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PRINCIPAL PARTS AND CANONICAL
FACTORISATION OF HYPOELLIPTIC POLYNOMIALS
IN TWO VARIABLES

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Introduction

The basis for this paper is Gorin's definition [5] of \( \begin{bmatrix} k \\ j \end{bmatrix} \)-hypoellipticity. A polynomial \( P(\xi) \), \( \xi = (\xi_1, ..., \xi_n) \), is called \( \begin{bmatrix} k \\ j \end{bmatrix} \)-hypoelliptic of type \( a_{jk} > 0 \), if

\[
P(\xi) = 0, \quad \text{Im} \, \xi_j = 0 \quad \text{for} \quad j \neq k \Rightarrow | \xi_j | \leq C(1 + | \text{Im} \, \xi_k |)^{a_{jk}}.
\]

It is always assumed that \( a_{ii} \geq 1 \). Then the inequality

\[
| (\partial/\partial \xi_k)P(\xi) | \leq C(1 + |P(\xi)|)(1 + |\xi_j|)^{-a_{jk}},
\]

(\( \xi \) real, \( i = 1, 2, ... \))

is a sufficient condition for (0.1) to hold, and also a necessary condition if \( a_{jk} \geq 1 \). If \( P(\xi) \) is \( \begin{bmatrix} k \\ j \end{bmatrix} \)-hypoelliptic for all \( k \) and \( j \), then it is hypoelliptic in the ordinary sense (Hörmander [6]); if it is \( \begin{bmatrix} k \\ j \end{bmatrix} \)-hypoelliptic for all \( k \) and for \( j = 1, ..., n' \) with \( n' > n \), then it is partially hypoelliptic in \( x' = (x_1, ..., x_{n'}) \) (See Friberg [2]), so that all solutions of \( P(D)u = 0 \), \( D = i^{-1}(\partial/\partial x_1, ..., \partial/\partial x_n) \) are sums of derivatives of functions, infinitely differentiable in the \( x' \)-variables. Finally, if \( P(\xi) \) is \( \begin{bmatrix} k \\ j \end{bmatrix} \)-hypoelliptic for \( k = n' + 1, ..., n \) and all \( j \), then it can be shown that all solutions

of $P(D)u = 0$, with support contained in a cylinder $|x'| \leq A$, are infinitely differentiable.

As in the related paper [3], we shall use the following notations. If $P(\xi) = \Sigma c_\alpha \xi^\alpha$, then we set the index set $(P) = \{\alpha; c_\alpha \neq 0\}$, while $(P)^*$ denotes the convex hull of $(P) \cup \{0\}$. The (upper) Newton surface (or polygon) $F(P)$ of $P$ is then the union of the flat $(n - 1)$ dimensional pieces of the boundary of $(P)^*$ bounding $(P)^*$ from above, i.e. which are not parts of the coordinate planes. If $F(P)$ consists of a single face segment, $F(P)$ will be called simple. Hypoelliptic polynomials with simple Newton polygon have been studied by Pini [7]. His main contribution was the introduction of a «principal part» containing all terms of $P(\xi)$ that are essential for the hypoellipticity. Cf. our Cor. 3.1. The purpose of the present paper is to extend and improve the results of Pini as far as possible for general $k$-hypoelliptic polynomial in two variables.

Our main tool in the two-dimensional case will be the construction of a polynomial $P'$, the $k$-hypoelliptic canonical factorization of $P = P(\xi_1, \xi_2)$, whose zeros $\xi_2 = \Phi(\xi_1)$ have a finite Puiseux expansion, obtained by a suitable truncation of the Puiseux expansions of the zeros $P(\xi)$. It can be shown that $P'$ is a product of simple Newton polygons with simple Newton polygons, possibly multiplied by a polynomial in $\xi_1$ only. This factorization allows us to derive precise lower estimates for $|P(\xi)|$, and to define a «minimal» principal part for $P(\xi)$. It turns out that the terms of $P(\xi)$ not belonging to this principal part are exactly the terms that are strictly weaker than $P(\xi)$, in the sense of Hörmander.

Since $2$- and $1$-hypoellipticity together imply hypoellipticity of $P(\xi)$, our results can be used to define a canonical factorization into hypoelliptic polynomials with simple Newton polygons, of any given hypoelliptic polynomial in two variables. We can also show that the $2$- and $1$-hypoelliptic principal parts coincide, hence define uniquely a hypoelliptic principal part. The special class of hypoelliptic polynomials in $\mathbb{R}^2$, for which the principal part contains only terms with indices corresponding to points on the Newton polygon, was discussed in our previous paper [3].

In the more-dimensional case it is no longer possible to define a
hypoelliptic canonical factorization. Consequently our results about principal parts for this case are still very incomplete.

1. The Newton algorithm.

Given a polynomial $P(\xi) = \sum c_\alpha \xi^\alpha$, $\xi \in \mathbb{R}^n$, with index set $(P) = \{\alpha; c_\alpha \neq 0\}$, we can always find integers $r, s, h > 0$, with $r$ and $s$ relatively prime, such that

\[ h = \max (r\alpha_1 + s\alpha_2), \quad \alpha = (\alpha_1, \alpha_2) \in (P). \]

Then every $\alpha \in (P)$ belongs to one of the lines $r\alpha_1 + s\alpha_2 = h - j$, $j = 0, 1, \ldots, h$. It follows that there are polynomials $q_{0k}(u)$, such that

\[ \tau^k P(\tau^{-r}, \tau^{-s}u) = f_0(\tau, u) = \sum_{k=0}^{h} q_{0k}(u)\tau^k. \]

We want to determine the Puiseux expansions for the zeros $u = u(\tau)$ of $f_0(\tau, u) = 0$ for $\tau$ near 0, using the method of the Newton polygon. (See Bliss [1]). Suppose $q_{00}(u)$ has a zero $c_0 \neq 0$ of multiplicity $\mu_0$. Let

\[ f_0(\tau, u) = f_0(\tau, c_0 + v) = g_0(\tau, v), \]

and construct the Newton polygon bounding the index set $(g_0)$ of $g_0$ from below. The polygon determines a finite set of couples of positive integers $(\beta_1, \gamma_1)$ and $(r_1, s_1)$, $r_1, s_1$ relatively prime, such that

\[ f_0(\tau_1^{r_1}, c_0 + \tau_1^{r_1}u_1) = g_0(\tau_1^{r_1}, \tau_1^{r_1}u_1) = \tau_1^{h_1}u_1^{\gamma_1}f_1(\tau_1, u_1), \]

\[ f_1(\tau_1, u_1) = \sum_{k=0}^{h_1} q_{1k}(u_1)\tau_1^k, \quad (h_1 > 0). \]

We can now repeat the process, determining a zero $c_1$ of $q_{10}(u_1)$ of multiplicity $\mu_1$, and so on. After $j$ iterations we have

\[ u = c_0 + \tau_1^{r_1}(c_1 + \tau_1^{r_1}(c_2 + \ldots (c_{j-1} + \tau_1 f_j u_1) \ldots)), \quad (\tau = \tau_1^{r_1\cdots r_1}). \]

If $\gamma_j \neq 0$, we may choose $u_j = 0$, which gives a zero with a finite Puiseux expansion. Also, it may or may not happen that the process terminates after a finite number of iterations, with an $h_j = 0$, hence with $u_j = e_j$, $q_{j0}(e_j) = 0$. If we introduce integers $\sigma_0, \sigma_1, \ldots$, such that

\[ \sigma_0 = 0; \quad \sigma_i = s_i + r_i\sigma_{i-1}, \quad i = 1, 2, \ldots, \]
we can write (1.4) as

\[ u = \sum_{i=0}^{\infty} c_i \tau^i + (u_0(\tau_i) - c_i) \tau^i, \quad (\tau = \tau_1^{i_1} \cdots \tau_n^{i_n}). \]

It can be shown (See Bliss [1]) that \( \tau_i = \mu_i = 1 \) for all \( i \) big enough, so that there is only a finite number of Puiseux expansions

\[ u = u(\tau) = \sum_{i=0}^{\infty} c_i \tau^i, \quad (\tau = \tau_1^{i_1} \cdots \tau_n^{i_n}), \]

generated by (1.6). Moreover, all these expansions converge for \( \tau \) small enough, and together they represent all zeros \( u = u(\tau) \) of \( f_0(\tau, u) \) near \( \tau = 0 \). Now let \( \omega_i \) be any one of unit the roots of order \( q_i = r_{i_1} \cdots r_{i_n} \), and let \( \omega_i = \omega_1^{q_1} \cdots \omega_n^{q_n} \), so that \( \omega_i \) is a unit root of order \( q_i = r_{i_1} \cdots r_{i_n} \). Then, in view of (1.2), every expansion (1.6), (1.7) defines exactly \( q_i \) zeros of \( P(\xi) = P(\xi_1, \xi_2) \), « conjugate at level \( j \). These are of the form

\[ \xi_2 = \varphi_i(\tau_1) = \sum_{i=0}^{\infty} c_i \tau_1^{q_i \alpha_i / q_1 - q_1} + \cdots, \quad (\omega_i \tau_1)^{-\alpha_i} = \xi_1, \]

or, if we define \( \xi_1^{1/\alpha_1} \) to be real for \( \xi_1 \) positive,

\[ \xi_2 = \sum_{i=0}^{\infty} c_i \omega_i^{q_i \alpha_i / q_1} \xi_1^{q_1} + \cdots, \quad \delta_i = s/r - \sigma_i/q_i, \quad (\xi_1 > 0). \]

It should be noticed here that, in view of (1.5),

\[ \delta_i = \delta_{i-1} - s_i/q_i = \cdots = \delta_0 - \sum_{k=1}^{i} s_k/q_k, \quad (\delta_0 = s/r), \]

so that \( \delta_0 > \delta_1 > \cdots \), all the \( \delta_i \) being rational numbers, negative for \( i \) big enough. In contrast,

\[ q_i \delta_i = sr_1 \cdots r_i - \sigma_i \]

is an integer for every value of \( i \).

Let now the couple \((r, s)\) take on all the possible positive values determined by the Newton polygon \( F'(P) \) bounding the index set \((P) \) « from above ». Then we obtain, through the algorithm just described, the Puiseux expansions of all those zeros \( \xi_2 = \Phi(\xi_1) \) of \( P(\xi) \), for which \( |\xi_2| \) tends to infinity with \( |\xi_1| \).
2. The canonical factorization.

Suppose that we want $P(\xi)$, $\xi = (\xi_1, \xi_2)$, to be $\binom{2}{1}$-hypoelliptic of type $a_{12}$ in the sense of Gorin [5], i.e. such that

\[(2.1) \quad P(\xi_1, \xi_2) = 0, \quad \text{Im } \xi_1 = 0 \Rightarrow |\xi_1| \leq C(1 + |\text{Im } \xi_2|)^{\alpha_1}, \quad (a_{12} > 0).\]

Then $|\text{Im } \xi_2| \to \infty$ as $\xi_1 \to \pm \infty$, $P(\xi_1, \xi_2) = 0$, and we see that (2.1) is satisfied if and only if every zero $\xi_2 = \Phi(\xi_1)$ of $P(\xi)$ is of the form

\[(2.2) \quad \xi_2 = \Phi(\xi_1) = \sum_{k=0}^{k} c_k \xi_1^k + O(1) \quad |\xi_1|^{\delta_2} = \sum_{l=0}^{l} c_l'(-\xi)^{\delta_l} + O(1) \quad |\xi_1|^{\delta_l},\]

with

\[(2.3) \quad \text{Im } c_k \neq 0, \quad \text{Im } c_l' \neq 0, \quad \text{for some } k, l \quad \text{with } \delta_k, \delta_l > 0.\]

**Definition 2.1:** Let (2.2) be defined by the Newton algorithm of section 1. Then \(c = c_0, c_1,...\) is called a minimal Newton sequence (of length \(J\)) if there are integers \(k, l\) with \(\max (k, l) = J\), such that

\[(2.4) \quad |\text{Im } \Phi_c(\xi)| = |\text{Im } c_k| \quad |\xi_1|^{\delta_k}(1 + O(1)) \quad \text{as } \xi_1 \to +\infty, \quad (\text{Im } c_k \neq 0),\]

\[(\text{Im } \Phi_c(\xi)) = |\text{Im } c_l'| \quad |\xi_1|^{\delta_l}(1 + O(1)) \quad \text{as } \xi_1 \to -\infty, \quad (\text{Im } c_l' \neq 0),\]

and if, for every zero \(\Phi_{c,\omega}(\xi)\) conjugate to \(\Phi_c(\xi)\) at level \(J\),

\[|\text{Im } \Phi_{c,\omega}(\xi)| = 0(1) \quad |\text{Im } \Phi_{c,\omega}(\xi)| \quad \text{as } \xi_1 \to \pm \infty.\]

(In this definition we have made use of the fact that there is a certain arbitrariness in the relation (2.2) between the sequences \(c_0, c_1,...\), and \(c_0', c_1', ...\)).

**Theorem 2.1:** The polynomial $P(\xi)$ is $\binom{2}{1}$-hypoelliptic if and only if for every minimal Newton sequence $c$ the critical exponent $\delta_2 = \min (\delta_k, \delta_l)$, given by (2.4), is strictly positive. If $P$ is $\binom{2}{1}$-hypoelliptic, then it is also $\binom{2}{2}$-hypoelliptic, and the types are

\[(2.5) \quad a_{12}(P) = \max_c (1/\delta_2), \quad a_{22}(P) = \max_c (\delta_0/\delta_2),\]

with each maximum taken over all minimal sequences.
PROOF. If \( c = c_0, c_1, \ldots \) is a minimal sequence, then it follows from (2.2), (2.4) with, to be specific, \( J = \max (k, l) = k \) that

\[
\begin{align*}
| \text{Re} \xi_2 | &= | \text{Re} c_0 | | \xi_1 | | \xi_2 | | \xi_1 |^d(1 + \mathcal{O}(1)), \quad \text{as } | \xi_1 | \to \infty \\
| \text{Im} \xi_2 | &= | \text{Im} c_2 | | \xi_1 | | \xi_2 | | \xi_1 |^d(1 + \mathcal{O}(1)), \quad \text{as } | \xi_1 | \to +\infty, \\
| \text{Im} \xi_2 | &\geq 0(1) | \xi_1 | | \xi_2 |^d \quad \text{as } | \xi_1 | \to +\infty. 
\end{align*}
\]

Since by construction \( \text{Im} c_j \neq 0 \), Theorem 2.1 is a direct consequence of (2.6) and the definition (0.1) of \( \binom{k}{j} \)-hypoellipticity. Note that \( \text{Re} c_0 \neq 0 \) except possibly when \( J = 0 \). But then (2.5) still holds because of the assumption that \( a_{ij} \geq 1 \).

Now let \( c \) be a minimal sequence for \( P(\xi) \) of length \( J \), with \( \delta_j > 0 \). Denote by

\[
\Phi_{c, \omega, J}(\xi_1) = \sum_0^J c_0 \omega^{\omega_j^{\delta_j}} \xi_1^{\delta_j}, \quad \omega \in U_J = \{ \omega ; \omega^{\omega_j} = 1 \},
\]

a Puiseux expansion, truncated at level \( J \), of any one of the zeros \( \xi_2 = \Phi(c, J) \) of \( P(\xi) \), conjugate to \( \Phi(c, J) \) at level \( J \). The product

\[
M(c, J)(\xi) = \prod_{\omega \in U_J} (\xi_2 - \Phi_{c, \omega, J}(\xi)) = \prod_{\omega \in U_J} (\xi_2 - \sum_0^J c_0 \omega^{\omega_j^{\delta_j}})
\]

is then a symmetric polynomial in the zeros \( \eta = \omega^{\xi_2^{\omega_j^{\delta_j}}} \) of the polynomial \( \eta^{\omega_j} - \xi_1 \), hence a polynomial in \( \xi_1 \) (and \( \xi_2 \)).

DEFINITION 2.2. Let every minimal Newton sequence \( c \) of length \( J \) for a \( \binom{2}{1} \)-hypoelliptic polynomial \( P(\xi) \) represent an equivalence class \( (c) \) of minimal sequences, namely the ones that define zeros conjugate to \( \Phi_{c, \omega, J}(\xi_1) \) at level \( J \). Construct, for every equivalence class \( (c) \), a polynomial \( M_{(c), J}(\xi) \) as in (2.8). Suppose that \( P(\xi) = P(\xi_1) \xi_2^{\omega_j} + \text{terms of lower degree in } \xi_2 \). Then the product

\[
P'(\xi) = P(\xi_1) \prod_{(c)} M_{(c), J}(\xi)
\]

is called the canonical \( \binom{2}{1} \)-hypoelliptic factorization of \( P(\xi) \), and the \( M_{(c), J} \) are called primitive \( \binom{2}{1} \)-hypoelliptic polynomials (of length \( J \)).
In general $P' \neq P$, but the notation « canonical factorization » is motivated by the following.

**Lemma 2.1:** Let $P'$ be the canonical $\binom{2}{1}$-hypoelliptic factorization of $P$. Then $P(\xi)$ and $P'(\xi)$ are strictly of the same strength, in the sense that $P(\xi)/P'(\xi) = 1 + O(1) |\xi|^{-\theta}$ as $|\xi| \to \infty$, $\xi$ real, for some $\theta > 0$. More exactly, let $\zeta = \min (\delta_j - \delta_{j+1})$, the minimum taken over all minimal sequences for $P$. Then

$$P(\xi)/P'(\xi) - 1 \leq C(|\xi_1| + |\xi_2|^{r/\theta})^{-\zeta} \text{ for } \xi \text{ real, } |\xi| \leq 1.$$

Moreover, $P'$ is $\binom{2}{1}$-hypoelliptic of the same types as $P$.

**Proof.** That $a_{ij}(P') = a_{ij}(P)$, $i = 1, 2$, follows from Theorem 2.1, because $P'$ and $P$ have the same minimal sequences.

To prove (2.10), we write every zero of $P(\xi)$ in the form $\Phi_{eJ} + \Phi_{jJ}$, supposing that the sequence $e$ is conjugate at level $J$ to a minimal sequence of length $J$. Then

$$P(\xi)/P'(\xi) = \prod_{c} \left\{ 1 - \Phi_e^{J}/(\xi_2 - \Phi_{eJ}) \right\} = 1 + O(1) \max_{c} |\Phi_{eJ}| / |\xi_2 - \Phi_{eJ}|.$$

But $|\xi_2 - \Phi_{eJ}| \geq A(|\xi_1| + |\xi_2|^{r/\theta})$ (this is a consequence of more accurate estimates given in the proof of Theorem 2.2), while

$$|\Phi_{eJ}| = |c_{J+1} \xi_1^{J+1} + \ldots| = O(1) |\xi_1|^{J+1}.$$

The lemma follows immediately.

**Corollary 2.1:** Let $P(\xi)$ be a $\binom{2}{1}$-hypoelliptic polynomial in two variables. Let $s^i \alpha_1 + r^i \alpha_2 = h^i$, $i = 1, \ldots, N$, be the equations for the sides $F^i(P)$ of the Newton polygon $F(P)$. Write the canonical factorization of $P$ as $P' = p(\xi_1) \Pi P^i(\xi)$, with $F(P')$ parallel to $F^i(P)$. Then

$$|\partial/\partial \xi_1^k P(\xi)| \leq C(1 + |P(\xi)|)^{1-b_2} \text{ for } \xi \text{ real, }$$

$k = 1, \ldots$, if we put

$$b_2 = \min_i (s^i/h^i)a_{22}(P^i).$$

**Proof.** Let us study the case $k = 1$, the general case offering no
additional difficulties. We have $P(\xi) = p(\xi_1) \Pi (\xi_2 - \Phi(\xi_1))$, hence

$$(\partial / \partial \xi_2 P(\xi))/P(\xi) = \Sigma(\xi_2 - \Phi(\xi_1))^{-1}.$$  

But $|\xi_2 - \Phi(\xi_1)| \geq A(\xi_1 |\xi_1|^s + |\xi_2|^r)^{b/\alpha}$, for $|\xi| < 1$, where $s$, $r$ and $\delta_\gamma$ depend on $\Phi$. Also, $|P(\xi)| \leq C(\xi_1 |\xi_1|^s + |\xi_2|^r)^{\beta/\alpha}$ for the corresponding value of $h$. Consequently, $|\partial / \partial \xi_2 P(\xi)| \leq (1 + |P(\xi)|)^{1-b}$, where $b_2 = \min r \delta_\gamma / h = \min (s/h)(\delta_\gamma / \delta_0)$. Since $a_{22}(P) = \max (\delta_0 / \delta_\gamma)$, for all $\Phi_i$ with $\delta_0 = s^i / r^i$, the proof of the lemma is complete.

Take as a simple example the following polynomial, used in a similar connection by Pini [7],

$$P(\xi) = \xi_1^4 + \xi_2^3 - i\xi_1^3 \xi_2.$$  

Here $s = 3$, $r = 4$, $c_0 = 1$; $r_1 = 1$, $s_1 = 2$, $c_1 = i / 3$, and the only minimal sequence is of length $J = k = 1$, with

$$\Phi_{c,1,0} = - \omega \xi_1^{5/3} + (i/3)\omega^3 \xi_1^{1/3}, \quad (\omega^3 = 1).$$

Then a simple computation gives

$$P'(\xi) = M_{c,1}(\xi) = P(\xi) - (i/3) \xi_1^4.$$  

We easily prove (Cf. Theorem 2.2) that, for some $A > 0$,

$$(1 + |P(\xi)|) \geq A(\xi_1 |\xi_1|^s + |\xi_2|^r)^{b/\alpha} \quad \text{(for $\xi$ real).}$$

Hence

$$P(\xi)/P'(\xi) = 1 + 0(1)(\xi_1 |\xi_1|^s + |\xi_2|^r)^{-1/\alpha} \quad \text{as } |\xi| \to \infty.$$  

Reversing the roles of $\xi_1$ and $\xi_2$, we find that (2.11) is also $\left(\frac{1}{2}\right)$-hypoelliptic.

In fact, we have then $s = 4$, $r = 3$, $c_0 = 1$; $r_1 = 1$, $s_1 = 2$, $c_1 = -i/4$; $J = l = 1$, and

$$\Phi_{c,1,0} = \omega^3 (\xi_1)^{3/4} - (i/4)\omega(-\xi_1)^{1/4}, \quad (\omega^4 = 1)$$

so that the corresponding $\left(\frac{1}{2}\right)$-hypoelliptic primitive polynomial is

$$P''(\xi) = P(\xi) + 2^{-s} \xi_2^s + 2^{-r} \xi_2,$$
obviously, like $P'$, strictly of the same strength as $P$. It now follows from Theorem 2.1 that (2.11) is hypoelliptic, with $a_{11} = 3$, $a_{21} = 4$; $a_{12} = 3/2$, $a_{22} = 2$, hence with $a_1 = 3$, $a_2 = 4$.

As a second example, let us consider polynomials parabolic in $\xi_2$ in the sense of Šilov, i.e. such that

$$P(\xi) = 0 \Rightarrow \text{Im } \xi_2 > C \mid \text{Re } \xi \mid^\theta - C_1 \quad (\text{some } \theta > 0).$$

Obviously such polynomials must be $\binom{2}{1}$-hypoelliptic. Let $\xi_2 = \Phi(\xi_1) = c_0 + \cdots + c_J \xi_1^J + \cdots$ be one of the zeros of $P(\xi)$, with $\text{Im } c_i = 0$ for $i < J$. Then it is clear that $\delta_0, \ldots, \delta_{J-1}$ must be integers, because otherwise some of the zeros conjugate to $\Phi(\xi_1)$ could not satisfy the parabolicity condition. Moreover $\delta_J$ must be an even integer, because otherwise $\text{Im } \Phi(\xi_1)$ and $\text{Im } \Phi(-\xi_1)$ could not both tend to $+\infty$ with $\xi_1$.

(These observations are originally due to V. M. Borok. See Gelfand-Šilov [4], p. 136). An immediate consequence is then, in view of Lemma 2.1, the following result

**Theorem 2.2:** Let $\xi = (\xi_1, \xi_2)$. Then $P(\xi)$ is parabolic in $\xi_2$ in the sense of Šilov if and only if it is strictly of the same strength as a product of polynomials of the type

$$S(\xi) = (\xi_2 - i\xi_1)^r + S_1(\xi_1),$$

where $S_1$ is an arbitrary real polynomial.

Let us now return to the case of a primitive polynomial. Then we have the following basic estimate.

**Lemma 2.2:** Let $M(\xi) = M(\xi)$ be a primitive $\binom{2}{1}$-hypoelliptic polynomial as in (2.8), with

$$\Phi(\xi) = \Phi(\xi) = \sum_{\alpha} c_i \omega \xi_1^{\alpha} (\xi_1 > 0), = \sum_{\alpha} c_i \omega \xi_1^{\alpha} (\xi_1 > 0).$$

(Here $i = \delta s/r - \sum \delta s_k/q_k$, and $\omega^2 = 1$, with $q = q_1 = r_1 \ldots r_3$. Then $M(\xi)$ is of degree $m_1 = s r_1 \ldots r_3 = \delta_0 q$, in $\xi_1$, and of degree $m_2 = r_1 \ldots r_3 = q$ in $\xi_2$. Moreover,

(2.13) $A(\xi_1 | m_1 + \xi_2 | m_2) \geq M(\xi) \geq A_1(\xi_1 | m_1 + \xi_2 | m_2)^{1/q}$

for $\xi$ real, $| \xi | \geq K$.}
with \( 0 \leq d < 1 \) or, more exactly,

\[(2.14) \quad d = \sum_{i=1}^{J} (\delta_{i-1} - \delta_{i})/\delta_{0}q_{i-1}.\]

Hence \( d = 0 \) for \( J = 0 \), but \( 0 < (1 - \delta_{J}/\delta_{0})/m_{2} \leq d \leq (1 - \delta_{J}/\delta_{0})/r < 1/r \) for \( J > 0 \).

If we add the restriction that, for some \( k < J \),

\[(2.15) \quad |\xi_{2} - \Phi_{\omega}(\xi)| > \varepsilon |\xi_{1}|^{\delta_{k}} \quad \text{for all } \omega, \xi \text{ real},\]

we get and improved estimate (2.13), with \( d \) replaced by

\[(2.16) \quad d_{k} = \sum_{i=1}^{J} (\delta_{i-1,k} - \delta_{i,k})/\delta_{0}q_{i-1}, \quad \delta_{i,k} = \max(\delta_{i}, \delta_{k}).\]

**Proof.** Suppose, for instance, that \( J = \max(k, l) = k \). Then we notice that, for some \( j = j(\omega), 0 < j + 1 \leq J \),

\[(2.17) \quad |\xi_{2} - \Phi_{\omega}(\xi)| \geq \max \left\{ |\xi_{2} - \sum_{i=1}^{J} \text{Re } e_{i,\omega}z_{i}|, \quad \sum_{j=1+1}^{J} \text{Im } e_{i,\omega}z_{i}^{j} \right\},

with \( \text{Im } c_{j+1,\omega} \neq 0 \). Set now, with \( \varepsilon \leq |c_{j}| \),

\[(2.18) \quad V_{\omega,i,\varepsilon} = \{ \xi \in \mathbb{R}^{2} ; |\xi_{2} - \Phi_{\omega}(\xi)| < \varepsilon |\xi_{1}|^{\delta_{j}} \}, \quad (j = j(\omega)).\]

Then \( |\xi_{2} - \Phi_{\omega}(\xi)| = o(1) |\xi_{1}|^{\delta_{j}} \) for \( \xi_{1} \rightarrow + \infty, \xi \in V_{\omega,i,\varepsilon} \), and only if \( e_{\omega,i} = e_{\omega,i} (= \text{Re } e_{\omega,i}) \) for \( i = 0, 1, ..., j - 1 \). There are exactly \( r_{j} \) \( ..., j = 0, q_{j-1} \) zeros \( \Phi_{\omega_{i}} \) of this kind. But since \( j < J \), exactly \( q_{j} \) of these zeros also satisfy \( |\xi_{2} - \Phi_{\omega_{i}}(\xi)| = o(1) |\xi_{1}|^{\delta_{j}} \), namely those for which \( e_{\omega,i} = e_{\omega,i} (= \text{Re } e_{\omega,i}) \). Hence the estimate

\[(2.19) \quad C_{1} |\xi_{1}|^{\delta_{j}} \leq |\xi_{2} - \Phi_{\omega_{i}}(\xi)| \leq C_{2} |\xi_{1}|^{\delta_{j}},\]

for some \( C_{1}, C_{2} > 0 \), is valid in the domain \( V_{\omega,i,\varepsilon} \) for exactly \( q_{j}/q_{j-1} - 1/q_{j} \) zeros \( \Phi_{\omega_{i}} \). On the other hand, outside the union \( \bigcup_{\omega} V_{\omega,i,\varepsilon}, j(\omega) = j \), the estimate (2.19) is valid for all the \( q_{j}/q_{j-1} \) zeros \( \Phi_{\omega_{i}} \). Consequently the best overall lower estimate for \( |M| \) under the restriction (2.15) is \( |M(\xi)| \leq B |\xi_{1}|^{\chi}, \) where

\[\chi = \sum_{j=0}^{J} \delta_{j,k}(v_{j} - v_{j+1}), \quad v_{j} = q_{j}/q_{j-1}, \quad v_{j+1} = 0.\]
(Notice that the domain (2.15) is the complement of $\bigcup_{\omega} V_{\omega,k,\epsilon}$). It follows that

\begin{align}
(2.20) \quad | M(\xi) | \geq B( | \xi_1 | + | \xi_2 |^{r/e})^x
\end{align}

if $| \xi_1 | \geq C | \xi_2 |^{r/e}$. But if $| \xi_1 | < C | \xi_2 |^{r/e}$, with $C$ small enough, then trivially

\begin{align}
| M(\xi) | \geq B | \xi_2 |^{m_2} \geq B_1( | \xi_1 | + | \xi_2 |^{r/e})^{\delta_0 m_2}.
\end{align}

Since $m_2 = q = \nu_0$ is the number of zeros $\Phi_0$, we see that $\nu \leq \delta_0 \Sigma(v_j - v_{j-1}) < \delta_0 m_2$, so that

\begin{align}
| M(\xi) | \geq B( | \xi_1 | + | \xi_2 |^{r/e})^x \equiv B_1( | \xi_1 |^{m_1} + | \xi_2 |^{m_2})^{x/m_1}
\end{align}

for $\xi$ real, $| \xi | > 1$. Since the upper estimate in (2.13) is trivial, it remains only to prove (2.14). But by partial summation

\begin{align}
\nu = \nu_k = \delta_0 \nu_0 - \sum_{j} (\delta_{j-1,k} - \delta_{j,k})v_j = \\
= m_1 \{1 - \sum_{j} (\delta_{j-1,k} - \delta_{j,k})/\delta_0 q_{j-1}\},
\end{align}

and (2.14) follows if we observe that $r = q_0 \leq q_1 \leq \ldots, \delta_j > 0$.

For instance, in the example (2.11) we have $\delta_0 = 4/3$, $\delta_1 = 2/3$, $J = 1$, and $r = 3$, so that $d = (1 - \delta_1/\delta_0)/r = 1/6$. Reversing the order of $\xi_1$ and $\xi_2$ we get $\delta_0 = 3/4$, $\delta_1 = 1/4$, $J = 1$, and $r = 4$, so that again $d = (1 - \delta_1/\delta_0)/r = 1/6$. (Cf. (2.12)).

Let now $P(\xi)$, $\xi \in R^2$, be a general $\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$-hypoelliptic polynomial, and let $c = c_0, c_1, \ldots$ be a minimal Newton sequence for $P$. Constructing $c$ by the Newton algorithm of section 1, we define $g_0, g_1, \ldots$ as in (1.3). The lower Newton polygon for $g_{j-1}$ then determines a number of couples of relatively prime integers $(r_{ij}, s_{ij}) > 0$, (not necessarily all different), one of which is $(r_0, s_0)$. To each couple $(r_{ij}, s_{ij})$ corresponds an exponent $\delta_{ij}$, and $\mu_{ij}$ (complex) zeros for $P(\xi)$ of the form

\begin{align}
(2.21) \quad \xi_2 = c_0 \xi_1^{r_0} + \ldots + c_{i-1} \xi_1^{r_{i-1}} + c_i \xi_1^{r_i} + \ldots
\end{align}

In case $P$ is primitive as in Lemma 2.1, the Newton polygon for each $g_{i-1}$ is simple so that there are only $r_i$ coefficients $c_{ij} = c_{i,\omega_j}$, all of multiplicity $\mu_i = r_i/\rho_i$ and with exponent $\delta_{ij} = \delta_i$. 

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DEFINITION 2.3: Let $P(\xi)$ be a $\binom{2}{1}$-hypoelliptic polynomial, $\xi \in \mathbb{R}^2$, with a simple Newton polygon $F(P)$, and such that, for some $\mu_0 > 1$, $r$ and $s$ relatively prime,

$$P(\xi) = (\xi_2^2 - c_0 \xi_1 \xi_1)^{\mu_0} + \Sigma' c_{\alpha} \xi_2^\alpha, \quad \alpha \notin F(P), \quad c_0 \neq 0.$$  

Then $P$ is called a simple $\binom{2}{1}$-hypoelliptic polynomial with leading part $P_0(\xi) = (\xi_2^2 - c_0 \xi_1 \xi_1)^{\mu_0}$.

Notice that every primitive $\binom{2}{1}$-hypoelliptic polynomial is simple, but the converse is not true. The importance of the simple polynomials is that if we use the canonical factorization (and Lemma 2.1) to write a given $\binom{2}{1}$-hypoelliptic $P(\xi)$ as equivalent to a product $\Pi S_\lambda$ of simple $\binom{2}{1}$-hypoelliptic polynomials, then in order to find a lower estimate for $|P(\xi)|$, it is sufficient to estimate each $|S_\lambda(\xi)|$ downwards. In contrast, the best lower estimate for a product of primitive $\binom{2}{1}$-hypoelliptic polynomials is in general better than the product of the lower estimates for each factor separately.

THEOREM 2.3: Let a $\binom{2}{1}$-hypoelliptic $P(\xi) = p(\xi_1)\xi_1^{2s} + ...$ have the canonical $\binom{2}{1}$-hypoelliptic factorization $P' = p(\xi_1)\Pi M_{\xi_\lambda}$, and group together the primitive factors to write $P'/p$ as a product $\Pi S_\lambda$ of simple $\binom{2}{1}$-hypoelliptic factors $S_\lambda = \Pi M_{\xi_\lambda}$, with relatively prime leading parts $(\xi_2^2 - c_0 \xi_1 \xi_1)^{\mu_0}$. Set

$$d_\lambda = \max_{(c_\lambda)} \left\{ \sum \sum (\delta_{i-1} - \delta_{i,s})/\mu_i r, \delta_{i,s} = \max (\delta_i, \delta_{is}) \right\}.$$  

Then

$$1 - \delta_{j,s}/\mu_{s}r < d_\lambda < (1 - \delta_{j,s}/\delta_0)/r,$$

and

$$A > \left| P(\xi) \right| \left/ \left\{ p(\xi) \prod_{\lambda} (1 \left| \xi_1 \right| |r' + |\xi_2|^s)^{\mu_\lambda} \right\} \right. A_1 H(\xi)$$

for $\xi$ real, $|\xi| < K$. 


where $H(\xi) = (|\xi| + |\xi_2|)^{-\mu_0d_2}$ when $|\xi_1| - c_0\xi_2 > \varepsilon |\xi_1|^{\varepsilon}$ for some real $c_0$ and some $\varepsilon$ small enough, while $H(\xi) = 1$ outside the union of all such sets.

Proof. In view of the proof of Lemma 2.1, it is enough to prove (2.25) for the case when $P = P'$. Further, it is evident that there are constants $A_{d_2}$ such that $|S_{d_2}(\xi)| \geq A_{d_2}(|\xi_1| + |\xi_2|)^{\mu_0}$ ($s$, $r$, and $\mu_0$ depending on $\lambda$) for all $\lambda$ except one, at most, at every real point $\xi$, $|\xi| \gg K$. Hence it is enough to prove that

$$|S_{d_2}(\xi)| \geq A_{d_2}(|\xi_1| + |\xi_2|)^{\mu(1-d_2)}, \quad |\xi| \gg K.$$  

This can be done easily by the same reasoning as in the proof of Lemma 2.2 (Cf. also (2.16)). We find that (2.26) is valid with

$$d_{d_2} = \max_{(\xi),i} \sum_{i=1}^{J} \sum'_{\xi \neq i} (\delta_0 - \delta_{i,\xi})\mu_{i,\xi}r_{\xi,i}/\mu_0s, \quad \delta_{i,\xi} = \max (\delta_{i,\xi}, \delta_{i,i}),$$

where $\Sigma'$ for $i > J$ means that the summation does not include the index $j$ for which $c_{ij} = c_i$. Since $\sum_{i} \mu_{i,\xi}r_{\xi,i} = \sum_{j} \mu_{j,\xi}r_{\xi,j} + \sum_{j} \mu_{i+1,\xi}r_{\xi+1,j}$ (2.26) clearly implies (2.23). Finally, we can derive the estimates for $d_{d_2}$ from (2.27), if we observe that $\delta_{i,\xi} \geq \delta_{\xi,\xi}$, and that $\Sigma' \mu_{i,\xi}r_{\xi,i} = \mu_0$.

Let us now recall (See Friberg [2]), that $P(\xi)$, $\xi = (\xi_1, \xi_2)$, is called partially hypoelliptic in $\xi_1$ if

$$P(\xi + i\eta) = 0, \quad |\xi_1| \to \infty \Rightarrow |\eta| \to \infty.$$  

An equivalent condition is that $P$ is both $\binom{2}{1}$- and $\binom{1}{1}$-hypoelliptic. (Gorin [5]). But if $P$ is $\binom{2}{1}$-hypoelliptic, then we know that $P$ is equivalent to a $\binom{2}{1}$-hypoelliptic polynomial $P' = p(\xi_1)P_1(\xi)$, where the Newton polygon $F(P_1)$ has only sides with positive normals. Hence all the zeros $\xi_1 = \Phi(\xi_2)$ of $P_1(\xi)$ are of the form $\xi_1 = \sum_{0}^{\infty} c_i\xi_2^i$, with $\delta_0 > 0$.

With $\eta_2 = 0$ in (2.28), we see that $P_1$, and then $P$, can be $\binom{1}{1}$-hypoelliptic only if Im $c_i \neq 0$ for some $i$ with $\delta_i > 0$, i.e. only if $P_1$ is $\binom{1}{2}$-hypoelliptic. It is now easy to complete the proof of the following

**Theorem 2.4:** Let $\xi = (\xi_1, \xi_2)$, $P(\xi) = p_i(\xi_1)\xi_2^i + \ldots$. Then $P$ is
partially hypoelliptic in \( \xi \), if and only if \( P \) is (strictly) of the same strength as a polynomial

\[(2.29) \quad P'(\xi) = p(\xi_1)P_1(\xi), \quad P_1(\xi) \text{ hypoelliptic.} \]

Polynomials of the type (2.29) have in fact been used earlier as examples of partially hypoelliptic polynomials (Friberg [2], Gorin [5]).

3. The principal part.

**DEFINITION 3.1:** Let a \( \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \)-hypoelliptic polynomial \( P(\xi) = \sum_{(P)} c_\alpha \xi^\alpha \) have the canonical factorization \( P' = p(\xi_1)P(\xi) \). Suppose that \( r^n\xi_1 + s^n\xi_2 = h^n \), \( n = 1, ..., N \), are the equations for the sides of the Newton polygon \( F(P) = F(P') \), with \( r^n, s^n \) relatively prime. For given \( n \), consider all \( \xi \) - \( \xi_1 \) with Newton polygon given by an equation \( r^n\xi_1 + s^n\xi_2 = h^n \), so that \( h^n = \sum h_{\alpha}^n \), and define \( d_i = d_2^i \) as in (2.28). Set

\[(3.1) \quad H(P) = \bigcup_{n=1}^{N} \{ \alpha \in (P); \ h^n \geq r^n\xi_1 + s^n\xi_2 \geq h^n - \max_\lambda h_{\lambda}^n d_{\lambda}^n \} \]

Then the polynomial

\[(3.2) \quad P_n(\xi) = \sum_{H(P)} c_\alpha \xi^\alpha \]

is called the \( \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \)-hypoelliptic principal part of \( P(\xi) \).

The definition is partly motivated by Theorem 2.3, which shows that \( P(\xi) - P_n(\xi) \) is strictly weaker than \( P_n(\xi) \). But we can prove more:

**THEOREM 3.1:** Let \( P, Q \) be \( \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \)-hypoelliptic, with coinciding principal parts, \( P_n = Q_n \). Then \( P \) and \( Q \) have identical minimal Newton sequences (and are consequently \( \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \)-hypoelliptic of the same type).

**PROOF:** It is enough to show that the minimal sequences of \( P \) depend only on the coefficients \( c_\alpha \) of \( P \) with \( \alpha \in H(P) \). Omitting the indices \( n \), let \( r\alpha_1 + s\alpha_2 = h \) be the equation of one side in \( F(P) \), and define \( f_i(\tau_i, u_i), \ i = 0, 1, ..., \) and \( \varphi_{ik}(u), \ i = 0, 1, ..., k = 0, ..., h_i, \) as in (1.2), (1.3). We notice that \( \varphi_{ik}(u) \) is determined entirely by the coefficients \( c_\alpha \) of \( P \) with \( r\alpha_1 + s\alpha_2 = h - k \). On the other hand, to compute the
coefficients $c_0, \ldots, c_j$ of a minimal Newton sequence, we need only know $q_{00}, \ldots, q_{20}$. But we have

\begin{equation}
\begin{aligned}
f_0(\tau_j^{\alpha_0^r}, c_0 + \ldots + \tau_j^{\alpha}u_0) = & \sum_{0}^{\pm} q_{00}(c_0 + \ldots + \tau_j^{\alpha}u_j)\tau_j^0 \\
= & (\tau_j^{\alpha_1}u_j^1) \ldots (\tau_j^{\alpha_j}u_j^j)f_0(\tau_j, u_j),
\end{aligned}
\end{equation}

Since $f_j(\tau_j, u_j) = q_{j0}(u_j) + O(1)\tau_j$, it follows that to determine $q_{j0}$, for instance, we need only know $q_{00}, \ldots, q_{01}$, where

\begin{equation}
I = \left[ r \sum_{i=1}^{J} \beta_i / q_i \right],
\end{equation}

the entire part of $r \sum_{i=1}^{J} \beta_i / q_i$. Now it is easy to check from the Newton polygon for $f_{i-1}(\tau_{i-1}, c_{i-1} + v_i)$ that

\begin{equation}
\beta_i / \rho_i = \sum_{j} \mu_{ij} r_{ij} \min(s_{ij}, s_{ij}/\rho_i).
\end{equation}

Since also, in view of (1.9), when $\delta_{i,j} = \max(\delta_i, \delta_{ij})$,

\begin{equation}
\delta_{i-1} - \delta_{ij} = (1/\delta_{i-1}) \min(s_{ij}/\rho_i, s_{ij}/\rho_i),
\end{equation}

we see on comparison of formulas (2.23) and (3.5) that

\begin{equation}
d_{ij} = \max (\{ \sum_{i} \beta_i / q_i \}) / \mu_{ij}.
\end{equation}

Consequently, in view of (3.3) to determine all the minimal sequences belonging to $S_i$, we must know $q_{ai}$ for $i \leq I_{\lambda}$, with

\begin{equation}
I_{\lambda} = \left[ \mu_{a0}\rho_{a}\lambda \right] = [h_{a0}\lambda].
\end{equation}

It follows that all minimal sequences for $P$ are determined by the coefficients $c_{\alpha}$ with $\alpha \in H(P)$.

**Corollary 3.1:** Let $r_0^\alpha + s^\alpha = h^n$, $n = 1, \ldots, N$, be the sides of the Newton polygon $F(P)$ for a given polynomial $P(\xi) = \sum_{\alpha} c_{\alpha}\xi^\alpha$, $\xi \in R^2$.

Suppose the maximal multiplicity of a zero of $q_{00}(u) = q_{00}^\alpha(u)$ is $\mu_{00}^\alpha$, $q_{00}$ being defined by (1.2). Let $Q(\xi) = \sum_{\alpha} c_{\alpha}\xi^\alpha$, with

\begin{equation}
Q = \bigcup_{n=1}^{N} \{ \alpha \in (P); h^n \gg r_0^\alpha + s^\alpha > h^n - \mu_{00}^\alpha \}.
\end{equation}

Then $P$ is $\binom{2}{1}$-hypoelliptic if and only if $Q$ is $\binom{2}{1}$-hypoelliptic.
PROOF. All we have to do is observe that, according to the estimates (2.24),
\[ \max_{\lambda} h^2 \tilde{d}^2 = \max_{\lambda} \mu_\theta^{r+s} \tilde{d}^2 \leq \mu_\theta^{s}. \]

**Theorem 3.2:** Let \( P(\xi) = \sum_{\alpha} c_\alpha \xi^\alpha, \xi \in \mathbb{R}^2, \) and set \( Q(\xi) = \sum_{\alpha} c_\alpha \xi^\alpha, \) with \( (2.24) \)
\[ (Q) = \bigcup_{n=1}^{N} \{ \alpha \in (P); h^n > r^n \alpha^1 + s^n \alpha^2 \geq h^n - \mu_0^* \min (s^n, r^n) \}. \]

Then \( P \) is both \( (2^2) \) and \( (2^1) \)-hypoelliptic if and only if the same is the case for \( Q. \) If this condition is satisfied, then \( P \) is also hypoelliptic in the ordinary sense and strictly of the strength as a product \( P' = \Pi S_1, \) where every \( S_1 \) is a simple \( (2^2) \)-hypoelliptic and \( (1^2) \)-hypoelliptic polynomial at the same time. Moreover, the \( (2^2) \)- and \( (1^2) \)-hypoelliptic principal parts of \( P \) coincide, as well as those of each \( S_1. \)

**Proof.** Suppose \( P(\xi), \xi \in \mathbb{R}^2, \) is both \( (2^2) \) and \( (2^1) \)-hypoelliptic. Then, in view of Theorem (2.1), \( P \) is also \( (2^2) \) and \( (1^2) \)-hypoelliptic, hence hypoelliptic in the ordinary sense. Now let \( P' = p(\xi_1)\Pi S_1(\xi) \) be the \( (2^2) \)-hypoelliptic factorization of \( P. \) Then since \( P \) and \( P' \) are strictly equally strong (Lemma 2.1), it follows that \( P' \) is hypoelliptic. Every factor of a hypoelliptic polynomial being hypoelliptic, this means that each \( S_1 \) is hypoelliptic, and that \( p(\xi_1) \) is a constant \( C. \) Let now \( d_1 \) be the maximum of the numbers \( d \) for which \( |S_2(\xi)| > A( |\xi_1|^r + |\xi_2|^s)^{\mu(1-d)} \) all real \( \xi, |\xi| > K, \) where \( s, r \) and \( \mu_0 \) are determined by the leading part \( (\xi_1^r - c_0^2 \xi_1^s)^{\mu_0} \) of \( S_1. \) Then, due to Theorem 2.3, we know that \( d_1 < 1/r. \) But \( S_1 \) is also simple \( (1^2) \)-hypoelliptic, hence we must also have \( d_1 < 1/s. \) Further, since every \( d_2 \) is the same, whether it is determined with start from the \( (2^2) \)-hypoellipticity or from the \( (1^2) \)-hypoellipticity, it follows that \( P_2(\xi) \) is not only the \( (2^2) \)-hypoelliptic but also the \( (1^2) \)-hypoelliptic principal part of \( P(\xi). \) Finally, the equivalence of \( P \) and \( Q \) is proved as in Corollary 3.1.
In view of Theorem 3.2, if $P(\xi), \xi \in \mathbb{R}^2,$ is hypoelliptic, we can justly call $P_h(\xi)$ the (hypoelliptic) principal part of $P,$ and, for instance, $P' = \Pi(S_1)_n$ a canonical hypoelliptic factorization of $P.$ It may be worth noticing, that in spite of the truncation of $S_1$ to $(S_1)_n$ (which is made for the sake of symmetry), in general $P_h \neq c\Pi(S_1)_n.$ This is obvious from the following example.

Suppose that $P$ has the hypoelliptic factorization $P' = c\Pi S_1,$ where each $S_1$ is equal to its leading part $(\xi_2^2 - c_0\xi_1^{p_1})^{p_1},$ i.e. suppose that every minimal sequence for $P$ (with respect to $(1)$ or $(2)$-hypoellipticity) is of length $J = 0.$ Then obviously $d_\lambda = 0,$ for all $\lambda,$ and (2.25) becomes

\begin{equation}
A \geq |P(\xi)| \left/ \prod_\lambda \left( |\xi_1|^\alpha + |\xi_2|^\beta \right) \right. > A_1, \quad \xi \text{ real}, \quad |\xi| \geq K.
\end{equation}

Also $P_h(\xi) = P_h' = \sum F(e_{2\xi})$, where $F = F(P)$ is the Newton polygon of $P.$ In general $P_h(\xi) \neq c\Pi S_1 = c\Pi(\xi_2^2 - c_0\xi_1^{p_1})^{p_1},$ as is easily checked. Now, (3.10) is exactly the definition of a multi-quasielliptic polynomial, in the sense of Friberg [3]. Hence we get from the preceding discussion and from Theorem 2.1 the following result (Cf. [3]).

**Theorem 3.3:** If $\xi \in \mathbb{R}^2,$ then $P(\xi)$ is multi-quasielliptic, i.e. satisfies an estimate (3.10), if and only if one of the following two (equivalent) conditions is satisfied:

i) $P_h = P_h',$

ii) $a_{11}(P) = a_{22}(P) = 1.$

4. **Sufficient conditions for hypoellipticity.**

Let us now drop the condition $\xi = (\xi_1, \xi_2).$ It is then no longer possible to extend the results of section 2 concerning the canonical $(\binom{k}{j})-$
hypoelliptic factorization of a $(\binom{k}{j})-$
hypoelliptic polynomial. Counter-examples were given in Friberg [3], all of them multiquasielliptic in the generalized sense that they satisfy an estimate of the type

\begin{equation}
A \geq |P(\xi)| \left/ \sum |\xi^{a_i}| \geq A_1 \geq 0 \text{ for } \xi \text{ real}, \quad |\xi| \geq K.
\end{equation}

(Here the $a_i$ are a finite number of multi-indices $\geq 0.$) Nevertheless, it is sometimes possible also in the more-dimensional case to find a prin-
principal part of a \( \binom{k}{j} \) hypoelliptic polynomial. For instance for a multi-
quasielliptic polynomial (4.1), the principal part is always \( P_x = \sum_{F(P)} c_\alpha z^\alpha \), as in the two-dimensional case. (Friberg [3]).

To simplify the exposition, we shall in what follows mostly restrict our attention to the case when \( P(\xi) \) has a simple Newton surface \( F(P) \), given by an equation

\[
(4.2) \quad \Sigma \alpha_i/m_i = 1 \quad \text{for all} \quad \alpha \in F(P).
\]

**Theorem 4.1:** Suppose \( F(P) \) is given by (4.2), and that \( P \) satisfies, for \( \xi \) real, \( |\xi| \geq K \), an estimate

\[
(4.3) \quad A \gg |P(\xi)| / \Sigma |\xi_i|^{m_i} \gg A_1(\Sigma |\xi|^p)^{-d}.
\]

Then \( P(\xi) \) is \( \binom{k}{j} \)-hypoelliptic for all \( j \) if \( d < 1/m_k \), hypoelliptic if \( d < \min(1/m_k) \).

The proof is trivial, because we have

\[
(4.4) \quad |(\partial/\partial \xi_k)^j P(\xi)| \ll C(\Sigma |\xi_i|^{m_i})^{1-1/m_k} \ll C_1(\Sigma |\xi|^p)^{d-1/m_k} |P(\xi)|
\]

for \( |\xi| \geq K \), when \( P \) satisfies (4.3). Clearly, in case of a non-simple \( F(P) \), (4.3) must be replaced by an estimate of the type

\[
(4.5) \quad A \ll |P(\xi)| / \Sigma |\xi^{\alpha} | \ll A_1(\Sigma |\xi^{\alpha} |)^{-d}, \quad A_1 > 0,
\]

with \( d > 0 \), but small enough. We notice that (4.5) defines a class of hypoelliptic polynomials slightly larger than the class of all multi-quasielliptic polynomials.

**Corollary 4.1:** Let \( \xi = (\xi_1, \xi_2) \), and suppose that \( S(\xi) = (\xi_2 - c_0^1 \xi_1) + \) + terms \( c_{\xi_2} \) with \( r\alpha_1 + s\alpha_2 < rs \). If \( S(\xi) \) is \( \binom{2}{1} \)-hypoelliptic, then it is simple \( \binom{2}{1} \)-hypoelliptic, with leading part of multiplicity \( \mu_0 = 1 \), and

\[
(4.5) \quad |S(\xi)| \gg A( |\xi_1|^s + |\xi_2|^r)^{1-d}, \quad d = (1 - \delta_j/\delta_0)/r > 0,
\]

for \( \xi \) real, \( |\xi| \geq K \).

Conversely, if \( S(\xi) \) satisfies (4.5), then \( S \) is \( \binom{2}{1} \)- and \( \binom{2}{2} \)-hypoelliptic, with

\[
(4.6) \quad a_{12}(S) = (s/r - sd)^{-1} = 1/\delta_j, \quad a_{22}(S) = (1 - rd)^{-1} = \delta_0/\delta_j.
\]
Moreover, if $\delta_0 - \delta_r < 1$, for instance if $s < r$, then (4.5) implies that $S$ is hypoelliptic, with

\begin{equation}
(4.7) \quad a_{31}(S) = (1 - s\alpha)^{-1}, \quad a_{11}(S) = (r/s - r\delta)^{-1}.
\end{equation}

\textbf{PROOF.} The value of $d$ follows from (2.24). Conversely, if $S$ satisfies (4.5), then we can prove as in (4.4) that for $|\xi| \geq 1$

\begin{align*}
|\partial/\partial \xi_2^1 P(\xi)| &\leq C(|\xi_1|^s + |\xi_2|^r)^{1/2}, \\
|\partial/\partial \xi_2^i P(\xi)| &\leq C(|\xi_1|^s + |\xi_2|^r)^{1/2}.
\end{align*}

The first estimate gives the values of $a_{12}$ and $a_{22}$, the same as were computed in Theorem 2.1. The second inequality implies the $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$- and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$-hypoellipticity, provided that $a_{11}$ and $a_{11}$, as given by (4.7), are positive.

A first example is the polynomial (2.11), for which $\delta_0 - \delta_r = 2/3$. As a second example, consider a primitive Silov-parabolic polynomial $S(\xi) = (\xi_1^2 + \xi_2^2 + S_1(\xi_1)) + S_1$ real, degree $S_1 = m_1$. Here $\delta_0 = \max(2p, m_1)$, and $\delta_r = 2p$. Hence $\delta_0 - \delta_r < 1$ if and only if $m_1 < 2p$. Consequently a Silov-parabolic polynomial is in general not hypoelliptic. This means that the definition of parabolicity given by Hörmander [6, p. 152] is more restrictive than Silov’s definition.

Let us return to the general more-dimensional case. Generalizing an observation due to Hörmander [6, p. 103] we have the following result, showing the existence of hypoelliptic polynomials with simple Newton surface and an almost arbitrary (real) leading part. Let

\begin{equation}
P = Q + iR_r, \quad Q = Q_1^\mu \ldots Q_r^\mu,
\end{equation}

where the $Q_i$ are real polynomials with every $F(Q_i)$ parallel to $F(P)$, $\Sigma a_i/m_i \leq 1$ for $\alpha \in (P)$, and where $R_r$ is real quasielliptic, $\Sigma \alpha_i/m_i = \gamma < 1$ for $\alpha \in (R_r)$. Then $P$ is $\begin{pmatrix} k \\ j \end{pmatrix}$-hypoelliptic for all $j$ provided that

\begin{equation}
(4.7) \quad \gamma > 1 - \mu/m_\gamma, \quad \mu = \min \mu_i.
\end{equation}

For the proof we first notice that, for instance,

\begin{equation}
(4.8) \quad |\partial/\partial \xi_2^i R_r(\xi)| = o(1) |R_r(\xi)| = o(1) |P(\xi)| \text{ as } |\xi| \to \infty, \xi \text{ real},
\end{equation}
because $|P| \geq \max (|Q|, |R|)$. Next we write $\mu_i = \varepsilon_i \mu$, $i = 1, \ldots, N$, so that $\min \varepsilon_i = 1$, and $|Q| = (\Pi |Q_i|^\nu)^\mu$. It follows that $|\partial / \partial \xi \xi Q(\xi)|$ can be estimated by a sum of terms like

$$ |Q \cdot \partial / \partial \xi \xi Q_i / Q_i| = |Q|^{1-1/\mu} \cdot \prod_{i=1}^{N} |Q_i|^{\varepsilon_i} \cdot |Q_i|^{\varepsilon_i-1} |\partial / \partial \xi \xi Q_i| \leq$$

$$ \leq A |Q|^{1-1/\mu}(\sum |\xi_j|^{m_j})^{1-1/\mu} \leq A |Q|^{1-1/\mu} |R_\gamma|^{(1/m-1/m_b)} \gamma$$

for $\xi$ real, $|\xi|$ large enough. Consequently

$$ |\partial / \partial \xi \xi Q| = O(1) |P| \text{ as } |\xi| \to \infty,$$

if we assume that $(1/\mu - 1/m_b)/\gamma < 1/\mu$, or $\gamma > 1 - \mu/m_b$, which is precisely (4.7). We notice that if $Q$ is not itself $\gamma$-hypoelliptic, then $R_\gamma$ must be considered to belong to the $\left(\begin{array}{c}k \\ j \end{array}\right)$-hypoelliptic principal part of $P$, for any sensible definition of the principal part in the more-dimensional case. (Cf. Pini [7], p. 11). It is also easy to find examples where $\gamma \leq 1 - \mu/m_b$ and $P$ is not $\left(\begin{array}{c}k \\ j \end{array}\right)$-hypoelliptic.

Consider now instead a polynomial $P = \sum_{0}^{N} P_i$, $P_i = \prod_{i} (Q_{i\mu})^{\alpha_i}$, $Q_i$ positive semi-definite, and suppose that $\sum_{i=1}^{N} \alpha_i/m_i \leq \gamma_i \leq \gamma_0$ for $\alpha \in (P_i)$. Set $\mu_i = \min \mu_{\alpha_i}$. Then the estimate

$$ (\sum |\xi_i|^{\nu})^{\gamma} \leq C(1 + |P(\xi)|) \gamma \geq \max_{i} (\gamma_i - \mu_i/m_i),$$

is a sufficient condition for $\left(\begin{array}{c}k \\ j \end{array}\right)$-hypoellipticity. (The proof is the same as in the preceding example). Notice that this result indicates that, as in the two-dimensional case, the form of the principal part of $P$ does not depend exclusively on the leading part $P_0$. 

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