OPERATIONAL FORMULAE FOR CERTAIN CLASSICAL POLYNOMIALS - III

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1. INTRODUCTION

In an earlier paper [1] we found the operational formula

$$\prod_{j=1}^{n} \{x^2D + (2j + a)x + b\}$$

(1.1)

$$= \sum_{r=0}^{n} \binom{n}{r} b^{n-r} x^{2r} y_{n-r}(x, a + 2r + 2, b) D^r.$$

where $y_n(x, a, b)$ is the generalised Bessel polynomials as defined by Krall and Frink [2]. In [1] we also noticed the following consequences of (1.1):

(1.2) $b^n y_n(x, a + 2, b) = \prod_{j=1}^{n} \{x^2D + (2j + a)x + b\} \cdot 1$

(1.3) $2^n y_n(x) = \prod_{j=1}^{n} (x^2D + 2jx + 2) \cdot 1$

where $y_n(x)$ is the special case of the polynomials $y_n(x, a, b)$.

*) L'ervenuta in redazione il 15 gennaio 1963.
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obtained by taking \( a = b = 2 \).

\[
b^x \{ y_{n+1}(x, a, b) - y_n(x, a, b) \} = (2n + a)x \{ by_n(x, a, b) + nxy_{n-1}(x, a + 2, b) \}
\]

which implies two well-known formulae:

(i) \( b \{ y_n(x, a + 1, b) - y_n(x, a, b) \} = nx y_{n-1}(x, a + 2, b) \)

(ii) \( b y_n'(x, a, b) = n(n + a - 1)y_{n-1}(x, a + 2, b) \);

\[
y_{n+m}(x, a, b) = \sum_{r=0}^{\min(m, n)} \binom{m}{r} \binom{n}{r} r! (m + 2n + a - 1, r, x/b)^r y_{n-r}(x, a + 2r, b).
\]

Later in a recent paper \([3]\) we have obtained the operational formula

\[
x^n \left[ D + \frac{2(nx + 1)}{x^2} \right]^n = \sum_{r=0}^{n} \binom{n}{r} 2^{n-r} r^r y_{n-r}(x, 2 + 2r, 2) D^r .
\]

which generalises the operational formula derived by Rajagopal \([4]\):

\[
x^n \left[ D + \frac{2(nx + 1)}{x} \right]^n \cdot 1 = 2^n y_n(x) .
\]

In \([3]\) we have also derived the following formulae:

\[
x^n \left[ D - \frac{2x + n + 1}{x} \right]^n = \sum_{r=0}^{n} \binom{n}{r} (-2)^{n-r} r^r \theta_{n-r}(x, 2 + r, 2) D^r .
\]

where \( \theta_n(x, a, b) \) are those polynomials defined by Burchnell \([5]\):

\[
\theta_n(x, a, b) = (-b)^{-n} e^{b x} x^{n+2n-1} D^n (x-a-n+1 e^{-bx})
\]

\[
\frac{x^n}{n!} \left[ D + \frac{\alpha + n - x}{x} \right]^n = \sum_{r=0}^{n} \frac{x^r}{r!} L_n^{(n-r)}(x) D^r .
\]
where $L^{(a)}_n(x)$ is the generalised Laguerre polynomials. In this connection we like to mention that we have been inspired by Carlitz's work [6]. The interesting result of Carlitz is

\begin{equation}
\prod_{i=1}^{n} (xD - x + \alpha + j) = n! \sum_{r=0}^{n} \frac{x^r}{r!} L^{(a+r)}_n(x) D^r.
\end{equation}

wherefrom he obtains $n! L^{(a)}_n(x) = \prod_{i=1}^{n} (xD - x + \alpha + j) \cdot 1$.

Thus far we have tried to give a systematic development of the operational formulae derived in [1] and [3], for certain classical polynomials. The object of this paper is to discuss in the same line the polynomials $\theta_n(x, a, b)$ as defined by Burchnall.

2. Burchnall defined the polynomials $\Phi_n(x, a, b)$ by

$$\Phi_n(x, a, b) = x^n y_n(x^{-1}, a, b).$$

He showed that $\Phi_n(x, a, b)$ was a solution of

\begin{equation}
\delta(\delta + 1 - a - 2n) z = bx(\delta - n) z; \quad (\delta \equiv xD)
\end{equation}

and that $e^{-bx} \Phi_n(x, a, b)$ was a solution of

\begin{equation}
\delta(\delta + 1 - a - 2n) \omega = -bx(\delta - n - a + 2) \omega.
\end{equation}

Further he showed that the equation (2.2) had the solution

\begin{equation}
\omega = (\delta - n - a + 1)(\delta - n - a) \ldots (\delta - 2n - a + 2)e^{-bx},
\end{equation}

wherefrom he deduced that

\begin{equation}
\Phi_n(x, a, b) = (-b)^{-n} e^{bx} x^{a+2n-1} D^n(x^{-a-n+1} e^{-bx}).
\end{equation}

In particular, when $a = b = 2$, he pointed out that

\begin{equation}
\theta_n(x) \equiv \Phi_n(x, 2, 2) = \left(-\frac{1}{2}\right)^n e^{2x} x^{2n+1} D^n(x^{n-1} e^{-2x}).
\end{equation}
We first mention that we shall write $\theta_n(x, a, b)$ for $\Phi_n(x, a, b)$ throughout this paper. Now we have for any arbitrarily differentiable function $y$ of $x$:

$$x^nD^n(x^{-a-n+1}y).$$

$$= \delta(\delta - 1) \ldots (\delta - n + 1)(x^{-a-n+1}y)$$

$$= x^{-a-n+1}(\delta - a - n + 1)(\delta - a - n) \ldots (\delta - a - 2n + 2)y$$

$$\therefore x^{a+2n-1}D^n(x^{-a-n+1}y)$$

$$(2.6) = (\delta - a - n + 1)(\delta - a - n) \ldots (\delta - a - 2n + 2)y$$

Now since the linear operators on the right of (2.6) are commutative, we can write (2.6) as

$$(2.7) x^{a+2n-1}D^n(x^{-a-n+1}y) = \prod_{j=1}^{n} (\delta - a - 2n + j + 1)y$$

Thus we easily have

$$x^{a+2n-1}D^n(x^{-a-n+1}e^{-bx}y)$$

$$(2.8) = \prod_{j=1}^{n} (\delta - a - 2n + j + 1)e^{-bx}y$$

But in [3] we have proved that

$$\theta^x x^{a+2n-1}D^n(x^{-a-n+1}e^{-bx}y)$$

$$(2.9) = \sum_{r=0}^{n} \binom{n}{r} (-b)^{n-r}x^r \theta_{n-r}(x, a + r, b)D^ry$$

It therefore follows from (2.8) and (2.9)

$$\theta^x \prod_{j=1}^{n} (\delta - a - 2n + j + 1)e^{-bx}y$$

$$(2.10) = \sum_{r=0}^{n} \binom{n}{r} (-b)^{n-r}x^r \theta_{n-r}(x, a + r, b)D^ry$$
As a special case of (2.10) we notice that

\[(2.11) \quad e^{bx} \prod_{j=1}^{n} (a - 2n + j + 1) e^{-bx} = (-b)^n \theta_n(x, a, b)\]

which may be compared with the remark made by Burchnall in (2.2) and (2.3).

Now we shall find a more interesting operational formula for \(\theta_n(x, a, b)\).

From (2.8) we again derive

\[(2.12) \quad e^{bx} e^{a+2n-1} D^n (x-a-n+1 e^{-bx} y)\]

\[= \prod_{j=1}^{n} (xD - bx - a - 2n + j + 1) y . \]

Now a comparison of (2.9) and (2.12) yields our desired result:

\[(2.13) \quad \prod_{j=1}^{n} (xD - bx - a - 2n + j + 1) y\]

\[= \sum_{r=0}^{n} \binom{n}{r} (-b)^{n-r} x^r \theta_{n-r}(x, a + r, b) D^r y . \]

When \(a = b = 2\), we get from (2.13)

\[(2.14) \quad \prod_{j=1}^{n} (xD - 2x - 2n + j + 1) y\]

\[= \sum_{r=0}^{n} \binom{n}{r} (-2)^{n-r} x^r \theta_{n-r}(x, 2 + r, 2) D^r y . \]

As particular cases of (2.13) and (2.14) we note that

\[(2.15) \quad (-b)^n \theta_n(x, a, b) = \prod_{j=1}^{n} (xD - bx - a - 2n + j + 1) \cdot 1\]

\[(2.16) \quad (-2)^n \theta_n(x) = \prod_{j=1}^{n} (xD - 2x - 2n + j + 1) \cdot 1\]

In this connection it is interesting to note that a comparison
Lastly we like to mention a consequence of the formula (2.16). To this end, we observe from (2.16)

\[
-2(xD - 2x - n)\theta_n(x) = (xD - 2x - 2n)(xD - 2x - 2n + 1)\theta_{n-1}(x)
\]

which implies

\[
2 \left\{ x\theta'_n - (2x + n)\theta_n \right\} + x^2\theta''_{n-1} - 2(2x^2 + 2nx - x)\theta'_{n-1}
\]
\[
+ 2 \left\{ 2x^2 + 2(2n - 1)x + 2n^2 - n \right\} \theta_{n-1} = 0 .
\]

To verify the truth of (2.19) we observe [5, formulae (15), (16)]

\[
\theta'_n - \theta_n = -x\theta_{n-1}
\]
\[
\theta_{n+1} - x^2\theta_{n-1} = (2n + 1)\theta_n
\]

At first we shall prove the formula

\[
x\theta'_{n-1} + \theta_n = (x + 2n - 1)\theta_{n-1}
\]

which is not mentioned in Burchnall's paper. For this, we easily notice from (2.20) and (2.21)

\[
\theta_{n+1} + x(\theta'_n - \theta_n) = (2n + 1)\theta_n
\]

or,

\[
x\theta'_n + \theta_{n+1} = (x + 2n + 1)\theta_n .
\]
Now we have
\[
2 \{ x\theta_n' - (2x + n)\theta_n \}
\]
(2.23)
\[
= 2x(\theta_n - x\theta_{n-1}) - 2(2x + n)\theta_n
\]
\[
= -2x^2\theta_n - 2(n + x)\theta_n
\]

Thus eliminating \(\theta_n\) between (2.19) and (2.22) with the help of (2.23), we obtain

(2.24) \[x\theta_n'' - 2(x + n - 1)\theta_n' + 2(n - 1)\theta_{n-1} = 0\]

which is the differential equation for \(\theta_{n-1}(x)\) and which may be compared with Burchnall’s form:

(2.25) \[\delta(\delta - 2n - 1)\theta_n = 2x(\delta - n)\theta_n\]

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REFERENCES