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ON A GENERALIZATION OF LAGUERRE POLYNOMIALS

Nota *) *di* S. K. CHATTERJEA (*a Calcutta*)

1. – In a recent paper [1], the writer has defined the polynomials $T_{kn}^{(\alpha)}(x)$ by the Rodrigues' formula

$$T_{kn}^{(\alpha)}(x) = \frac{1}{n\,!}\, x^{-\alpha} e^{x^k} D^n (x^{\alpha+n} e^{-x^k}) \,, \tag{1.1}$$

where k is a natural number. The polynomials $T_{kn}^{(\alpha)}(x)$ are of exactly degree kn $(n = 0, 1, 2, \ldots)$. They satisfy the operational formula

$$\prod_{j=1}^{n} (xD - kx^k + \alpha + j) = n\,! \sum_{r=0}^{n} \frac{x^r}{r\,!}\, T_{k(n-r)}^{(\alpha+r)}(x) D^r \,. \tag{1.2}$$

The following are the consequences of the operational formula (1.2):

$$nT_{kn}^{(\alpha)}(x) = (xD - kx^k + \alpha + n) T_{k(n-1)}^{(\alpha)}(x) \tag{1.3}$$

$$\binom{m+n}{m} T_{k(m+n)}^{(\alpha)}(x) = \sum_{r=0}^{\min(m,n)} \frac{x^r}{r\,!}\, T_{k(m-p)}^{(\alpha+n+r)}(x) D^r T_{kn}^{(\alpha)}(x) \,. \tag{1.4}$$

The polynomials $T_{kn}^{(\alpha)}(x)$ are generated by the function

$$(1 - t)^{-\alpha-1} \exp\,[x^k u(t)] = \sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x) t^n \tag{1.5}$$

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where

$$u(t) = 1 - (1 - t)^{-k}.$$

In the same paper the writer has also proved the following properties

$$\sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} (n+1-r) T_{k(n+1-r)}^{(\alpha)}(x)$$

$$= (\alpha + 1) \sum_{r=0}^{k} (-1)^r \binom{k}{r} T_{k(n-r)}^{(\alpha)}(x) - kx^k T_{kn}^{(\alpha)}(x) \tag{1.6}$$

$$\sum_{r=0}^{k} (-1)^r \binom{k}{r} D T_{k(n-r)}^{(\alpha)}(x) = kx^{k-1} \sum_{r=1}^{k} (-1)^r \binom{k}{r} T_{k(n-r)}^{(\alpha)}(x) \tag{1.7}$$

$$T_{kn}^{(\alpha)}(x) = \sum_{r=0}^{n} \frac{(\alpha - \beta)_r}{r\,!} T_{k(n-r)}^{(\beta)}(x) \tag{1.8}$$

This work of the writer generalizes some properties of the Laguerre polynomials $L_n^{(\alpha)}(x)$. Indeed, when $k = 1$, $T_{kn}^{(\alpha)}(x) \equiv \equiv L_n^{(\alpha)}(x)$. A similar generalization viz., $T_{kn}^{(0)}(x)$, has been previously studied by Palas [2]. The purpose of this paper is to discuss a more general class of Laguerre polynomials.

2. Definition: We first make the definition

$$T_{kn}^{(\alpha)}(x, p) = \frac{1}{n\,!} x^{-\alpha} e^{px^k} D^n (x^{\alpha+n} e^{-px^k}) \tag{2.1}$$

where k is a natural number.

We now show that the polynomial $T_{kn}^{(\alpha)}(x, p)$ is of exactly degree kn $(n = 0, 1, 2, \ldots)$. In this connection we know the result [3], for which I must thank Prof. H. W. Gould:

$$D_z^s(z) = \sum_{k=0}^{s} \frac{(-1)^k}{k\,!} D_z^k f(z) \sum_{j=0}^{k} (-1)^j \binom{k}{j} z^{k-j} D_z^s z^j \tag{2.2}$$

Thus we obtain from (2.1)

$$T_{kn}^{(\alpha)}(x, p) =$$

$$= \frac{1}{n\,!} x^{-\alpha} e^{px^k} \sum_{s=0}^{n} \binom{n}{s} (D^{n-s} x^{\alpha+n})(D^s e^{-px^k}) =$$

$$= \frac{1}{n!} e^{px^k} \sum_{s=0}^{n} \binom{n}{s} \binom{\alpha + n}{n - s} (n - s)!\, x^s (D^s e^{-px^k})$$

$$= \sum_{s=0}^{n} \binom{\alpha + n}{n - s} \sum_{i=0}^{s} \frac{p^i}{i!} x^{ki} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{kj}{s}$$

$$= \sum_{i=0}^{n} \frac{p^i}{i!} x^{ki} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \sum_{s=i}^{n} \binom{\alpha + n}{n - s} \binom{kj}{s}$$

Now we know that

$$\sum_{j=0}^{i} (-1)^j \binom{i}{j} \sum_{s=0}^{i-1} \binom{\alpha + n}{n - s} \binom{kj}{s} = 0 .$$

Thus we finally obtain

$$T_{kn}^{(\alpha)}(x, p) = \sum_{i=0}^{n} \frac{p^i}{i!} x^{ki} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{\alpha + n + kj}{n}, \tag{2.3}$$

which is the explicit formula for $T_{kn}^{(\alpha)}(x, p)$.

In particular, when $k = 1$, and $p = 1$, we derive

$$T_{n}^{(\alpha)}(x, 1) = \sum_{i=0}^{n} \frac{x^i}{i!} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{\alpha + n + j}{n} = \tag{2.4}$$

$$= \sum_{i=0}^{n} \frac{x^i}{i!} \cdot (-1)^i \binom{\alpha + n}{n - i}$$

which is the explicit formula for the general Laguerre polynomials $L_n^{(\alpha)}(x)$. Thus $T_n^{(\alpha)}(x, 1) \equiv L_n^{(\alpha)}(x)$.

3. – Operational formulae: Recently we [4] have derived the general operational formula

$$x^{-\alpha} D^n (x^{kn+\alpha} Y) = \prod_{j=1}^{n} \{x^{k-1}(z + \alpha + kj)\} Y , \tag{3.1}$$

$$(k = 1, 2, 3, \ldots),$$

where $z \equiv xD$ and Y is any sufficiently differentiable function of x. The operators on the right of (3.1) commute only when $k = 1$.

Thus we derive

$$(3.2)\qquad x^{-\alpha}e^{px^k}D^n(x^{\alpha+n}e^{-px^k}Y) = \prod_{j=1}^{n}(xD - pkx^k + \alpha + j)Y$$

Again we observe

$$D^n(x^{\alpha+n}e^{-px^k}Y) = \sum_{r=0}^{n}\binom{n}{r}D^{n-r}(x^{\alpha+n}e^{-px^k})D^rY =$$

$$= n\,!\,x^{\alpha}e^{-px^k}\sum_{r=0}^{n}\frac{x^r}{r\,!}T_{k(n-r)}^{(\alpha+r)}(x,\,p)D^rY,$$

whence we obtain

$$(3.3)\qquad \frac{1}{n\,!}x^{-\alpha}e^{px^k}D^n(x^{\alpha+n}e^{-px^k}Y) = \sum_{r=0}^{n}\frac{x^r}{r\,!}T_{k(n-r)}^{(\alpha+r)}(x,\,p)D^rY$$

It therefore follows from (3.2) and (3.3) that

$$(3.4)\qquad \prod_{j=1}^{n}(xD - pkx^k + \alpha + j)Y = n\,!\sum_{r=0}^{n}\frac{x^r}{r\,!}T_{k(n-r)}^{(\alpha+r)}(x,\,p)D^rY$$

If we set $Y = 1$, we derive from (3.4)

$$(3.5)\qquad n\,!\,T_{kn}^{(\alpha)}(x,\,p) = \prod_{j=1}^{n}(xD - pkx^k + \alpha + j)\cdot 1$$

Further if $k = 1$, and $p = 1$, we obtain from (3.4)

$$(3.6)\qquad \prod_{j=1}^{n}(xD - x + \alpha + j)Y = n\,!\sum_{r=0}^{n}\frac{x^r}{r\,!}T_{n-r}^{(\alpha+r)}(x,\,1)D^rY;$$

which may be compared with the operational formula for the general Laguerre polynomials, derived by Carlitz [5].

In a recent paper [6], Gould and Hopper have generalized the Hermite polynomials by the definition

$$(3.7)\qquad H_n^k(x,\,\alpha,\,p) = (-1)^n x^{-\alpha}e^{px^k}D^n(x^{\alpha}e^{-px^k})$$

We remark that $x^n H_n^k(x, \alpha, p)$ yeilds a generalized class of polynomials of exactly degree kn $(n = 0, 1, 2, \ldots)$, provided k is a natural number. Consequently if we write

$$x^n H_n^k(x, \alpha, p) = H_{kn}^{(\alpha)}(x, p)$$

then

$$H_{kn}^{(\alpha)}(x, p) = (-1)^n x^{n-\alpha} e^{px^k} D^n (x^\alpha e^{-px^k}) \tag{3.8}$$

Thus the polynomials $H_{kn}^{(\alpha)}(x, p)$ are related to our polynomials by

$$H_{kn}^{(\alpha)}(x, p) = (-1)^n n!\, T_{kn}^{(\alpha-n)}(x, p). \tag{3.9}$$

Now returning to the operational formula (3.5) we obtain

$$(-1)^n H_{kn}^{(\alpha)}(x, p) = \prod_{j=1}^{n} (xD - pkx^k + \alpha - n + j) \cdot 1 \tag{3.10}$$

More generally we have from (3.4)

$$\begin{aligned} \prod_{j=1}^{n} (xD - pkx^k + \alpha - n + j) Y &= \\ &= \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} x^r H_{k(n-r)}^{(\alpha)}(x, p) D^r Y \end{aligned} \tag{3.11}$$

This operational formula viz., (3.11) seems to be of particular interest. Indeed, using $k = 2$, $p = 1$, and $\alpha = 0$, we have

$$\begin{aligned} \prod_{j=1}^{n} (xD - 2x^2 - n + j) Y &= \\ &= \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} x^r H_{2(n-r)}^{(0)}(x, 1) D^r Y \end{aligned}$$

Now noticing that

$$H_{2(n-r)}^{(0)}(x, 1) = x^{n-r} H_{n-r}(x),$$

where $H_n(x)$ denotes the ordinary Hermite polynomials defined by

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2},$$

we obtain

$$\prod_{j=1}^{n}(xD - 2x^2 - n + j)Y = x^n \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r} H_{n-r}(x)D^r Y$$

Now we note that

$$(3.12) \qquad x^{-n}\prod_{j=1}^{n}(xD - 2x^2 - n + j) \equiv (D - 2x)^n.$$

For, (3.12) is evidently true for $n = 1$. Next assume that (3.12) is true for $n = m$. Then we have

$$\begin{aligned} x^{-(m+1)}\prod_{j=1}^{m+1}(xD - 2x^2 - m - 1 + j) &= \\ &= x^{-(m+1)}(xD - 2x^2 - m)\prod_{j=1}^{m}(xD - 2x^2 - m + j) = \\ &= x^{-(m+1)}(xD - 2x^2 - m)x^m(D - 2x)^m = \\ &= x^{-(m+1)} \cdot x^m(xD - 2x^2)(D - 2x)^m = \\ &= (D - 2x)^{m+1}. \end{aligned}$$

Hence by induction (3.12) is true for all positive integers n. Thus we finally derive

$$(3.13) \qquad (D - 2x)^n = \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r} H_{n-r}(x)D^r,$$

a formula which Burchnall [7] derived some years ago.

4. – SOME APPLICATIONS OF THE OPERATIONAL FORMULA: From (3.5) we note that

$$(4.1) \qquad nT_{kn}^{(\alpha)}(x, p) = (xD - pkx^k + \alpha + n)T_{k(n-1)}^{(\alpha)}(x, p).$$

In particular, when $k = 1$, and $p = 1$, we derive

$$(4.2) \qquad nT_{n}^{(\alpha)}(x, 1) = (xD - x + \alpha + n)T_{n-1}^{(\alpha)}(x, 1)$$

which is well-known for the Laguerre polynomials $L_n^{(\alpha)}(x)$.

Again in terms of the polynomials of Gould and Hopper, (4.1) stands thus

$$(4.3) \qquad H_{kn}^{(\alpha)}(x, p) + (xD - pkx^k + \alpha) H_{k(n-1)}^{(\alpha-1)}(x, p) = 0.$$

Next we consider

$$(m + n)!\, T_{k(m+n)}^{(\alpha)}(x, p) =$$

$$= \prod_{j=1}^{m} (xD - pkx^k + \alpha + n + j) \prod_{i=1}^{n} (xD - pkx^k + \alpha + i) \cdot 1$$

$$= n! \prod_{j=1}^{m} (xD - pkx^k + \alpha + n + j) \cdot T_{kn}^{(\alpha)}(x, p)$$

$$= m!\, n! \sum_{r=0}^{m} \frac{x^r}{r!} T_{k(m-r)}^{(\alpha+n+r)}(x, p) D^r T_{kn}^{(\alpha)}(x, p);$$

which implies that

$$(4.4) \qquad \binom{m+n}{m} T_{k(m+n)}^{(\alpha)}(x, p) =$$

$$= \sum_{r=0}^{\min(m,n)} \frac{x^r}{r!} T_{k(m-r)}^{(\alpha+n+r)}(x, p) D^r T_{kn}^{(\alpha)}(x, p).$$

The formula (4.4) readily yields the corresponding formula for the polynomials of Gould and Hopper:

$$(4.5) \quad H_{k(m+n)}^{(\alpha+m+n)}(x, p)$$

$$= \sum_{r=0}^{\min(m,n)} (-1)^r \binom{m}{r} x^r H_{k(m-r)}^{(\alpha+m+n)}(x, p) D^r H_{kn}^{(\alpha+n)}(x, p).$$

5. - Generating function: We shall now show that the polynomials $T_{kn}^{(\alpha)}(x, p)$ are generated by

$$(5.1) \qquad g(x, t) = (1 - t)^{-\alpha-1} \exp\,[px^k u(t)] = \sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x, p) t^n,$$

where

$$u(t) = 1 - (1 - t)^{-k}.$$

From the definition (2.1) we observe

$$(5.2) \qquad T^{(\alpha)}_{kn}(x, p) = e^{px^k} \sum_{r=0}^{\infty} \frac{(-p)^r}{r!} \binom{kr + \alpha + n}{n} x^{kr}.$$

It may be noted that (2.3) is a consequence of (5.2).

Now we notice that

$$(5.3) \qquad T^{(\alpha)}_{kn}(x, p) = \frac{1}{n!} \left[\frac{\partial^n}{\partial t^n} g(x, 0) \right]$$

Also

$$(5.4) \quad \left[\frac{\partial^n}{\partial t^n} \{(1 - t)^{-\alpha-1} \exp (px^k u(t))\} \right]_{t=0}$$

$$= e^{px^k} \left[\frac{\partial^n}{\partial t^n} \left\{ (1 - t)^{-\alpha-1} \exp \left(- p \left(\frac{x}{1 - t} \right)^k \right) \right\} \right]_{t=0}$$

$$= n!\, e^{px^k} \sum_{r=0}^{\infty} \frac{(-p)^r}{r!} \binom{kr + \alpha + n}{n} x^{kr}$$

Thus a comparison of (5.3) and (5.4) with (5.2) confirms (5.1).

Now from the generating function (5.1) we easily derive the following multiplication formula:

$$(5.5) \qquad T^{(\alpha)}_{kn}(xm^{1/k}, p) = T^{(\alpha)}_{kn}(x, mp)\,,$$

which, in terms of the polynomials of Gould and Hopper, shapes into

$$(5.6) \qquad H^{(\alpha)}_{kn}(xm^{1/k}, p) = H^{(\alpha)}_{kn}(x, mp)\,,$$

which may well be compared with (3.9) of [6, p. 54].

It is also interesting to note from (5.5) that

$$(5.7) \qquad T^{(\alpha)}_{n}(x, m) = L^{(\alpha)}_{n}(mx)\,.$$

Again we observe

$$(1 - t)^{-\alpha-1} \exp [px^k\{1 - (1 - t)^{-k}\}]$$

$$= (1 - t)^{(\alpha-\beta)} (1 - t)^{-\beta-1} \exp [px^k\{1 - (1 - t)^{-k}\}]$$

whence we obtain

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x, p)t^n = (1 - t)^{-(\alpha-\beta)} \sum_{n=0}^{\infty} T_{kn}^{(\beta)}(x, p)t^n .$$

Now comparing the coefficients of t^n on both sides we get

$$T_{kn}^{(\alpha)}(x, p) = \sum_{r=0}^{n} \frac{(\alpha - \beta)_r}{r!} T_{k(n-r)}^{(\beta)}(x, p) , \tag{5.8}$$

where α and β are arbitrary real numbers.

Next we notice that

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_{kn}^{(\alpha+\beta+1)}(x, p + q)t^n \\
&\quad = (1 - t)^{-\alpha-1} e^{px^k\{1-(1-t)^{-k}\}} \cdot (1 - t)^{-\beta-1} e^{qx^k\{1-(1-t)^{-k}\}} \\
&\quad = \sum_{m=0}^{\infty} T_{km}^{(\alpha)}(x, p)t^m \cdot \sum_{n=0}^{\infty} T_{kn}^{(\beta)}(x, q)t^n \\
&\quad = \sum_{n=0}^{\infty} \sum_{m=0}^{n} T_{km}^{(\alpha)}(x, p)T_{k(n-m)}^{(\beta)}(x, q)t^n .
\end{aligned}$$

Thus we obtain the following 'doubly-additive' addition formula

$$T_{kn}^{(\alpha+\beta+1)}(x, p + q) = \sum_{m=0}^{n} T_{km}^{(\alpha)}(x, p)T_{k(n-m)}^{(\beta)}(x, q) . \tag{5.9}$$

In particular, when $p = q = 1$, and $k = 1$, we derive

$$T_{n}^{(\alpha+\beta+1)}(x, 2) = \sum_{m=0}^{n} L_{m}^{(\alpha)}(x)L_{n-m}^{(\beta)}(x) . \tag{5.10}$$

It follows therefore from (5.7) and (5.10) that

$$L_{n}^{(\alpha+\beta+1)}(2x) = \sum_{m=0}^{n} L_{m}^{(\alpha)}(x)L_{n-m}^{(\beta)}(x) , \tag{5.11}$$

which is implied by the well-known formula of the Laguerre polynomials

(5.12) $$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{m=0}^{n} L_m^{(\alpha)}(x) L_{n-m}^{(\beta)}(y) .$$

Again returning to (5.1) we obtain

$$(1-t)^{k+1} \frac{\partial g(x,t)}{\partial t} = [(\alpha+1)(1-t)^k - pkx^k] g(x,t)$$

whence we notice

$$(1-t)^{k+1} \sum_{n=1}^{\infty} n t^{n-1} T_{kn}^{(\alpha)}(x,p) = [(\alpha+1)(1-t)^k - pkx^k] \cdot \sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x,p) t^n .$$

Performing the indicated multiplication on both sides and comparing coefficients of t^n on both sides, we derive

(5.13) $$\sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} (n+1-r) T_{k(n+1-r)}^{(\alpha)}(x,p) .$$
$$= (\alpha+1) \sum_{r=0}^{k} (-1)^r \binom{k}{r} T_{k(n-r)}^{(\alpha)}(x,p) - pkx^k T_{kn}^{(\alpha)}(x,p) .$$

Lastly we observe

$$(1-t)^k \frac{\partial g(x,t)}{\partial x} = pkx^{k-1}\{(1-t)^k - 1\} g(x,t) ,$$

whence we obtain in like manner

(5.14) $$\sum_{r=0}^{k} (-1)^r \binom{k}{r} D T_{(n-r)}^{(\alpha)}(x,p)$$
$$= pkx^{k-1} \sum_{r=1}^{k} (-1)^r \binom{k}{r} T_{(n-r)}^{(\alpha)}(x,p) .$$

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