S. K. CHATTERJEA

On a generalization of Laguerre polynomials

Rendiconti del Seminario Matematico della Università di Padova, tome 34 (1964), p. 180-190

<http://www.numdam.org/item?id=RSMUP_1964__34__180_0>
ON A GENERALIZATION OF LAGUERRE POLYNOMIALS

Nota *) di S. K. CHATTERJEA (a Calcutta)

1. - In a recent paper [1], the writer has defined the polynomials $T_{kn}^{(a)}(x)$ by the Rodrigues' formula

$$T_{kn}^{(a)}(x) = \frac{1}{n!} x^{-2k}e^{x}D^{n}(x^{\alpha+n}e^{-x}) ,$$

where $k$ is a natural number. The polynomials $T_{kn}^{(a)}(x)$ are of exactly degree $kn$ ($n = 0, 1, 2, ...$). They satisfy the operational formula

$$\prod_{j=1}^{n} (xI - kx^{k} + \alpha + j) = n! \sum_{r=0}^{n} \frac{x^{r}}{r!} T_{k(n-1)}^{(a+r)}(x)D^{r} .$$

The following are the consequences of the operational formula (1.2):

$$nT_{kn}^{(a)}(x) = (xD - kx^{k} + \alpha + n)T_{k(n-1)}^{(a)}(x)$$

$$\binom{m+n}{m} T_{k(n+s)}^{(a)}(x) = \sum_{r=0}^{\min(m,n)} \frac{x^{r}}{r!} T_{k(n-s-r)}^{(a)}(x)D^{r}T_{kn}^{(a)}(x) .$$

The polynomials $T_{kn}^{(a)}(x)$ are generated by the function

$$(1 - t)^{-a-1} \exp \{ x^{a}u(t) \} = \sum_{n=0}^{\infty} T_{kn}^{(a)}(x)t^{n}$$

*) Pervenuta in redazione il 17 giugno 1963.

Indirizzo dell'A.: Department of mathematics. Bangabasi College, Calcutta (India).
where
\[ u(t) = 1 - (1 - t)^{-k}. \]

In the same paper the writer has also proved the following properties
\[
\sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} (n+1-r) T_{k+1-r}^{(a)}(x) = \alpha + 1 \sum_{r=0}^{k} (-1)^r \binom{k}{r} T_{k-r}^{(a)}(x) - kx^k T_{kn}^{(s)}(x)
\]
(1.6)
\[
\sum_{r=0}^{k} (-1)^r \binom{k}{r} DT_{k-r}^{(a)}(x) = kx^{k-1} \sum_{r=1}^{k} (-1)^r \binom{k}{r} T_{k-r}^{(a)}(x)
\]
(1.7)
\[
T_{kn}^{(a)}(x) = \sum_{r=0}^{n} \frac{\alpha - \beta}{r!} T_{k-n-r}^{(s)}(x)
\]
(1.8)

This work of the writer generalizes some properties of the Laguerre polynomials \( L_n^{(a)}(x) \). Indeed, when \( k = 1 \), \( T_{kn}^{(a)}(x) \equiv L_n^{(a)}(x) \). A similar generalization viz., \( T_{kn}^{(a)}(x) \), has been previously studied by Palas [2]. The purpose of this paper is to discuss a more general class of Laguerre polynomials.

2. DEFINITION: We first make the definition

\[
T_{kn}^{(a)}(x, p) = \frac{1}{n!} x^{-\alpha - \beta} D^n(x^a + \beta e^{-\beta k})
\]
(2.1)

where \( k \) is a natural number.

We now show that the polynomial \( T_{kn}^{(a)}(x, p) \) is of exactly degree \( kn \) \( (n = 0, 1, 2, ...) \). In this connection we know the result [3], for which I must thank Prof. H. W. Gould:

\[
D^k(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} D^k(x) \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^{k-j} D^j x^j
\]
(2.2)

Thus we obtain from (2.1)

\[
T_{kn}^{(a)}(x, p) =
\]

\[ = \frac{1}{n!} x^{-\alpha - \beta} \sum_{s=0}^{n} \binom{n}{s} (D^{n-xa} + \beta) (D^x e^{-\beta k}) =
\]
Now we know that
\[
\sum_{j=0}^{i} (-1)^j \binom{i}{j} \sum_{s=0}^{i-1} \binom{\alpha + n}{n-s} = 0.
\]

Thus we finally obtain

\[
\begin{align*}
T^{(a)}_{km}(x, p) &= \sum_{i=0}^{n} \frac{p^i}{i!} x^{ki} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{\alpha + n + kj}{n} \\
&= \sum_{i=0}^{n} \frac{x^i}{i!} \sum_{j=0}^{i} (-1)^j \binom{\alpha + n + j}{n-j}
\end{align*}
\]

which is the explicit formula for \(T^{(a)}_{km}(x, p)\).

In particular, when \(k = 1\), and \(p = 1\), we derive

\[
T^{(a)}_{n, 1}(x, 1) = \sum_{i=0}^{n} \frac{x^i}{i!} \sum_{j=0}^{i} (-1)^j \binom{\alpha + n + j}{n-j} = \sum_{i=0}^{n} \frac{x^i}{i!} \cdot (-1)^i \binom{\alpha + n}{n-i}
\]

which is the explicit formula for the general Laguerre polynomials \(L^{(a)}_{n}(x)\). Thus \(T^{(a)}_{n, 1}(x, 1) \equiv L^{(a)}_{n}(x)\).

3. Operational formulae: Recently we [4] have derived the general operational formula

\[
x^{-a} D^n (x^{n+a} Y) = \prod_{j=1}^{n} \{x^{k-1}(z + \alpha + kj)\} Y,
\]

\((k = 1, 2, 3, ...),\)

where \(z \equiv x D\) and \(Y\) is any sufficiently differentiable function of \(x\). The operators on the right of (3.1) commute only when \(k = 1\).
Thus we derive

\[(3.2) \quad x^{-z}e^{z}D^n(x^ne^{-pz}Y) = \prod_{j=1}^{n} (xD - pkx^j + \alpha + j)Y\]

Again we observe

\[D^n(x^ne^{-pz}Y) = \sum_{r=0}^{n} \binom{n}{r} D^{n-r}(x^ne^{-pz})D^rY =\]

\[= n! x^{-z}e^{z} \sum_{r=0}^{n} \frac{x^r}{r!} T_{k(a-r)}^{(a+\gamma)}(x, p)D^rY,\]

whence we obtain

\[(3.3) \quad \frac{1}{n!} x^{-z}e^{z}D^n(x^ne^{-pz}Y) = \sum_{r=0}^{n} \frac{x^r}{r!} T_{k(a-r)}^{(a+\gamma)}(x, p)D^rY\]

It therefore follows from (3.2) and (3.3) that

\[(3.4) \quad \prod_{j=1}^{n} (xD - pkx^j + \alpha + j)Y = n! \sum_{r=0}^{n} \frac{x^r}{r!} T_{k(a-r)}^{(a+\gamma)}(x, p)D^rY\]

If we set \(Y = 1\), we derive from (3.4)

\[(3.5) \quad n! T_{k(a)}^{(a+\gamma)}(x, p) = \prod_{j=1}^{n} (xD - pkx^j + \alpha + j)\cdot 1\]

Further if \(k = 1\), and \(p = 1\), we obtain from (3.4)

\[(3.6) \quad \prod_{j=1}^{n} (xD - x + \alpha + j)Y = n! \sum_{r=0}^{n} \frac{x^r}{r!} T_{a-r}^{(a+\gamma)}(x, 1)D^rY;\]

which may be compared with the operational formula for the general Laguerre polynomials, derived by Carlitz [5].

In a recent paper [6], Gould and Hopper have generalized the Hermite polynomials by the definition

\[(3.7) \quad H_n^p(x, \alpha, p) = (-1)^n x^{-z}e^{z}D^n(x^ne^{-pz})\]
We remark that $x^n H_{n}(x, \alpha, p)$ yields a generalized class of polynomials of exactly degree $kn$ ($n = 0, 1, 2, ...$), provided $k$ is a natural number. Consequently if we write

$$x^n H_{n}(x, \alpha, p) = H^{(\alpha)}_{n}(x, p)$$

then

$$H^{(\alpha)}_{n}(x, p) = (-1)^n x^{n-\alpha^2 x^2} D^n(x^2 e^{-x^2})$$

Thus the polynomials $H^{(\alpha)}_{n}(x, p)$ are related to our polynomials by

$$H^{(\alpha)}_{n}(x, p) = (-1)^n n! T^{(n-\alpha)}_{n}(x, p).$$

Now returning to the operational formula (3.5) we obtain

$$(-1)^n H^{(\alpha)}_{n}(x, p) = \prod_{j=1}^{n} (xD - pkx^2 + \alpha - n + j) \cdot 1$$

More generally we have from (3.4)

$$\prod_{j=1}^{n} (xD - pkx^2 + \alpha - n + j) Y = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} x^r H^{(\alpha)}_{n-r}(x, p) D^r Y$$

This operational formula viz., (3.11) seems to be of particular interest. Indeed, using $k = 2$, $p = 1$, and $\alpha = 0$, we have

$$\prod_{j=1}^{n} (xD - 2x^2 - n + j) Y = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} x^r H^{(\alpha)}_{n-r}(x, 1) D^r Y$$

Now noticing that

$$H^{(\alpha)}_{n-r}(x, 1) = x^{n-r} H_{n-r}(x),$$

where $H_{n}(x)$ denotes the ordinary Hermite polynomials defined by

$$H_{n}(x) = (-1)^n e^{x^2} D^n e^{-x^2},$$
we obtain
\[
\prod_{j=1}^{n} (xD - 2x^2 - n + j) Y = \left(\sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} H_{n-r}(x) D^r Y \right)
\]

Now we note that

\[
(3.12) \quad x^{-n} \prod_{j=1}^{n} (xD - 2x^2 - n + j) \equiv (D - 2x)^n.
\]

For, (3.12) is evidently true for \(n = 1\). Next assume that (3.12) is true for \(n = m\). Then we have

\[
x^{-1} \prod_{j=1}^{m+1} (xD - 2x^2 - m - 1 + j) = \\
\quad = x^{-(m+1)} (xD - 2x^2 - m) \prod_{j=1}^{m} (xD - 2x^2 - m + j) = \\
\quad = x^{-(m+1)} (xD - 2x^2 - m) x^m (D - 2x)^m = \\
\quad = x^{-(m+1)} x^m (xD - 2x^2) (D - 2x)^m = \\
\quad = (D - 2x)^{m+1}.
\]

Hence by induction (3.12) is true for all positive integers \(n\). Thus we finally derive

\[
(3.13) \quad (D - 2x)^n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} H_{n-r}(x) D^r,
\]

a formula which Burchnall [7] derived some years ago.

4. SOME APPLICATIONS OF THE OPERATIONAL FORMULA:

From (3.5) we note that

\[
n T_{\alpha}^{(n)}(x, p) = (xD - p k x^\alpha + \alpha + n) T_{\alpha}^{(n-1)}(x, p).
\]

In particular, when \(k = 1\), and \(p = 1\), we derive

\[
n T_{\alpha}^{(n)}(x, 1) = (xD - x + \alpha + n) T_{\alpha}^{(n-1)}(x, 1)
\]

which is well-known for the Laguerre polynomials \(L_{n}^{(\alpha)}(x)\).
Again in terms of the polynomials of Gould and Hopper, (4.1) stands thus

\begin{equation}
H_{\frac{m}{n}}^{(x)}(x, p) + (xD - pxk + \alpha)H_{\frac{m}{n} - 1}(x, p) = 0.
\end{equation}

Next we consider

\[
(m + n)! T_{\frac{m}{n}}^{(x)}(x, p) =
\]

\[
= \prod_{j=1}^{m} (xD - pxk + \alpha + n + j) \prod_{i=1}^{n} (xD - pxk + \alpha + i) \cdot 1
\]

\[
= n! \prod_{j=1}^{m} (xD - pxk + \alpha + n + j) \cdot T_{\frac{m}{n}}^{(x)}(x, p)
\]

\[
= m! n! \sum_{r=0}^{m} \frac{x^r}{r!} T_{\frac{m}{n} - r}^{(x)}(x, p) D^r T_{\frac{m}{n}}^{(x)}(x, p);
\]

which implies that

\begin{equation}
\binom{m + n}{m} T_{\frac{m}{n}}^{(x)}(x, p) = \sum_{r=0}^{\min(m, n)} \frac{x^r}{r!} T_{\frac{m}{n} - r}^{(x)}(x, p) D^r T_{\frac{m}{n}}^{(x)}(x, p).
\end{equation}

The formula (4.4) readily yields the corresponding formula for the polynomials of Gould and Hopper:

\begin{equation}
H_{\frac{m}{n}}^{(x) + (x)}(x, p)
\end{equation}

\[
= \sum_{r=0}^{\min(m, n)} (-1)^r \binom{m}{r} x^r H_{\frac{m}{n} - r}^{(x) + (x)}(x, p) D^r H_{\frac{m}{n}}^{(x) + (x)}(x, p).
\]

5. - GENERATING FUNCTION: We shall now show that the polynomials \(T_{\frac{m}{n}}^{(x)}(x, p)\) are generated by

\begin{equation}
g(x, t) = (1 - t)^{x-1} \exp \left[ px^k u(t) \right] = \sum_{n=0}^{\infty} T_{\frac{m}{n}}^{(x)}(x, p) t^n,
\end{equation}

where

\[ u(t) = 1 - (1 - t)^{-k}. \]
From the definition (2.1) we observe

\begin{equation}
T_{\nu}(x, p) = e^{x^*t} \sum_{r=0}^{\infty} \frac{(-p)^r}{r!} \left( \frac{k_r + \alpha + n}{n} \right) x^r.
\end{equation}

It may be noted that (2.3) is a consequence of (5.2).

Now we notice that

\begin{equation}
T_{\nu}^{(a)}(x, p) = \frac{1}{n!} \left[ \frac{d^n}{dx^n} g(x, 0) \right]
\end{equation}

Also

\begin{align*}
\left[ \frac{d^n}{dt^n} \left( (1 - t)^{-x-1} \exp(p x^* u(t)) \right) \right]_{t=0} \\
= e^{x^*t} \left[ \frac{d^n}{dt^n} \left( (1 - t)^{-x-1} \exp \left( -p \left( \frac{x}{1-t} \right)^k \right) \right) \right]_{t=0} \\
= n! e^{x^*t} \sum_{r=0}^{\infty} \frac{(-p)^r}{r!} \left( \frac{k_r + \alpha + n}{n} \right) x^r
\end{align*}

Thus a comparison of (5.3) and (5.4) with (5.2) confirms (5.1).

Now from the generating function (5.1) we easily derive the following multiplication formula:

\begin{equation}
T_{\nu}^{(a)}(x m^1/n, p) = T_{\nu}^{(a)}(x, mp),
\end{equation}

which, in terms of the polynomials of Gould and Hopper, shapens into

\begin{equation}
H_{\nu}^{(a)}(x m^1/n, p) = H_{\nu}^{(a)}(x, mp),
\end{equation}

which may well be compared with (3.9) of [6, p. 54].

It is also interesting to note from (5.5) that

\begin{equation}
T_{\nu}^{(a)}(x, m) = I_{\nu}^{(a)}(x, m).
\end{equation}

Again we observe

\begin{align*}
(1 - t)^{-x-1} \exp \left[ p x^* t \{1 - (1 - t)^{-t}\} \right] \\
= (1 - t)^{(x - r) \left( 1 - t \right)^{-x-1} \exp \left[ p x^* t \{1 - (1 - t)^{-t}\} \right]}
\end{align*}
whence we obtain

\[ \sum_{n=0}^{\infty} T^{(a)}_m(x, p)t^n = (1 - t)^{(a - \beta)} \sum_{n=0}^{\infty} T^{(b)}_m(x, p)t^n. \]

Now comparing the coefficients of \( t^n \) on both sides we get

\[ T^{(a)}_m(x, p) = \sum_{r=0}^{n} \frac{(\alpha - \beta)_r}{r!} T^{(b)}_{a-r}(x, p), \]

(5.8)

where \( \alpha \) and \( \beta \) are arbitrary real numbers.

Next we notice that

\[ \sum_{n=0}^{\infty} T^{(a+b+1)}_m(x, p+q)t^n \]

\[ = (1 - t)^{a-1}e^{xqz}(1-(1-t)^{q-k}) \cdot (1 - t)^{b-1}e^{xqz}(1-(1-t)^{-k}) \]

\[ = \sum_{m=0}^{\infty} T^{(a)}_m(x, p)t^m \cdot \sum_{n=0}^{\infty} T^{(b)}_n(x, q)t^n \]

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{n} T^{(a)}_m(x, p)T^{(b)}_{a-m}(x, q)t^n. \]

Thus we obtain the following 'doubly-additive' addition formula

\[ T^{(a+b+1)}_m(x, p+q) = \sum_{m=0}^{n} T^{(a)}_m(x, p)T^{(b)}_{a-m}(x, q). \]

(5.9)

In particular, when \( p = q = 1 \), and \( k = 1 \), we derive

\[ T^{(a+b+1)}_m(x, 2) = \sum_{m=0}^{n} L^{(a)}_m(x)L^{(b)}_{a-m}(x). \]

(5.10)

It follows therefore from (5.7) and (5.10) that

\[ L^{(a+b+1)}(2x) = \sum_{m=0}^{n} L^{(a)}_m(x)L^{(b)}_{a-m}(x), \]

(5.11)
which is implied by the well-known formula of the Laguerre polynomials

\[(5.12) \quad L_{n}^{(x+\beta+1)}(x + y) = \sum_{m=0}^{n} L_{n}^{(x)}(x) L_{m-n}^{(\beta)}(y).\]

Again returning to (5.1) we obtain

\[ (1 - t)^{k+1} \frac{\partial g(x, t)}{\partial t} = [(\alpha + 1)(1 - t)^k - p k x^k] g(x, t) \]

whence we notice

\[ (1 - t)^{k+1} \sum_{n=1}^{\infty} n t^{n-1} T_n^{(x)}(x, p) = [(\alpha + 1)(1 - t)^k - p k x^k] \sum_{n=0}^{\infty} T_n^{(x)}(x, p) t^n. \]

Performing the indicated multiplication on both sides and comparing coefficients of \( t^n \) on both sides, we derive

\[(5.13) \quad \sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} (n + 1 - r) T_{n+r}^{(x+1-r)}(x, p). \]

\[= (\alpha + 1) \sum_{r=0}^{k} (-1)^r \binom{k}{r} T_{n+r}^{(x-r)}(x, p) - p k x^k T_n^{(x)}(x, p). \]

Lastly we observe

\[ (1 - t)^{k} \frac{\partial g(x, t)}{\partial x} = p k x^{k-1} (1 - t)^k - 1) g(x, t), \]

whence we obtain in like manner

\[(5.14) \quad \sum_{r=0}^{k} (-1)^r \binom{k}{r} D T_{n+r}^{(x-r)}(x, p) \]

\[= p k x^{k-1} \sum_{r=1}^{k} (-1)^r \binom{k}{r} T_{n+r}^{(x-r)}(x, p). \]
REFERENCES


