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# STRUCTURE THEORY FOR GEOMETRIC LATTICES

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## 1. Introduction.

A geometric lattice (Birkhoff [1], and in Jónsson [5], a matroid lattice) is a lattice which is complete, atomistic, continuous, and semimodular.

A sublattice of a geometric lattice need not be geometric. Consequently, any, categorical analysis of geometric lattices considered as algebras with two operators will most likely be inconclusive.

It is possible, however, to define a geometric lattice as a set  $L$ , together with an operator  $\sup$  (*supremum or join*), defined on arbitrary subsets of  $L$  and taking values in  $L$ , and with a binary relation  $\downarrow$  (*covers, or is equal to*). In writing the axioms, it is convenient to write  $0 = \sup \emptyset$ ,  $x \vee y = \sup \{x, y\}$ , and  $x \leq y$  if and only if  $x \vee y = y$ . Two of the axioms may be taken to be

$\alpha)$   $y \downarrow x$  if and only if  $x \leq y$  and  $x < z \leq y$  implies  $z = y$ .

$\beta)$   $\forall x < y \quad \exists p \downarrow 0 \cdot \exists \cdot x < x \vee p \leq y$ .

Axiom  $\alpha$  expresses the connection between  $\sup$  and  $\downarrow$ , while axiom  $\beta$  indicates that the atoms separate the lattice elements.

In this way, geometric lattices form an axiomatic model class [2] of relational structures. A substructure  $P$  of a geometric lattice  $Q$

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is also a geometric lattice if axioms  $\alpha$  and  $\beta$  apply in  $P$ . A relation-homomorphic image  $Q$  of a geometric lattice  $P$  is also a geometric lattice if axiom  $\alpha$  holds in  $Q$ .

In section 2, we show how mappings preserving join and cover correspond to the strong maps of geometries introduced by Higgs [4]. Images of geometric lattices are studied in section 3. The kernel of such a map is shown to be a closure with the exchange property, acting on a geometric lattice. We are indebted to Professor G.-C. Rota for his recommendation of this approach.

## 2. Strong maps.

We write  $y \downarrow x$  if an element  $y$  covers or is equal to an element  $x$  in a lattice. A function  $f$  from a lattice  $P$  to a lattice  $Q$  is *cover-preserving* if and only if  $y \downarrow x$  implies  $f(y) \downarrow f(x)$ , for all elements  $x, y$  in  $P$ .

**PROPOSITION 1.** Any lattice-epimorphism is cover-preserving.

*Proof.* Assume  $y \downarrow x$  in  $P$ . For any element  $c$  such that  $f(x) \leq c \leq f(y)$ , choose a preimage  $z \in P$ . Then  $c = (f(z) \wedge f(y)) \vee f(x) = f((z \wedge y) \vee x)$ . Since  $x \leq (z \wedge y) \vee x \leq y$ , and  $y \downarrow x$ ,  $(z \wedge y) \vee x$  is equal either to  $x$  or to  $y$ . Thus  $c$  is equal either to  $f(x)$  or to  $f(y)$ , and  $f(y) \downarrow f(x)$ . ■

A *join-homomorphism* from a geometric lattice  $P$  into a geometric lattice  $Q$  is any function  $f: P \rightarrow Q$  such that  $f(\sup X) = \sup f(X)$  for every subset  $X \subseteq P$ . Such a join-homomorphism  $f$  determines and is determined by the restriction  $f|_A$  of  $f$  to the set  $A$  of atoms of  $P$ .

**PROPOSITION 2.** A join-homomorphism  $f$  from a geometric lattice  $P$  into a geometric lattice  $Q$  is cover-preserving if and only if the image of each atom of  $P$  is either 0 or else an atom of  $Q$ .

*Proof:* If  $f$  is cover-preserving, and  $p \downarrow 0$  in  $P$ , then  $f(p) \downarrow f(0)$  in  $Q$ . But  $f(0) = f(\sup \emptyset) = \sup f(\emptyset) = \sup \emptyset = 0$ . Conversely, if  $p \downarrow 0$

in  $P$  implies  $f(p) \downarrow 0$  in  $Q$ , and if  $y \downarrow x$  in  $P$ , choose an atom  $p \in P$  such that  $x \vee p = y$ . Then  $f(p) \downarrow 0$  in  $Q$ , and  $f(y) = f(x \vee p) = f(x) \vee \vee f(p)$ , so  $f(y) \downarrow f(x)$  in  $Q$ , by semimodularity. ■

A join-homomorphism from a lattice  $P$  to a lattice  $Q$  is *non-singular* if and only if  $f(x) = 0$  implies  $x = 0$ . A *strong map* from a geometric lattice  $P$  into a geometric lattice  $Q$  is any non-singular cover-preserving join-homomorphism from  $P$  to  $Q$ . Higgs [4] coined the term «strong map» for those functions  $f$  from a geometry  $G_1$  to a geometry  $G_2$  such that  $f(\overline{X}) \subseteq \overline{f(X)}$ , for each subset  $X \subseteq G_1$ . In other words,  $f$  is a «strong map» if and only if the inverse image of each  $G_2$ -closed set is  $G_1$ -closed. Proposition 3, below, justifies our usage of the term.

If  $A$  is the set of atoms of a geometric lattice  $P$ , then  $\overline{X} = \{p \in A; p \leq \sup X\}$  defines a closure operator with the exchange property (MacLane [6]), and a geometry, in the sense of Higgs [3]. This geometry we shall denote  $G(P)$ .

**PROPOSITION 3.** If  $f$  is a non-singular strong map from a geometric lattice  $P$ , with atom set  $A$ , into a geometric lattice  $Q$ , then  $f|A$  is a strong map from the geometry  $G(P)$  into the geometry  $G(Q)$ . Conversely, if  $g$  is a strong map from  $G(P)$  into  $G(Q)$ , then  $g = f|A$  for a unique non-singular strong map  $f: P \rightarrow Q$ .

*Proof:* A strong map  $f$  carries atoms of  $P$  into atoms of  $Q$ , so  $f|A$  is a function from  $G(P)$  into  $G(Q)$ . If  $p \in \overline{X}$ , for a subset  $X \subseteq G(P)$ , then  $p \leq \sup X$ , and  $f(p) \leq f(\sup X) = \sup f(X)$ . Thus  $f(p) \in \overline{f(X)}$ , and  $f(\overline{X}) \subseteq \overline{f(X)}$ .

Conversely, if  $g$  is a strong map from  $G(P)$  into  $G(Q)$ , and if  $f$  is a join-preserving function from  $P$  into  $Q$  which extends  $g$ ,  $f$  must satisfy, for each  $x \in P$ ,  $f(x) = f(\sup \{p; p \text{ an atom, } p \leq x\})$ , because  $P$  is atomistic, and must satisfy

$$(1) \quad f(x) = \sup \{g(p); p \text{ an atom, } p \leq x\}$$

because  $f$  preserves join. The assumption that  $g$  is a strong map is required for the following proof that  $f$ , defined by equation (1),

is join-preserving. If  $Y$  is a subset of the lattice  $P$ ,

$$\begin{aligned}
 f(\sup Y) &= \sup \{g(p); p \text{ an atom, } p \leq \sup Y\} \\
 &\geq \sup \{g(p); p \text{ an atom, } p \leq y \text{ for some } y \in Y\} \\
 &= \sup_{y \in Y} \sup \{g(p); p \text{ an atom, } p \leq y\} \\
 &= \sup f(Y).
 \end{aligned}$$

If  $p$  is an atom beneath  $\sup Y$ ,  $p$  is in the closure of the set of atoms  $\{q; q \leq y \text{ for some } y \in Y\}$ ,  $g(p)$  is in the closure of the set of atoms  $\{g(q); q \leq y \text{ for some } y \in Y\}$  because  $g$  is a strong map, and  $g(p) \leq \sup f(Y)$ . Thus  $f(\sup Y) = \sup f(Y)$ . ■

The function  $f$ , defined by equation (1), is non-singular, and is cover-preserving, because  $P$  is atomistic,  $f$  is join-preserving, and  $Q$  is semimodular.

Non-singularity, join-preservation, and cover-preservation are properties preserved under composition of functions. On each geometric lattice  $P$ , the identity function is a strong map. Therefore geometric lattices and strong maps are the objects and morphisms respectively, of an abstract category, which we shall denote  $\mathcal{G}$ .

**PROPOSITION 4.** A substructure  $P$  of a geometric lattice  $Q$  is a geometric lattice if axioms  $\alpha$  and  $\beta$  hold in  $P$ .

*Proof:* If  $P$  is such a substructure of  $Q$ , then  $P$  is a subset of  $Q$ , closed with respect to arbitrary join. Thus  $P$  is a complete lattice.  $y \downarrow x$  in  $P$  if and only if  $y \downarrow x$  in  $Q$ , and the axiom for definition of  $\downarrow$  in terms of  $\sup$  is satisfied.  $0 = \sup \Phi$  is an element of  $P$ , so the atoms of  $P$  are precisely those atoms of  $Q$  contained in the subset  $P$ . Since axiom  $\beta$  holds in  $P$ ,  $P$  is atomistic. Since the atoms of  $P$  are a subset of the atoms of  $Q$ ,  $P$ , being atomistic, is continuous. If  $p \downarrow 0$  in  $P$ , and  $x \in P$ , then  $x \vee p$ , the supremum in  $Q$ , is an element of  $P$ , and  $x \vee p \downarrow x$  in  $Q$  implies  $x \vee p \downarrow x$  in  $P$ . Thus  $P$  is semimodular. It is clear that the injection map preserves join and cover, and is non-singular. ■

**COROLLARY TO PROPOSITION 4.** A subset  $P$  of a geometric lattice  $Q$  is a substructure of  $Q$  if and only if  $P$  is the set of arbitrary joins of subsets of some subset of the atoms of  $Q$ . ■

In the following section, we investigate the dual notion: images of geometric lattices.

### 3. Images.

Given any strong map  $f: P \rightarrow Q$  in the category  $\mathcal{G}$  of geometric lattices, the image of  $P$  in  $Q$  is also geometric. The operator  $J$  on  $P$  defined by  $J(x) = \sup \{y; f(y) = f(x)\}$  is a finitistic closure operator with the exchange property. If  $R$  is the natural map from  $J$ -closed elements of  $P$  into the lattice  $P/J$  of all  $J$ -closed elements of  $P$ , then  $RJ: P \rightarrow P/J$  is a strong map, and  $P/J \simeq \text{Im } f$ . Thus any strong map  $f: P \rightarrow Q$  may be factored  $f = f_3 f_2 f_1$ , where  $f_1$  is the strong map from  $P$  onto  $P/f$ ,  $f_2$  is an isomorphism of  $P/f$  with  $\text{Im } f$ , and  $f_3$  is a one-one strong map from  $\text{Im } f$  into  $Q$ . These facts are proven below.

**PROPOSITION 5.** If  $f$  is a strong map from a geometric lattice  $P$  into a complete, continuous lattice  $Q$ , then  $\text{Im } f$ , the image of  $P$  in  $Q$ , is also geometric.

*Proof:* Since  $f$  is a join-homomorphism,  $\text{Im } f$  is closed with respect to join, and is a complete lattice, with order induced by that on  $Q$ . If  $y \in \text{Im } f$ , choose a preimage  $x$  of  $y$ , and express  $x$  as a join of atoms in  $P$ . Then  $y$  is the join of the images of those atoms, and  $\text{Im } f$  is atomistic. If an atom  $p$  is beneath the supremum of a set  $X$  of atoms of  $\text{Im } f$ , consider  $p$  and the atoms in  $X$  as elements of  $Q$ . Since  $Q$  is continuous, there is a finite subset  $X' \subseteq X$  such that  $p \leq \sup X'$ . But  $p \leq \sup X'$  in  $\text{Im } f$ , so  $\text{Im } f$ , being atomistic, is continuous [5]. If  $q$  is an atom of  $\text{Im } f$ , and  $q \leq y$  for an element  $y \in \text{Im } f$ , choose preimages  $x$  of  $y$  and  $z$  of  $q$ . For each atom  $p \in P$  such that  $p \leq z$ , either  $f(p) = 0$  or  $f(p) = q$ . Since  $q = \sup \{f(p); p \leq z\}$ ,  $f(p) = q$  for some atom  $p \in P$ . Then  $p \neq 0$ ,  $p \vee x \downarrow x$ ,  $q \vee y = f(p) \vee f(x) = f(p \vee x) \downarrow f(x) = y$ , and  $\text{Im } f$  is semi-modular [5]. ■

A *join-congruence* on a complete lattice  $L$  is an equivalence relation  $\sim$  on  $L$  such that for any subset  $X \subseteq L$  and any function  $h: L \rightarrow L$  dominated by  $\sim$  (ie:  $x \sim h(x)$  for all  $x \in L$ ),  $\sup X \sim \sup h(X)$ .

LEMMA TO PROPOSITION 6. If  $\circ$  is a join-congruence on a complete lattice  $L$ , then the operator  $J$  defined by  $J(x) = \sup \{y; y \circ x\}$  is a closure operator.

*Proof:* Since  $x \circ x$ ,  $x \leq J(x)$ . If  $x \leq y$ , and  $z \circ x$ , then  $y \vee z \circ y \vee x = y$ , and  $z \leq y \vee z \leq J(y)$ . Thus  $J(x) \leq J(z)$ .  $J(x) = \sup \{y; y \circ x\} \circ \sup \{x\} = x$ , so  $z \circ J(x)$  if and only if  $z \circ x$ , and  $JJ(x) = J(x)$ . ■

A closure operator acting on a set, ie: on the Boolean algebra of all subsets of that set, has a lattice of closed subsets which is geometric if the closure is finitary and has the exchange property [3]. These two properties may be rephrased for closure operators on more general lattices. A closure  $J$  on a complete lattice  $L$  is *finitary* if and only if, for each directed subset  $X \subseteq L$ ,  $J(\sup X) = \sup J(X)$ .

PROPOSITION 6.<sup>(2)</sup> A closure  $J$  on a complete atomistic continuous lattice  $L$  is finitary if and only if, for every atom  $p \in L$  and every element  $x \in L$ , such that  $p \leq J(x)$ , there exists a finite set  $Y$  of atoms beneath  $x$ , such that  $p \leq J(\sup Y)$ .

*Proof:* Assume  $J$  is finitary,  $p$  an atom, and  $p \leq J(x)$  for some  $x \in L$ . Let  $X$  be the set of joins of finite sets of atoms beneath  $x$ .  $X$  is directed, so  $J(x) = J(\sup X) = \sup J(X)$ .  $J(X)$  is also directed, in a continuous lattice, so  $p \leq \sup J(X)$  implies  $p \leq J(y)$  for some  $y \in X$ . Conversely, assume  $X$  is a directed subset of  $L$ . Then  $J(\sup X) \geq \sup J(X)$ . Let  $p$  be any atom beneath  $J(\sup X)$ . Select a finite set  $Y$  of atoms beneath  $\sup X$ , such that  $p \leq J(\sup Y)$ . For each atom  $q \in Y$ , select an element  $x_q$  above  $q$  in the directed set  $X$ . Then choose an element  $x \in X$  such that  $x_q \leq x$  for all  $q \in Y$ .  $q \leq J(x) \leq \sup J(X)$ , so  $J(\sup X) = \sup J(X)$ . ■

A closure  $J$ , on a complete atomistic lattice  $L$ , has the *exchange property* if and only if, for any atoms  $p, q \in L$  and any element  $x \in L$ ,  $p \leq J(x)$  and  $p \leq J(x \vee q)$  imply  $q \leq J(x \vee p)$ .

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<sup>2)</sup> cf. Cohn [2], Theorem II.1.2: A closure system on a set is algebraic if and only if it is inductive.

**PROPOSITION 7.** If  $f$  is a strong map from a geometric lattice  $P$  into a geometric lattice  $Q$ , the operator  $J$  defined on  $P$  by

$$J(x) = \sup \{y ; f(y) = f(x)\}$$

is a finitary closure operator with the exchange property.

*Proof:* Consider the equivalence relation  $\sim$  defined by  $x \sim y$  if and only if  $f(x) = f(y)$ . If  $h : P \rightarrow P$  is any function dominated by  $\sim$ , and if  $X$  is any subset of  $P$ ,  $f(\sup h(X)) = \sup f(h(X)) = \sup f(X) = f(\sup X)$ , so  $\sup X \sim \sup h(X)$ , and  $\sim$  is a join-congruence. By the above lemma,  $J$  is a closure operator on  $P$ . If  $p$  is an atom of  $P$ , and  $p \leq J(x)$  for some  $x \in P$ ,  $f(p) \downarrow 0$  in the finitistic lattice  $Q$ , and  $f(x) = \sup f(X)$ , where  $X = \{q ; q \downarrow 0, q \leq x \text{ in } P\}$ . We may select a finite subset  $X' \subseteq X$  such that  $f(p) \leq \sup f(X')$ . Then  $p \leq J(\sup X')$  because  $p \leq p \vee \sup X'$ , and  $f(p \vee \sup X') = f(p) \vee \sup f(X') = \sup f(X') = f(\sup X')$ . Thus  $J$  is finitistic. Note that  $x \leq J(y)$  if and only if  $f(x) \leq f(y)$ . If  $p$  and  $q$  are atoms of  $P$ , and  $x$  is an element of  $P$  such that  $p \leq J(x)$ ,  $p \leq J(x \vee q)$ , then  $f(x) \vee f(p)$  covers  $f(x)$ . Since  $f(p) \leq f(x \vee q) = f(x) \vee f(q)$ , and  $f(x) \vee f(q) \downarrow f(x)$ , we have  $f(x \vee p) = f(x \vee q)$ ,  $f(q) \leq f(x \wedge p)$ , and  $q \leq J(x \vee p)$ . ■

**LEMMA TO PROPOSITION 8.** If  $J$  is a closure operator on a lattice  $L$ , and if  $R$  is the natural injection of  $J$ -closed elements of  $L$  into the lattice  $L/J$  of all  $J$ -closed elements of  $L$ , then the composition  $RJ : L \rightarrow L/J$  is a join-homomorphism.

*Proof:* If  $X$  is a subset of  $L$ ,  $\sup RJ(X)$  is the image in  $L/J$  of the least closed element of  $L$  lying above  $J(x)$ , for all  $x \in X$ , ie: the least closed element lying above  $x$ , for all  $x \in X$ , ie:  $J(\sup X)$ . Thus  $\sup RJ(X) = RI(\sup X)$ . ■

**PROPOSITION 8.** If  $J$  is a finitistic closure operator with the exchange property on a geometric lattice  $P$ , then the lattice  $P/J$  is geometric. If  $J(\Phi) = \Phi$ , and if  $R$  is the natural injection of the  $J$ -closed elements of  $P$  into  $P/J$ , then  $RJ$ , is a strong map.

*Proof:* Let  $y$  be any element of  $P/J$ , and  $X$  any directed set in  $P/J$ . By proposition 6,  $\sup_{P/J} X = J(\sup_P X) = \sup_P J(X) = \sup_P X$ . Thus  $y \wedge \sup_P X = \sup_P y \wedge X \leq \sup_{P/J} y \wedge X \leq y \wedge \sup_{P/J} X = y \wedge \sup_P X$ , and  $\sup_{P/J} y \wedge X = y \wedge \sup_{P/J} X$ , so the complete lattice  $P/J$  is continuous. By the above lemma,  $RJ$  is a join-homomorphism. If  $J(\Phi) = \Phi$ ,  $RJ$  is non-singular.

Assume  $y \downarrow x$  in  $P$ , and assume  $z$  is closed, with  $J(x) < z \leq J(y)$ . Choose an atom  $p$  such that  $x \vee p = y$ , and let  $q$  be any atom such that  $J(x) < J(x) \vee q \leq z$ . Then  $q \leq J(x)$ ,  $q \leq J(x \vee p)$ , so  $p \leq J(x \vee q) \leq J(z) = z$ . Thus  $x \vee p \leq z$ , and  $J(y) = z$ , so  $RJ$  is cover-preserving. By proposition 4,  $P/J$ , the image of a geometric lattice in a complete continuous lattice, is geometric. ■

**PROPOSITION 9.** Any map  $f: P \rightarrow Q$  in the category  $G$  of geometric lattices and strong maps has a factorization  $f = f_3 f_2 f_1$  in  $G$ , where  $f_1$  is onto,  $f_2$  is an isomorphism, and  $f_3$  is one-one.

*Proof:* Let  $J$  be the closure determined by  $f$ . Then the natural map  $f_1 = RJ: P \rightarrow P/J$  is onto.  $f_2 = fR^{-1}$ , cut down to  $\text{Im } f$ , is clearly one-one, onto, and order-preserving. Since the statement  $x \leq J(y) \iff f(x) \leq f(y)$  holds, in particular, for closed elements of  $P$ ,  $f_2$  is an order isomorphism, and therefore a strong map.  $f_3$  is then the natural embedding of  $\text{Im } f$  into  $Q$ . Since the order and cover relations on  $\text{Im } f$  are those induced by  $Q$ ,  $f_3$  is a strong map. ■

## BIBLIOGRAPHY

- [1] G. BIRKHOFF, *Abstract Linear Dependence and Lattices*, Amer. J. 57 (1935) p. 800-804.
- [2] P. M. COHN, *Universal Algebra*, Harper & Row, N. Y., 1965.
- [3] D. A. HIGGS, *Maps of Geometries*, J. London Math. Soc. 41 (1966), p. 612-618.
- [4] D. A. HIGGS, *Strong Maps of Geometries*, J. Combinatorial Theory, to appear. (seen in preparation).
- [5] B. JONSSON, *Lattice-theoretical Approach to Projective and Affine Geometry*, Symposium on the Axiomatic Method, p. 188-203.
- [6] S. MACLANE, *A Lattice Formulation for Transcendence Degrees and  $p$ -Bases*, Duke J. 4 (1938), p. 455-468.

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