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## REMARKS ON SOME OPERATIONAL FORMULAS

*Nota \*) di W. W. A. AL-SALAM (a Lubboch)\*\**

Let  $y_n(x, a, b)$  be the Bessel polynomial of Krall and Frink defined by means of

$$y_n(x, a, b) = b^{-n} x^{2-a} e^{b/2} D^n (x^{2n+a-2} e^{-b/x}).$$

where  $D = d/dx$ . Chatterjea proved in [1] the operational formula

$$(1) \quad \prod_{j=1}^n (x^2 D + (2j + a)x + b) = \sum_{r=0}^n \binom{n}{r} b^{n-r} x^{2r} y_{n-r}(x, a + 2r + 2, b) \cdot D^r$$

where the terms of the operator on the left hand side of (1) do not commute among themselves and the operator must be interpreted to mean

$$\left\{ \prod_{j=1}^{n-1} (x^2 D + (2j + a)x + b) \right\} \cdot (x^2 D + (2n + a)x + b).$$

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He also gave in [2] the formula

$$(2) \quad x^{2n} \left[ D + \frac{2(nx + 1)}{x^2} \right]^n = \sum_{r=0}^n \binom{n}{r} 2^{n-r} x^{2r} y_{n-r}(x, 2 + 2r, 2) D^r .$$

We remark that (2) can be generalized to involve the generalized Bessel polynomial  $y_n(x, a, b)$ . In fact we prove first the following theorem.

**THEOREM 1.**

$$(3) \quad x^{2n} \left\{ D + \frac{(a + 2n)x + b}{x^2} \right\}^n = \prod_{j=1}^n (x^2 D + (2j + a)x + b) .$$

To prove this theorem we first note that

$$(4) \quad \left\{ x^\lambda D + g(x) \right\}^n x^v Y = x^v \left\{ x^\lambda D + g(x) + \frac{v}{x} \right\}^n Y$$

which can be proved by induction.

It is now easy to prove (3) by induction making use of (4).

Combining (1) and (3) we get

$$(5) \quad x^{2n} \left\{ D + \frac{(a + 2n)x + b}{x^2} \right\}^n = \sum_{r=0}^n \binom{n}{r} b^{n-r} x^{2r} y_{n-r}(x, a + 2r + 2, b) D^r .$$

We next show that (1) and (5) (and hence (2)) are special cases of some results of Gould and Hopper [3].

Let

$$(6) \quad \mathfrak{D} = D - px^{r-1} + \frac{a}{x}$$

and

$$(7) \quad H_n^r(x, a, p) = (-1)^n x^{-a} e^{px^r} D^n (x^a e^{-px^r}) .$$

Gould and Hopper proved, among other results,

$$(8) \quad \mathfrak{D}^n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} H_{n-k}^r(x, a, p) D^k$$

and

$$(9) \quad x^n \mathfrak{D}^n = \prod_{j=0}^{n-1} (xD - p + x^j - a - j).$$

It is easy to see that (7) and

$$(10) \quad y_n(x, a, b) = b^{-n} x^{2-a} e^{b/x} D^n (x^{2n+a-2} e^{-b/x})$$

imply the relation

$$y_n(x, a, b) = (-1)^n b^{-n} x^{2n} H_n^{-1}(x, an + a - 2, b)$$

so that

$$(11) \quad y_{n-r}(x, a + 2r + 2, b) = (-1)^{n-r} b^{r-n} x^{2n-2r} H_{n-r}^{-1}(x, 2n + a, b).$$

Consequently putting  $r = -1$  and  $a = 2n + a$  in formula (8) we get

$$\left( D + \frac{(2n + \alpha)x + b}{x^2} \right)^n = \sum_{r=0}^n \binom{n}{r} b^{r-n} x^{2r-2n} y_{n-r}(x, \alpha + 2r + 2, b) D^r$$

which is formula (5). Putting  $\alpha = 0$ ,  $b = 2$  we get (2).

Another operational formula similar to (3) is contained in the following theorem.

**THEOREM 2.**

$$(12) \quad (x^{v+1}D + x^v g(x))^n = x^{nv} \prod_{j=1}^n (xD + g(x) + (n-j)v).$$

**Proof.** (By induction). Formula (12) is obviously true for  $n = 1$ . Assume it is true for  $n = m$  and consider

$$\begin{aligned}
(x^{v+1}D + x^v g(x))^{m+1} Y &= \\
&= x^v (xD + g(x)) x^{mv} \prod_{j=1}^m (xD + g(x) + (m-j)v) Y = \\
&= x^{(m+1)v} (xD + g(x) + mv) \prod (xD + g(x) + (m-j)v) Y = \\
&= x^{(m+1)v} \prod_{j=0}^n (xD + g(x) + (m-j)v) Y = \\
&= x^{(m+1)v} \prod_{j=0}^{m+1} (xD + g(x) + (m+1-j)v) Y .
\end{aligned}$$

Thus (12) is also true for  $n = m + 1$ . This completes the proof of (12).

One interesting special case of (12), in terms of the Gould-Hopper operator  $\mathfrak{D}$ , is obtained by taking  $g(x) = a - px^r$  we get

$$(13) \quad (x^{v+1}\mathfrak{D})^n = x^{nv} \prod_{j=1}^n (xD - px^r + a + (n-j)v) .$$

which may be compared with (9).

#### REFERENCE

- [1] CHATTERCEA S. K.: *Operational formulae for certain classical polynomials*. Quarterly Journal of Mathematics, Oxford, (2), vol. 14 (1963), pp. 241-246.
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- [3] GOULD H. W. and HOPPER A. T.: *Operational Formulas connected with two generalizations of Hermite polynomials*. Duke Mathematical Journal, vol. 29 (1962), pp. 51-64.