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ON THE STRUCTURE OF CLIFFORD ALGEBRAS

Nota *) *di* GUEORGUIY V. BABIKOV (*a Sverdlovsk*)

The principal result of the paper by G. B. Rizza published in this journal [1]) may be defined as following:

« Clifford algebra C_n of index n over the field of real numbers is direct sum of 2^{n-2} fields of quaternions. »

This result is wrong. The mistake may be found in the inattentive application of the method of mathematical induction.

After the above given result has been considered and proved for $n = 3$ the process of provement for $n > 3$ follows in such a way.

Let $x = \alpha_1 + \alpha_2 i_n \in C_n$ and $\alpha_1, \alpha_2 \in C_{n-1}$.

Suppose that $C_{n-1} = C_2^{(1)} \dot{+} C_2^{(2)} \dot{+} \ldots \dot{+} C_2^{(2^{n-3})}$ where $C_2^{(h)}$ is isomorphic to a field of quaternions.

Then $x = \sum_{1}^{2^{n-3}}{}_h [\alpha_1^{(h)} + \alpha_2^{(h)} i_n]$ and the sets $\{\alpha_1^{(h)} + \alpha_2^{(h)} i_n\}$ are isomorphic to C_3 when $\alpha_1^{(h)}$ and $\alpha_2^{(h)}$ run through all possible real quaternions. This last statement is wrong for $n \geqslant 4$. If a set $\{\alpha_1^{(h)} + \alpha_2^{(h)} i_n\}$ is isomorphic to $C_3 = C_2^{(s)} \dot{+} C_2^{(t)}$, then i_n and base units of $C_2^{(h)}$ have specific forms depending on base units of $C_2^{(s)}$ and $C_2^{(t)}$.

The result by Rizza disproves also the following contradictory example.

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Suppose we have algebra C_n ($n > 3$) over the field of real numbers. Let us take an element $x = i_1 + i_2 i_3 i_4$.

Let $x = q_1 \dot{+} q_2 \dot{+} \dots \dot{+} q_{2^{n-2}}$, where q_k belongs to some field of quaternions $C_2^{(k)}$.

Then $x^2 = q_1^2 \dot{+} q_2^2 \dot{+} \dots \dot{+} q_{2^{n-2}}^2 = (i_1 + i_2 i_3 i_4)^2 = 0$. As the above sum is direct, we have $q_1^2 = q_2^2 = \dots = q_{2^{n-2}}^2 = 0$. Consequently $q_1 = q_2 = \dots = q_{2^{n-2}} = 0$, i.e. $x = 0$.

We come to a contradiction.

The structure of Clifford algebras defined by the following theorem [2]):

« Clifford algebras over the arbitrary field F of characteristic $\neq 2$ under different n (with exactness to isomorfism) are exhausted by the following list:

1) C_{8m} is algebra of matrices of power 2^{4m} over F.

2) C_{8m+1} is direct product $A \times K$, where A is algebra of matrices of power 2^{4m} over F and K is algebra of complex numbers over F.

3) C_{8m+2} is direct product $A \times Q$, where A is algebra of matrices of power 2^{4m} over F and Q is algebra of quaternions over F.

4) C_{8m+3} is algebra $Q \times A \dot{+} Q \times B$, where A and B are algebras of matrices of power 2^{4m} over F and Q is algebra of quaternions over F.

5) C_{8m+4} is direct product $A \times Q$, where A is algebra of matrices of power 2^{4m+1} over F and Q is algebra of quaternions over F.

6) C_{8m+5} is direct product $A \times K$, where A is algebra of matrices of power 2^{4m+2} over F and K is algebra of complex numbers over F.

7) C_{8m+6} is algebra of matrices of power 2^{4m+3} over F.

8) C_{8m+7} is direct sum $A \dot{+} B$, where A and B are algebras of matrices of power 2^{4m+3} over F. »

Relying upon this it's possible to solve the question about the distribution of divisors of zero in Clifford algebras.

[1] RIZZA G. B.: *Sulla struttura delle algebre di Clifford*. Rend. sem. mat. Univ. Padova, 1954, 23, N. 1, 91-99.

[2] BABIKOV G. V.: *Predstavleniya algebr Clifforda nad proisvolnimi polyami*. Uchenie sapiski Uralskogo Universiteta, 1960, Vip. 23 N. 2, tetr. 3, s. 19-26