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H. M. SRIVASTAVA

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and Legendre functions - II**

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SOME INTEGRALS INVOLVING PRODUCTS OF BESSEL AND LEGENDRE FUNCTIONS - II

di H. M. SRIVASTAVA (a Jodhpur) *)

Sunto: In the present paper the integral

$$\int_0^c x^{q-1} (c^2 - x^2)^{-(1/2)\sigma} R\left(\alpha, \beta, \gamma, \frac{x^2}{a^2}\right) R\left(\lambda, \mu, \eta, \frac{x^2}{b^2}\right) P_\nu^\sigma\left(\frac{x}{c}\right) dx,$$

where c is real, *non-zero* and finite, $\text{Re } (q) > 0$, $\text{Re } (\sigma) < 1$, is evaluated in terms of a double hypergeometric series and its several interesting special cases are discussed. A number of known results are also exhibited as necessary consequences of this integral.

1. In an earlier paper [5] we have proved that, for real, *non-zero* and finite values of c , the integral [loc. cit. p. 419]

$$\begin{aligned} (1.1) \quad & \int_0^c x^{q-1} (c^2 - x^2)^{-(1/2)\sigma} J_\lambda\left(\frac{2x}{a}\right) J_\mu\left(\frac{2x}{b}\right) P_\nu^\sigma\left(\frac{x}{c}\right) dx \\ &= \frac{2^{\sigma-1} c^{\delta-\sigma} \Gamma\left(\frac{1}{2} \delta\right) \Gamma\left(\frac{1}{2} \delta + \frac{1}{2}\right)}{a^\lambda b^\mu \Gamma(\lambda + 1) \Gamma(\mu + 1) \Gamma\left(\frac{1}{2} (\delta - \nu - \sigma + 1)\right) \Gamma\left(\frac{1}{2} (\delta + \nu - \sigma + 2)\right)} \cdot \\ & \cdot F\left[\begin{matrix} \frac{1}{2} \delta, & \frac{1}{2} \delta + \frac{1}{2} & : & - & ; & - \\ \frac{1}{2} (\delta - \nu - \sigma + 1), & \frac{1}{2} (\delta + \nu - \sigma + 2) : & \lambda + 1; & \mu + 1; & -\frac{c^2}{a^2}, & -\frac{c^2}{b^2} \end{matrix}\right], \end{aligned}$$

*) Indirizzo dell'A.: Department of Mathematics, The University, Jodhpur, India.

where $\delta = \lambda + \mu + \varrho$, $\operatorname{Re}(\delta) > 0$, $\operatorname{Re}(\sigma) < 1$. Here as well as in what follows the notation for double hypergeometric functions is due to Burch-nall and Chaundy [3, p. 112] in preference, for the sake of brevity, to that of Kampé de Fériet [2, p. 150] introduced earlier.

Put

$$(1.2) \quad R(\lambda, \mu, \nu, z) = \sum_{m=0}^{\infty} \frac{(-)^m (\lambda + m + 1)_m}{m! \Gamma(\mu + m + 1) \Gamma(\nu + m + 1)} z^m,$$

then the known formula [7, p. 151]

$$\left(\frac{1}{2} z\right)^{\mu+\nu} = \frac{\Gamma(\mu+1)\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} \sum_{n=0}^{\infty} \frac{(\mu+\nu+2n)\Gamma(\mu+\nu+n)}{n!} J_{\mu+n}(z) J_{\nu+n}(z)$$

admits of the generalization

$$(1.3) \quad \sum_{n=0}^{\infty} \frac{(\lambda+2n)\Gamma(\lambda+n)}{n!} z^n R(\lambda+2n, \mu+n, \nu+n, z) = \\ = \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\nu+1)},$$

since we observe that [7, p. 147]

$$(1.4) \quad J_{\mu}(2z)J_{\nu}(2z) = z^{\mu+\nu}R(\mu+\nu, \mu, \nu, z^2).$$

For a detailed discussion of the various properties of the entire function $R(\lambda, \mu, \nu, z)$ see [1] and [6].

Now restrict c in the manner stated earlier and make use of the formula [4, p. 314]

$$\int_0^1 x^{\lambda-1} (1-x^2)^{-(1/2)\mu} P_{\nu}^{\mu}(x) dx = \\ = \frac{\pi^{1/2} 2^{\mu-\lambda} \Gamma(\lambda)}{\Gamma\left\{\frac{1}{2}(\lambda-\mu-\nu+1)\right\} \Gamma\left\{\frac{1}{2}(\lambda-\mu+\nu+2)\right\}},$$

$$\operatorname{Re}(\lambda) > 0, \quad \operatorname{Re}(\mu) < 1;$$

so that

$$\int_0^c x^{\varrho-1} (c^2 - x^2)^{-(1/2)\sigma} R\left(\alpha, \beta, \gamma, \frac{x^2}{a^2}\right) R\left(\lambda, \mu, \eta, \frac{x^2}{b^2}\right) P_{\nu}^{\sigma}\left(\frac{x}{c}\right) dx$$

$$\begin{aligned}
&= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-)^{r+s} \left(\frac{1}{a}\right)^{2r} \left(\frac{1}{b}\right)^{2s} (\alpha + r + 1)_r (\lambda + s + 1)_s}{r! s! \Gamma(\beta + r + 1) \Gamma(\gamma + r + 1) \Gamma(\mu + s + 1) \Gamma(\eta + s + 1)} \\
&\quad \cdot \int_0^c x^{\varrho+2r+2s-1} (c^2 - x^2)^{-(1/2)\sigma} P_{\nu}^{\sigma} \left(\frac{x}{c}\right) dx, \\
&= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-)^{r+s} 2^{\sigma-1} c^{\varrho-\sigma} (\alpha + r + 1)_r (\lambda + s + 1)_s}{r! s! \Gamma(\beta + r + 1) \Gamma(\gamma + r + 1) \Gamma(\mu + s + 1) \Gamma(\eta + s + 1)} \\
&\quad \cdot \frac{\Gamma\left\{\frac{1}{2}(\varrho + 2r + 2s)\right\} \Gamma\left\{\frac{1}{2}(\varrho + 2r + 2s + 1)\right\} \left(\frac{c}{a}\right)^{2r} \left(\frac{c}{b}\right)^{2s}}{\Gamma\left\{\frac{1}{2}(\varrho - \sigma - \nu + 2r + 2s + 1)\right\} \Gamma\left\{\frac{1}{2}(\varrho - \sigma + \nu + 2r + 2s + 2)\right\}},
\end{aligned}$$

and therefore

$$\begin{aligned}
(1.5) \quad &\int_0^c x^{\varrho-1} (c^2 - x^2)^{-(1/2)\sigma} R\left(\alpha, \beta, \gamma, \frac{x^2}{4a^2}\right) R\left(\lambda, \mu, \eta, \frac{x^2}{4b^2}\right) P_{\nu}^{\sigma}\left(\frac{x}{c}\right) dx \\
&= \frac{2^{\sigma-1} c^{\varrho-\sigma} \Gamma\left(\frac{1}{2}\varrho\right) \Gamma\left(\frac{1}{2}\varrho + \frac{1}{2}\right)}{\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\mu+1)\Gamma(\eta+1)\Gamma\left\{\frac{1}{2}(\varrho-\sigma-\nu+1)\right\} \Gamma\left\{\frac{1}{2}(\varrho-\sigma+\nu+2)\right\}} \\
&\quad \cdot F\left[\begin{matrix} \frac{1}{2}\varrho, \frac{1}{2}\varrho + \frac{1}{2} : \frac{1}{2}\alpha + \frac{1}{2}, \frac{1}{2}\alpha + 1; \frac{1}{2}\lambda + \frac{1}{2}, \frac{1}{2}\lambda + 1; \\ \frac{1}{2}(\varrho-\sigma-\nu+1), \frac{1}{2}(\varrho-\sigma+\nu+2) : \alpha+1, \beta+1, \gamma+1; \lambda+1, \mu+1, \eta+1; \\ -\frac{c^2}{a^2}, -\frac{c^2}{b^2} \end{matrix}\right],
\end{aligned}$$

provided that $\text{Re}(\varrho) > 0$ and $\text{Re}(\sigma) < 1$.

For $\sigma = 0$, (1.5) gives us the formula

$$\begin{aligned}
(1.6) \quad &\int_0^c x^{\varrho-1} R\left(\alpha, \beta, \gamma, \frac{x^2}{4a^2}\right) R\left(\lambda, \mu, \eta, \frac{x^2}{4b^2}\right) P_{\nu}\left(\frac{x}{c}\right) dx = \\
&= \frac{c^{\varrho} \Gamma\left(\frac{1}{2}\varrho\right) \Gamma\left(\frac{1}{2}\varrho + \frac{1}{2}\right)}{2\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\mu+1)\Gamma(\eta+1)\Gamma\left\{\frac{1}{2}(\varrho-\nu+1)\right\} \Gamma\left\{\frac{1}{2}(\varrho+\nu+2)\right\}}.
\end{aligned}$$

$$\cdot F \left[\begin{array}{c} \frac{1}{2} \varrho, \frac{1}{2} \varrho + \frac{1}{2} : \frac{1}{2} \alpha + \frac{1}{2}, \frac{1}{2} \alpha + 1; \frac{1}{2} \lambda + \frac{1}{2}, \frac{1}{2} \lambda + 1; \\ \frac{1}{2} (\varrho - \nu + 1), \frac{1}{2} (\varrho + \nu + 2) : \alpha + 1, \beta + 1, \gamma + 1; \lambda + 1, \mu + 1, \eta + 1; \\ -\frac{c^2}{a^2}, -\frac{c^2}{b^2} \end{array} \right],$$

where, as in the earlier case, $\text{Re}(\varrho) > 0$.

2. Set $\alpha = \beta + \gamma$, $\lambda = \mu + \eta$ and change the notation slightly. In view of (1.4), the formula (1.5) will then express an integral involving the product

$$J_\lambda \left(\frac{x}{a} \right) J_\mu \left(\frac{x}{a} \right) J_\xi \left(\frac{x}{b} \right) J_\eta \left(\frac{x}{b} \right) P_\nu^\sigma \left(\frac{x}{c} \right)$$

in terms of a hypergeometric series, and we have

$$\begin{aligned} (2.1) \quad & \int_0^c x^{\varrho-1} (c^2 - x^2)^{-(1/2)\sigma} J_\lambda \left(\frac{x}{a} \right) J_\mu \left(\frac{x}{a} \right) J_\xi \left(\frac{x}{b} \right) J_\eta \left(\frac{x}{b} \right) P_\nu^\sigma \left(\frac{x}{c} \right) dx \\ &= \frac{2^{\sigma-\delta+\varrho-1} c^{\delta-\sigma} \Gamma \left(\frac{1}{2} \delta \right) \Gamma \left(\frac{1}{2} \delta + \frac{1}{2} \right)}{a^{\lambda+\mu} b^{\xi+\eta} \Gamma(\lambda+1) \Gamma(\mu+1) \Gamma(\xi+1) \Gamma(\eta+1) \Gamma \left\{ \frac{1}{2} (\delta - \sigma - \nu + 1) \right\} \Gamma \left\{ \frac{1}{2} (\delta - \sigma + \nu + 2) \right\}} \\ & \cdot F \left[\begin{array}{c} \frac{1}{2} \delta, \frac{1}{2} \delta + \frac{1}{2} : \frac{1}{2} (\lambda + \mu + 1), \frac{1}{2} (\lambda + \mu + 2); \\ \frac{1}{2} (\delta - \sigma - \nu + 1), \frac{1}{2} (\delta - \sigma + \nu + 2) : \lambda + 1, \mu + 1, \lambda + \mu + 1; \\ \frac{1}{2} (\xi + \eta + 1), \frac{1}{2} (\xi + \eta + 2); \\ \xi + 1, \eta + 1, \xi + \eta + 1 \end{array} ; -\frac{c^2}{a^2}, -\frac{c^2}{b^2} \right], \end{aligned}$$

where $\delta = \lambda + \mu + \xi + \eta + \varrho$, $\text{Re}(\delta) > 0$ and $\text{Re}(\sigma) < 1$.

Similarly, (1.6) is reduced to the form

$$\begin{aligned}
 (2.2) \quad & \int_0^c x^{\varrho-1} J_\lambda\left(\frac{x}{a}\right) J_\mu\left(\frac{x}{a}\right) J_\xi\left(\frac{x}{b}\right) J_\eta\left(\frac{x}{b}\right) P_\nu^\sigma\left(\frac{x}{c}\right) dx = \\
 & = \frac{2^{\varrho-\delta-1} c^\delta \Gamma\left(\frac{1}{2} \delta\right) \Gamma\left(\frac{1}{2} \delta + \frac{1}{2}\right)}{a^{\lambda+\mu} b^{\xi+\eta} \Gamma(\lambda+1) \Gamma(\mu+1) \Gamma(\xi+1) \Gamma(\eta+1) \Gamma\left\{\frac{1}{2}(\delta-\nu+1)\right\} \Gamma\left\{\frac{1}{2}(\delta+\nu+2)\right\}} \\
 & \cdot F \left[\begin{array}{c} \frac{1}{2} \delta, \frac{1}{2} \delta + \frac{1}{2} \quad : \frac{1}{2}(\lambda + \mu + 1), \frac{1}{2}(\lambda + \mu + 2); \\ \frac{1}{2}(\delta - \nu + 1), \frac{1}{2}(\delta + \nu + 2) : \lambda + 1, \mu + 1, \lambda + \mu + 1 \quad ; \\ \frac{1}{2}(\xi + \eta + 1), \frac{1}{2}(\xi + \eta + 2); \\ \xi + 1, \eta + 1, \xi + \eta + 1 \quad ; \end{array} \quad -\frac{c^2}{a^2}, -\frac{c^2}{b^2} \right],
 \end{aligned}$$

valid if $\text{Re}(\delta) > 0$.

3. When $a = b$, the double series in (1.5) equals

$$\begin{aligned}
 & \sum_{k=0}^{\infty} \left\{ \frac{\left(\frac{1}{2} \varrho\right)_k \left(\frac{1}{2} \varrho + \frac{1}{2}\right)_k \left(\frac{1}{2} \lambda + \frac{1}{2}\right)_k \left(\frac{1}{2} \lambda + 1\right)_k \left(-\frac{c^2}{a^2}\right)^k}{k! \left(\frac{1}{2} \varrho - \frac{1}{2} \sigma - \frac{1}{2} \nu + \frac{1}{2}\right)_k \left(\frac{1}{2} \varrho - \frac{1}{2} \sigma + \frac{1}{2} \nu + 1\right)_k (\lambda+1)_k (\mu+1)_k (\eta+1)_k} \right. \\
 & \cdot \left. \sum_{r=0}^k \frac{(-k)_r \left(\frac{1}{2} \alpha + \frac{1}{2}\right)_r \left(\frac{1}{2} \alpha + 1\right)_r (-\lambda - k)_r (-\mu - k)_r (-\eta - k)_r}{r! (\alpha + 1)_r (\beta + 1)_r (\gamma + 1)_r \left(\frac{1}{2} - \frac{1}{2} \lambda - k\right)_r \left(-\frac{1}{2} \lambda - k\right)_r} \right\},
 \end{aligned}$$

and this simplifies as ${}_4F_5$ if we further set

$$\beta = \gamma + \frac{1}{2} = \frac{1}{2} \alpha \quad \text{and} \quad \mu = \eta + \frac{1}{2} = \frac{1}{2} \lambda.$$

Therefore, a special case of (1.5) is

$$\begin{aligned}
 (3.1) \quad & \int_0^c x^{q-1} (c^2 - x^2)^{-(1/2)\sigma} R \left(\lambda, \frac{1}{2} \lambda, \frac{1}{2} \lambda - \frac{1}{2}, \frac{x^2}{16a^2} \right) \cdot \\
 & \cdot R \left(\mu, \frac{1}{2} \mu, \frac{1}{2} \mu - \frac{1}{2}, \frac{x^2}{16a^2} \right) P_{\nu}^{\sigma} \left(\frac{x}{c} \right) dx = \\
 & = \frac{2^{\lambda+\mu+\sigma-1} c^{q-\sigma} \Gamma \left(\frac{1}{2} q \right) \Gamma \left(\frac{1}{2} q + \frac{1}{2} \right)}{\pi \Gamma(\lambda+1) \Gamma(\mu+1) \Gamma \left\{ \frac{1}{2} (q - \sigma - \nu + 1) \right\} \Gamma \left\{ \frac{1}{2} (q - \sigma + \nu + 2) \right\}} \cdot \\
 & \cdot {}_4F_5 \left[\begin{matrix} \frac{1}{2} q, \frac{1}{2} q + \frac{1}{2}, \frac{1}{2} (\lambda + \mu + 1), \frac{1}{2} (\lambda + \mu + 2) \\ \frac{1}{2} (q - \sigma - \nu + 1), \frac{1}{2} (q - \sigma + \nu + 2), \lambda + 1, \mu + 1, \lambda + \mu + 1 \end{matrix} ; -\frac{c^2}{a^2} \right],
 \end{aligned}$$

where $\operatorname{Re}(q) > 0$ and $\operatorname{Re}(\sigma) < 1$.

The last formula when re-written by virtue of the relation [1, p. 911]

$$(3.2) \quad R \left(2\nu, \nu, \nu - \frac{1}{2}, z^2 \right) = \frac{z^{-2\nu}}{\pi^{1/2}} J_{2\nu}(4z),$$

gives us the known result [5, p. 420 (2.1)]

$$\begin{aligned}
 (3.3) \quad & \int_0^c x^{q-1} (c^2 - x^2)^{-(1/2)\sigma} J_{\lambda} \left(\frac{x}{a} \right) J_{\mu} \left(\frac{x}{a} \right) P_{\nu}^{\sigma} \left(\frac{x}{c} \right) dx = \\
 & = \frac{2^{\sigma-1} c^{q-\sigma} \Gamma \left(\frac{1}{2} \delta \right) \Gamma \left(\frac{1}{2} \delta + \frac{1}{2} \right)}{a^{\lambda+\mu} \Gamma(\lambda+1) \Gamma(\mu+1) \Gamma \left\{ \frac{1}{2} (\delta - \sigma - \nu + 1) \right\} \Gamma \left\{ \frac{1}{2} (\delta - \sigma + \nu + 2) \right\}} \cdot \\
 & \cdot {}_4F_5 \left[\begin{matrix} \frac{1}{2} \delta, \frac{1}{2} \delta + \frac{1}{2}, \frac{1}{2} (\lambda + \mu + 1), \frac{1}{2} (\lambda + \mu + 2) \\ \frac{1}{2} (\delta - \sigma - \nu + 1), \frac{1}{2} (\delta - \sigma + \nu + 2), \lambda + 1, \mu + 1, \lambda + \mu + 1 \end{matrix} ; -\frac{c^2}{a^2} \right],
 \end{aligned}$$

which generalizes Bailey's integral [4, p. 338] and is valid when $\operatorname{Re}(\delta) > 0$, $\delta = \lambda + \mu + q$, $\operatorname{Re}(\sigma) < 1$.

In a similar way, (1.6) reduces to the formula [5, p. 421 (2.2)]

$$\begin{aligned}
 (3.4) \quad & \int_0^c x^{\varrho-1} J_\lambda \left(\frac{x}{a} \right) J_\mu \left(\frac{x}{a} \right) P_\nu \left(\frac{x}{c} \right) dx \\
 &= \frac{c^\delta \Gamma \left(\frac{1}{2} \delta \right) \Gamma \left(\frac{1}{2} \delta + \frac{1}{2} \right)}{2a^{\lambda+\mu} \Gamma(\lambda+1) \Gamma(\mu+1) \Gamma \left\{ \frac{1}{2} (\delta - \nu + 1) \right\} \Gamma \left\{ \frac{1}{2} (\delta + \nu + 2) \right\}} \\
 & \cdot {}_4F_5 \left[\begin{matrix} \frac{1}{2} \delta, \frac{1}{2} \delta + \frac{1}{2}, \frac{1}{2} (\lambda + \mu + 1), \frac{1}{2} (\lambda + \mu + 2) & ; & -\frac{c^2}{a^2} \\ \frac{1}{2} (\delta - \nu + 1), \frac{1}{2} (\delta + \nu + 2), \lambda + 1, \mu + 1, \lambda + \mu + 1 & ; & \end{matrix} \right], \\
 & \text{Re } (\delta) > 0;
 \end{aligned}$$

whose particular case in which $a = 1$, $\varrho = \lambda$ and ν is an integer corresponds to Bose's result [4, p. 337 (31)].

Since [7, p. 150]

$$(3.5) \quad J_\mu(z) J_\nu(z) = \frac{2}{\pi} \int_0^{(1/2)\pi} J_{\mu+\nu}(2z \cos \theta) \cos(\mu - \nu)\theta \, d\theta,$$

from the formulae (3.3) and (3.4) we also have

$$\begin{aligned}
 (3.6) \quad & {}_4F_5 \left[\begin{matrix} \frac{1}{2} \delta, \frac{1}{2} \delta + \frac{1}{2}, \frac{1}{2} (\lambda + \mu + 1), \frac{1}{2} (\lambda + \mu + 2) & ; & -\frac{c^2}{a^2} \\ \frac{1}{2} (\delta - \sigma - \nu + 1), \frac{1}{2} (\delta - \sigma + \nu + 2), \lambda + 1, \mu + 1, \lambda + \mu + 1 & ; & \end{matrix} \right] \\
 &= \frac{\Gamma(\lambda+1) \Gamma(\mu+1) \Gamma \left\{ \frac{1}{2} (\delta - \sigma - \nu + 1) \right\} \Gamma \left\{ \frac{1}{2} (\delta - \sigma + \nu + 2) \right\}}{2^{\sigma-1} c^{\delta-\sigma} \Gamma \left(\frac{1}{2} \delta \right) \Gamma \left(\frac{1}{2} \delta + \frac{1}{2} \right)} \\
 & \cdot \frac{a^{\lambda+\mu}}{\pi} \int_0^c \int_0^{(1/2)\pi} t^{\varrho-1} (c^2 - t^2)^{-(1/2)\sigma} J_{\lambda+\mu} \left(\frac{2t}{a} \cos \theta \right) P_\nu^\sigma \left(\frac{t}{c} \right) \cos(\lambda - \mu)\theta \, dt \, d\theta,
 \end{aligned}$$

and

$$\begin{aligned}
 (3.7) \quad {}_4F_5 & \left[\begin{array}{c} \frac{1}{2} \delta, \frac{1}{2} \delta + \frac{1}{2}, \frac{1}{2} (\lambda + \mu + 1), \frac{1}{2} (\lambda + \mu + 2) ; \\ \frac{1}{2} (\delta - \nu + 1), \frac{1}{2} (\delta + \nu + 2), \lambda + 1, \mu + 1, \lambda + \mu + 1 ; \end{array} - \frac{c^2}{a^2} \right] \\
 &= \frac{\Gamma(\lambda + 1) \Gamma(\mu + 1) \Gamma\left\{\frac{1}{2} (\delta - \nu + 1)\right\} \Gamma\left\{\frac{1}{2} (\delta + \nu + 2)\right\}}{c^\delta \Gamma\left(\frac{1}{2} \delta\right) \Gamma\left(\frac{1}{2} \delta + \frac{1}{2}\right)} \\
 &\quad \cdot \frac{4a^{\lambda+\mu}}{\pi} \int_0^c \int_0^{(1/2)\pi} t^{\varrho-1} J_{\lambda+\mu}\left(\frac{2t}{a} \cos \theta\right) P_\nu\left(\frac{t}{c}\right) \cos (\lambda - \mu)\theta \, dt \, d\theta,
 \end{aligned}$$

respectively, provided $\operatorname{Re}(\lambda + \mu) > -1$ in addition to the conditions already stated.

4. Following the method illustrated in § 1 we also have

$$\begin{aligned}
 (4.1) \quad & \int_0^a x^{\varrho-1} (a^2 - x^2)^{-(1/2)\mu} P_\nu^\mu\left(\frac{x}{a}\right) R\left(\alpha, \beta, \gamma, \frac{x^2}{4b^2}\right) dx \\
 &= \frac{2^{\mu-1} a^{\varrho-\mu} \Gamma\left(\frac{1}{2} \varrho\right) \Gamma\left(\frac{1}{2} \varrho + \frac{1}{2}\right)}{\Gamma(\beta + 1) \Gamma(\gamma + 1) \Gamma\left\{\frac{1}{2} (\varrho - \mu - \nu + 1)\right\} \Gamma\left\{\frac{1}{2} (\varrho - \mu + \nu + 2)\right\}} \\
 &\quad \cdot {}_4F_5 \left[\begin{array}{c} \frac{1}{2} \varrho, \frac{1}{2} \varrho + \frac{1}{2}, \frac{1}{2} \alpha + \frac{1}{2}, \frac{1}{2} \alpha + 1 ; \\ \frac{1}{2} (\varrho - \mu - \nu + 1), \frac{1}{2} (\varrho - \mu + \nu + 2), \alpha + 1, \beta + 1, \gamma + 1 ; \end{array} - \frac{a^2}{b^2} \right],
 \end{aligned}$$

provided that $\operatorname{Re}(\varrho) > 0$ and $\operatorname{Re}(\mu) < 1$, and the corresponding formula for $\mu = 0$ is

$$\begin{aligned}
 (4.2) \quad & \int_0^a x^{\varrho-1} P_\nu \left(\frac{x}{a} \right) R \left(\alpha, \beta, \gamma, \frac{x^2}{4b^2} \right) dx = \\
 & = \frac{a^\varrho \Gamma \left(\frac{1}{2} \varrho \right) \Gamma \left(\frac{1}{2} \varrho + \frac{1}{2} \right)}{2\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma \left\{ \frac{1}{2} (\varrho - \nu + 1) \right\} \Gamma \left\{ \frac{1}{2} (\varrho + \nu + 2) \right\}} \cdot \\
 & \cdot {}_4F_5 \left[\begin{matrix} \frac{1}{2} \varrho, \frac{1}{2} \varrho + \frac{1}{2}, \frac{1}{2} \alpha + \frac{1}{2}, \frac{1}{2} \alpha + 1 & ; & -\frac{a^2}{b^2} \\ \frac{1}{2} (\varrho - \nu + 1), \frac{1}{2} (\varrho + \nu + 2), \alpha + 1, \beta + 1, \gamma + 1 \end{matrix} \right],
 \end{aligned}$$

valid when $\operatorname{Re}(\varrho) > 0$.

It is not difficult to exhibit (4.1) and (4.2) as the limiting cases of the formulae (1.5) and (1.6) respectively, when $b \rightarrow \infty$.

In the special case $\alpha = \beta + \gamma$ of these integrals if we make use of the relation (1.4) we shall again have the formulae (3.3) and (3.4).

On the other hand, if in (4.1) we set $\beta = \gamma + \frac{1}{2} = \frac{1}{2} \alpha$, and employ the relation (3.2), we get

$$\begin{aligned}
 (4.3) \quad & \int_0^a x^{\varrho-1} (a^2 - x^2)^{-(1/2)\mu} J_\lambda \left(\frac{2x}{\xi} \right) P_\nu^\mu \left(\frac{x}{a} \right) dx \\
 & = \pi^{1/2} \frac{\left(\frac{1}{2} a \right)^{\varrho+\lambda-\mu}}{\xi^\lambda} R \left(\varrho + \lambda - 1, \frac{1}{2} - \mu, \lambda, \frac{a^2}{4\xi^2} \right),
 \end{aligned}$$

where $\operatorname{Re}(\varrho + \lambda) > 0$, $\operatorname{Re}(\mu) < 1$ and the parameters are constrained by means of

$$\varrho + \lambda + \mu - \nu = 2.$$

Choose $\varrho = \frac{3}{2} - \mu$ so that $\lambda = \nu + \frac{1}{2}$ and make use of the relation (1.4). The special case $\xi = 2$ of (4.3) then leads to the known formula [4, p. 337]

$$\begin{aligned}
 (4.4) \quad \int_0^a x^{(1/2)-\mu} (a^2 - x^2)^{-(1/2)\mu} P_\nu^\mu \left(\frac{x}{a} \right) J_{\nu+(1/2)}(x) dx = \\
 = \left(\frac{1}{2} \pi \right)^{1/2} a^{1-\mu} J_{(1/2)-\mu} \left(\frac{1}{2} a \right) J_{\nu+(1/2)} \left(\frac{1}{2} a \right),
 \end{aligned}$$

which holds whenever $\operatorname{Re}(\mu - \nu) < 2$ and $\operatorname{Re}(\mu) < 1$.

By virtue of (3.5) the last formula gives us the interesting result

$$\begin{aligned}
 (4.5) \quad \int_0^{(1/2)\pi} J_{\nu-\mu+1}(a \cos \theta) \cos(\mu + \nu)\theta d\theta = \\
 = a^{\mu-1} \left(\frac{1}{2} \pi \right)^{1/2} \int_0^a x^{(1/2)-\mu} (a^2 - x^2)^{-(1/2)\mu} P_\nu^\mu \left(\frac{x}{a} \right) J_{\nu+(1/2)}(x) dx,
 \end{aligned}$$

which holds under the constraints stated earlier.

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