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ON THE APPROXIMATION OF CONTINUOUS
FUNCTIONS BY POLYNOMIALS ON $(-\infty, \infty)$
AND $(0, \infty)$ IN TERMS OF EXPONENTIAL
WEIGHT FACTOR

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I. Introduction

On studying the completeness of the Hermite and Laguerre polynomials, one is led to the consideration of the following more general problem.

Suppose

$X = \{f \mid f \text{ a real-valued function continuous over the real line; } \lim_{x \rightarrow \pm\infty} |f(x)| e^{-\delta|x|^\alpha} = 0, \alpha > 0, \delta > 0\}$,

$Y = \{p \mid p \text{ a polynomial in } x\}$.

Define, for $f \in X$

$$\|f\| = \sup_{-\infty < x < \infty} |f(x)e^{-\delta|x|^\alpha}|.$$

The problem is: can any f in X be approximated by polynomials in the above defined norm $\|\cdot\|$? In this paper, we show that the above mentioned f can be approximated by polynomials in this norm if $\alpha \geq 1$, but not for $0 < \alpha < 1$. The demarcation is shifted to $\alpha = 1/2$, if the norm is defined as

$$\|f\|_{1/2} = \sup_{x \geq 0} \{ |f(x)| e^{-\delta|x|^\alpha} \}.$$

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The proof is based on reducing this problem to a corresponding problem in the theory of moments. From these results we find that a theorem of Carlson [1] follows as a natural consequence. It is also clear that convergence in the norm $\| \cdot \|$ implies convergence in $L^p(\mu)$, $1 \leq p < \infty$, for any measure μ on $(-\infty, \infty)$ satisfying

$$\int_{-\infty}^{\infty} e^{\varepsilon|x|^\alpha} d\mu(x) < \infty, \quad \varepsilon = p\delta.$$

II. The Main Theorem

The following is our main theorem where X , Y and the norm are defined as in the introduction.

THEOREM 1: *If $\alpha \geq 1$, then Y is dense in X . However this conclusion is false if $0 < \alpha < 1$.*

Before presenting the proof, we need the following well known theorem of Riesz [5, p. 115].

THEOREM 2 (Riesz): *Y is dense in X if and only if there does not exist a nontrivial finite signed measure μ such that*

$$\text{I) } \int_{-\infty}^{\infty} x^n d\mu(x) = 0, \quad n = 0, 1, 2, \dots,$$

$$\text{II) } \int_{-\infty}^{\infty} e^{\delta|x|^\alpha} d|\mu(x)| < \infty.$$

We also need

LEMMA 3: *If μ is a finite signed measure satisfying the conditions I) and II) of theorem 2, and if $\alpha \geq 1$, $\delta > 0$, then μ is trivial, i.e.*

$$\int_{-\infty}^{\infty} f d\mu = 0, \quad \forall f \in X.$$

Proof.: Let μ^+ and μ^- be the positive and negative part of μ respectively. Then from I),

$$\text{III) } \int_{-\infty}^{\infty} x^n d\mu^+ = \int_{-\infty}^{\infty} x^n d\mu^-, \quad n = 0, 1, 2, \dots$$

From II) we have

$$\text{IV) } \int_{-\infty}^{\infty} e^{\delta|x|^\alpha} d\mu^+ < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} e^{\delta|x|^\alpha} d\mu^- < \infty.$$

By a theorem of Carleman [2], a sufficient condition that two positive measures μ^+ and μ^- satisfying III) are equal is that $\sum_1^{\infty} \mu_{2n}^{-1/2n}$ diverges, where

$$\mu_n = \int_{-\infty}^{\infty} x^n d\mu^+ = \int_{-\infty}^{\infty} x^n d\mu^-, \quad n = 1, 2, \dots$$

An easy estimation, based on IV), shows that $\mu_{2n} = 0[(2n)^{2n}]$ and therefore $\sum_1^{\infty} \mu_{2n}^{-1/2n}$ diverges. (The sufficiency of condition IV) is due to Hardy [3]. For more details, see [6, p. 19]).

Hence $\mu^+ = \mu^-$, i.e.

$$\int f d\mu^+ = \int f d\mu^-, \quad \forall f \in X,$$

or

$$\int f d\mu = 0, \quad \forall f \in X.$$

This means μ is trivial and the lemma is proved.

Now we are ready to prove the main theorem.

Proof of theorem 1: If $\alpha \geq 1$, $\delta > 0$, then by lemma 3, there does not exist a nontrivial measure satisfying I) and II) of theorem 2. Hence by the sufficiency of theorem 2, Y is dense in X .

However, for $0 < \alpha < 1$, there does exist a nontrivial finite measure $g(t)dt$ satisfying I) and II): namely let

$$g(x) = \operatorname{Re} \left[e^{-\left(\frac{x}{i}\right)^\beta} \right] = e^{-(\cos \frac{\beta\pi}{2})|x|^\beta} \cos \left[\left(\sin \frac{\beta\pi}{2} \right) |x|^\beta \right]$$

where $\alpha < \beta < 1$. Hence, by the same theorem, Y is not dense in X if $0 < \alpha < 1$, and the proof is complete.

As an application, we give an independent proof of the following theorem due to Carlson [1].

THEOREM 4: *If $f(z)$ is regular and of the form $0(e^{k|z|})$ for $\operatorname{Im}(z) \geq 0$ and*

$$f(z) = 0(e^{-a|z|^\alpha}), \quad a > 0, \quad \alpha \geq 1,$$

on the line $\operatorname{Im}(z) = 0$, then $f(z) = 0$ identically.

Proof.: Let $g(z) = e^{imzf(z)}$, where $m > k$, and

$$\bar{g}(\beta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{i\beta x} dx .$$

Then by the Paley-Wiener Theorem [4], the derivatives

$$\bar{g}^{(n)}(0) = 0, \quad n = 0, 1, 2, \dots,$$

i.e.

$$\int_{-\infty}^{\infty} t^n g(t) dt = 0, \quad n = 0, 1, 2, \dots .$$

Therefore $g(t)dt$ satisfies the conditions of lemma 3, provided the constant δ is chosen less than a . By lemma 3 $g(t)dt$ is a trivial measure, i.e. $g(x) = 0$ for every x . Thus by the unique continuation theorem $g(z) \equiv 0$; hence $f(z) \equiv 0$ and the theorem is proved.

In the following, we shall give the result corresponding to theorem 1 for the case where

$\tilde{X} = \{f \mid f \text{ a real-valued function continuous over } x \geq 0,$

$$\lim_{x \rightarrow \infty} |f(x)| e^{-\delta x^\alpha} = 0, \alpha > 0, \delta > 0\},$$

$\tilde{Y} = \{p \mid p \text{ a polynomial in } x, x \geq 0\},$

and

$$\|f\|_{1/2} = \sup_{x \geq 0} |f(x)e^{-\delta x^\alpha}|.$$

THEOREM 5: *If $\alpha \geq 1/2$, then \tilde{Y} is dense in \tilde{X} . However this conclusion is false if $0 < \alpha < 1/2$.*

The proof follows from a modified form of Carleman's theorem (see [6, p. 20]), and is otherwise exactly similar to the proof of theorem 1.

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