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A SUMMATION THEOREM FOR DOUBLE HYPERGEOMETRIC SERIES

by L. CARLITZ *)

1. In contrast with the situation for ordinary hypergeometric series, not many summation formulas are known for double series of hypergeometric type. The writer [2], [3] has proved the following results.

$$(1.) \quad \sum_{s=0}^m \sum_{r=0}^n \frac{(-m)_r (-n)_s (\alpha)_{r+s} (\beta)_r (\beta')_s}{r! s! (\gamma)_{r+s} (\delta)_r (\delta')_s} \\ = \frac{(\beta + \beta' - \alpha)_{m+n} (\beta')_m (\beta)_n}{(\beta + \beta')_{m+n} (\beta' - \alpha)_m (\beta - \alpha)_n},$$

provided

$$(1.2) \quad \begin{cases} \gamma + \delta = \alpha + \beta - m + 1 \\ \gamma + \delta' = \alpha + \beta' - n + 1 \\ \gamma = \beta + \beta'; \end{cases}$$

$$(1.3) \quad \sum_{r+s \leq n} \frac{(-n)_{r+s} (\alpha)_r (\alpha')_s (\beta)_r (\beta')_s}{r! s! (\gamma)_{r+s} (\delta)_r (\delta')_s} \\ = \frac{(\beta)_n (\beta')_n (\gamma - \alpha - \alpha')_n}{(\gamma)_n (\beta - \alpha')_n (\beta' - \alpha)_n},$$

provided

$$(1.4) \quad \begin{cases} \gamma + \delta = \alpha + \beta - n + 1 \\ \gamma + \delta' = \alpha' + \beta' - n + 1 \\ \gamma = \beta + \beta'. \end{cases}$$

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In the present note we consider the sum

$$(1.5) \quad S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m (a')_n (b)_{m+n} (c)_m (c')_n}{m! n! (b)_m (b)_n (c + c')_{m+n}}$$

which is absolutely convergent for

$$(1.6) \quad \begin{cases} R(c + c' - b) \geq 0 \\ R(b - a - c) > 0 \\ R(b - a' - c') > 0. \end{cases}$$

We shall show that

$$(1.7) \quad S = \frac{\Gamma(c - a') \Gamma(c' - a) \Gamma(b) \Gamma(b - a - a') \Gamma(c + c')}{\Gamma(c + c' - a - a') \Gamma(b - a) \Gamma(b - a') \Gamma(c) \Gamma(c')}.$$

This result was originally obtained by using a reduction formula [1, p. 81] for the Appell function F_2 . However we shall give a direct proof below.

2. By Vandermonde's theorem

$$\frac{(b)_{m+n}}{(b)_m (b)_n} = \sum_{k=0}^{\min(m,n)} \frac{(-m)_k (-n)_k}{k! (b)_k}.$$

Thus (1.5) becomes

$$\begin{aligned} S &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m (a')_n (c)_m (c')_n}{m! n! (c + c')_{m+n}} \sum_{k=0}^{\min(m,n)} \frac{(-m)_k (-n)_k}{k! (b)_k} \\ &= \sum_{k=0}^{\infty} \frac{(a)_k (a')_k (c)_k (c')_k}{k! (b)_k (c + c')_{2k}} \sum_{m=0}^{\infty} \frac{(a + k)_m (c + k)_m}{m! (c + c' + 2k)_m} \sum_{n=0}^{\infty} \frac{(a' + k)_n (c' + k)_n}{n! (c + c' + 2k + m)_m}. \end{aligned}$$

By Gauss's theorem the sum on the extreme right is equal to

$$\begin{aligned} &\frac{\Gamma(c + c' + 2k + m) \Gamma(c - a' + m)}{\Gamma(c + c' - a' + k + m) \Gamma(c - k - m)} = \\ &= \frac{\Gamma(c + c') \Gamma(c + a')}{\Gamma(c + c' - a') \Gamma(c)} \frac{(c + c')_{2k+m} (c - a')_m}{(c + c' - a')_{k+m} (c)_{k+m}}. \end{aligned}$$

It follows that

$$\begin{aligned}
 S &= \frac{\Gamma(c + c')\Gamma(c - a')}{\Gamma(c + c' - a')\Gamma(c)} \sum_{k=0}^{\infty} \frac{(a)_k(a')_k(c)_k(c')_k}{k! (b)_k(c + c')_{2k}} \\
 &\cdot \sum_{m=0}^{\infty} \frac{(a + k)_m(c + k)_m}{m! (c + c' + 2k)_m} \frac{(c + c')_{2k+m}(c - a')_m}{(c + c' - a')_{k+m}(c)_{k+m}} \\
 &= \frac{\Gamma(c + c')\Gamma(c - a')}{\Gamma(c + c' - a')\Gamma(c)} \sum_{k=0}^{\infty} \frac{(a)_k(a')_k(c')_k}{k! (b)_k(c + c' - a')_k} \cdot \sum_{m=0}^{\infty} \frac{(a + k)_m(c - a')_m}{m! (c + c' - a' + k)_m}.
 \end{aligned}$$

Applying Gauss's theorem to the inner sum, we get

$$\begin{aligned}
 S &= \frac{\Gamma(c + c')\Gamma(c - a')}{\Gamma(c + c' - a')\Gamma(c)} \sum_{k=0}^{\infty} \frac{(a)_k(a')_k(c')_k}{k! (b)_k(c + c' - a')_k} \frac{\Gamma(c + c' - a' + k)\Gamma(c' - a)}{\Gamma(c + c' - a - a')\Gamma(c' + k)} \\
 &= \frac{\Gamma(c + c')\Gamma(c - a')\Gamma(c' - a)}{\Gamma(c)\Gamma(c')\Gamma(c + c' - a - a')} \sum_{k=0}^{\infty} \frac{(a)_k(a')_k}{k! (b)_k} \\
 &= \frac{\Gamma(c + c')\Gamma(c - a')\Gamma(c' - a)}{\Gamma(c)\Gamma(c')\Gamma(c + c' - a - a')} \frac{\Gamma(b)\Gamma(b - a - a')}{\Gamma(b - a)\Gamma(b - a')}.
 \end{aligned}$$

by another application of Gauss's theorem. This evidently completes the proof of (1.7).

3. If we take $a = -m, a' = -n$, where m, n are nonnegative integers, (1.7) reduces to

$$(3.1) \quad \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-m)_r(-n)_s(b)_{r+s}(c)_r(c')_s}{r! s! (b)_r(b)_s(c + c')_{r+s}} = \frac{(b)_{m+n}(c')_m(c)_n}{(b)_m(b)_n(c + c')_{m+n}}.$$

If we put

$$u_{m,n}(b; c, c') = \frac{(b)_{m+n}(c)_m(c')_n}{(b)_m(b)_n(c + c')_{m+n}},$$

we may write (3.1) in the form

$$(3.2) \quad \sum_{s=0}^m \sum_{r=0}^n (-1)^{r+s} \binom{m}{r} \binom{n}{s} u_{r,s}(b; c, c') = u_{m,n}(b; c', c).$$

It also follows from (3.1) that

$$\begin{aligned} & \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_{m+n} (c)_m (c')_n}{m! n! (b)_m (b)_n (c + c')_{m+n}} x^m y^n \\ &= \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n}{m! n!} x^m y^n \sum_{r=0}^m \sum_{s=0}^n \frac{(-m)_r (-n)_s (b)_{r+s} (c)_r (c')_s}{r! s! (b)_r (b)_s (c + c')_{r+s}} \\ &= \sum_{r,s=0}^{\infty} (-1)^{r+s} \frac{(a)_r (a')_s (b)_{r+s} (c)_r (c')_s}{r! s! (b)_r (b)_s (c + c')_{r+s}} x^r y^s \sum_{m=0}^{\infty} \frac{(a+r)_m}{m!} x^m \sum_{n=0}^{\infty} \frac{(a'+s)_n}{n!} y^n \\ &= \sum_{r,s=0}^{\infty} (-1)^{r+s} \frac{(a)_r (a')_s (b)_{r+s} (c)_r (c')_s}{r! s! (b)_r (b)_s (c + c')_{r+s}} \frac{x^r}{(1-x)^{a+r}} \frac{y^s}{1-y^{a'+s}}. \end{aligned}$$

Hence if we put

$$F(a, a'; b; c, c'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_{m+n} (c)_m (c')_n}{m! n! (b)_m (b)_n (c + c')_{m+n}} x^m y^n$$

it is clear that

$$\begin{aligned} (3.3) \quad & F(a, a'; b; c, c'; x, y) = \\ & = (1-x)^{-a} (1-y)^{-a'} F\left(a, a'; b; c, c'; \frac{x}{1-x}, \frac{y}{1-y}\right). \end{aligned}$$

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