)^{\frac{1}{2}}$$

and using the results (2.2), (2.3) and definition of $P_n^m(x)$ as (1.1) we get

$$\begin{aligned} \sum_{n=0}^{\infty} A t^n &= \frac{\Gamma(m)}{2} \left(\frac{x+1}{x-1} \right)^{\frac{1}{2}m} \cdot \frac{2}{1-t+\varrho} \cdot \left[1 - \frac{1-t-\varrho}{1-t+\varrho} \right]^{-1-m} \times \\ &\quad \times {}_2F_1 \left[\begin{matrix} -m, 1+m; \\ 1-m; \end{matrix} \frac{1}{2} - \frac{1}{2} \left(\frac{1-t}{\varrho} \right) \right] \\ (3.3) \quad &= \frac{\Gamma(m)\Gamma(1-m)}{2} \varrho^{-1-m} \left[\frac{1}{2}(x+1)t \right]^{\frac{1}{2}m} P_m^m \left(\frac{1-t}{\varrho} \right). \end{aligned}$$

Again

$$\begin{aligned} B &= \frac{\Gamma(1+m+n)\Gamma(-m)}{\Gamma(1+n-m)} 2^{-1-n} (x+1)^{-\frac{1}{2}m+n} (x-1)^{\frac{1}{2}m} {}_2F_1 \left[\begin{matrix} -n, -n+m; \\ 1+m; \end{matrix} \frac{x-1}{x+1} \right] \\ &= \frac{\Gamma(1+m)\Gamma(-m)}{2\Gamma(1-m)} \left(\frac{x-1}{x+1} \right)^{\frac{1}{2}m} \frac{(1+m)_n}{(1-m)_n} \sum_{k=0}^n \frac{(1)_n (1-m)_n}{k! (n-k)! (1+m)_k (1-m)_{n-k}} \times \\ &\quad \times \left[\frac{1}{2}(x-1) \right]^k \left[\frac{1}{2}(x+1) \right]^{n-k}. \end{aligned}$$

Multiplying both sides by t^n and summing as indicated, we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} B t^n &= \frac{\Gamma(1+m) \Gamma(-m)}{2 \Gamma(1-m)} \left(\frac{x-1}{x+1} \right)^{\frac{1}{2}m} \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(1)_n (1+m)_n}{k! (n-k)! (1+m)_k (1-m)_{n-k}} \times \\
 &\quad \times \left[\frac{1}{2} (x-1) t \right]^k \left[\frac{1}{2} (x+1) t \right]^{n-k} \\
 &= \frac{\Gamma(1+m) \Gamma(-m)}{2 \Gamma(1-m)} \left(\frac{x-1}{x+1} \right)^{\frac{1}{2}m} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(1)_{n+k} (1+m)_{n+k}}{k! n! (1+m)_k (1-m)_n} \times \\
 &\quad \times \left[\frac{1}{2} (x-1) t \right]^k \left[\frac{1}{2} (x+1) t \right]^n.
 \end{aligned}$$

and using the result (2.1)

$$(3.4) = \frac{\Gamma(1+m) \Gamma(-m)}{2 \Gamma(1-m)} \left(\frac{x-1}{x+1} \right)^{\frac{1}{2}m} F_4 \left[1, \quad 1+m; \frac{1}{2} (x-1)t, \frac{1}{2} (x+1)t \right].$$

Proceeding as above and using result (2.2), (2.3) and (1.1) we get

$$(3.5) \sum_{n=0}^{\infty} B t^n = \frac{p(1+m)p(-m)}{2} \left(\frac{x-1}{x+1} \right)^m e^{-1-m} \left[\frac{1}{2} (1+t) t \right]^{\frac{1}{2}m} P_m^m \left(\frac{1-t}{e} \right).$$

Hence combining (3.3) and (3.5) we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} e^{-im\pi} Q_n^m(x) t^n &= \frac{p(1+m)p(-m)}{2} \left[\left(\frac{x-1}{x+1} \right)^m - 1 \right] \cdot e^{-1-m} \times \\
 &\quad \times \left[\frac{1}{2} (x+1) t \right]^{\frac{1}{2}m} P_m^m \left(\frac{1-t}{e} \right)
 \end{aligned}$$

where $P_m^m \left(\frac{1-t}{e} \right)$ is Legendre function in which degree and order are equal.

4. PROOF OF (1.4). In the proof, I shall use the generating function for Legendre polynomial as [3]

$$(4.1) \quad \varrho^{-1-n} \left[\frac{1}{2} t (x+1) \right]^{\frac{1}{2}m} P_m^m \left(\frac{1-t}{\varrho} \right) = \sum_{n=0}^{\infty} P_n^m(x) t^n$$

where

$$\varrho = (1 - 2xt + t^2)^{\frac{1}{2}}$$

Putting

$$x = \frac{X-T}{G}, \quad t = \frac{-TY}{G}$$

where

$$G = (1 - 2XT + T^2)^{\frac{1}{2}}$$

(4.1) becomes

$$\begin{aligned} & [1 - 2XT(1-Y) + T^2(1-Y)]^{-\frac{1}{2} - \frac{1}{2}m} \left[-\frac{1}{2} TY(X-T+G) \right]^{\frac{1}{2}m} \times \\ & \times P_m^m \left\{ \frac{G + TY}{[1 - 2XT(1-Y) + T^2(1-Y)^2]^{\frac{1}{2}}} \right\} \\ & = \sum_{n=0}^{\infty} G^{-1-n} P_n^m \left(\frac{X-T}{G} \right) (-TY)^n \end{aligned}$$

with (2.4), the right-hand side of above becomes

$$\begin{aligned} & = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n-m+k}{k} P_{n+k}^m(X) (-TY)^n T^k \\ & = \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n-m}{k} P_n^m(X) (-TY)^{n-k} T^k \\ & = \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n-m}{n-k} P_n^m(X) T^n (-Y)^k \\ & = \sum_{n=0}^{\infty} \frac{(1-m)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 1 \\ 1-m \end{matrix}; Y \right] P_n^m(X) T^n. \end{aligned}$$

which completes the proof of (1.4)

5. PROOF OF (1.5) In the proof, I shall use a generating function for Jacobi Polynomial defined as [2; p. 262]

$$\begin{aligned}
 (5.1) \quad & \sum_{n=0}^{\infty} \frac{P_n^{(\alpha, \beta)}(x) t^n}{(1+\alpha)_n (1+\beta)_n} = \\
 & = \Gamma(1+\alpha) \Gamma(1+\beta) \left(\frac{1}{2} t\right)^{-\frac{1}{2}\alpha - \frac{1}{2}\beta} (1-x)^{-\frac{1}{2}\alpha} (1+x)^{-\frac{1}{2}\beta} \times \\
 & \times J_{\alpha} \{[2t(1-x)]^{\frac{1}{2}}\} I_{\beta} \{[2t(1+x)]^{\frac{1}{2}}\}.
 \end{aligned}$$

Putting $\beta = -\alpha$ and using (2.5), we get

$$\begin{aligned}
 (5.2) \quad & \sum_{n=0}^{\infty} \frac{P_n^{-\alpha}(x) t^n}{n! (1+\alpha)_n} = \\
 & = (-1)^{\frac{1}{2}\alpha} \Gamma(1-\alpha) J_{\alpha} \{[2t(1-x)]^{\frac{1}{2}}\} I_{\beta} \{[2t(1+x)]^{\frac{1}{2}}\}
 \end{aligned}$$

Taking

$$x = \frac{X-T}{G}, \quad t = \frac{-TY}{G} \text{ and } m = -\alpha$$

where

$$G = (1 - 2XT + T^2)^{\frac{1}{2}}$$

(5.2) becomes

$$\begin{aligned}
 & \frac{e^{\frac{1}{2}im\pi}}{\Gamma(1+m)} J_{-m} \left\{ \left[\frac{2TY(X+T-G)}{G^2} \right]^{\frac{1}{2}} \right\} I_m \left\{ \left[\frac{-2TY(G+X-T)}{G^2} \right]^{\frac{1}{2}} \right\} \\
 & = \sum_{n=0}^{\infty} \frac{1}{n! (1+m)_n} \cdot P_n^m \left(\frac{X-T}{G} \right) \left(\frac{-TY}{G} \right)^n.
 \end{aligned}$$

With (2.4), right-hand side of above becomes

$$= G \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{n! (1+m)_n} \binom{n-m+k}{k} P_{n+k}^m(X) T^{n+k} (-Y)^n$$

$$\begin{aligned}
&= G \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{(n-k)!(1+m)_{n-k}} \binom{n-m}{k}_i P_n^m(X) T^n(-Y)^{n-k} \\
&= G \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{k!(1+m)_k} \binom{n-m}{n-k} P_n^m(X) T^n(-Y)^k \\
&= G \sum_{n=0}^{\infty} \frac{(1-m)_n}{n!} {}_1F_2 \left[\begin{matrix} -n; \\ 1+m, 1-m; \end{matrix} Y \right] P_n^m(X) T^n
\end{aligned}$$

which proves (1.5).

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