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On irregular varieties which contain cyclic involutions

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ON IRREGULAR VARIETIES WHICH
CONTAIN CYCLIC INVOLUTIONS

Nota (*) di LEONARD ROTH (a Londra)

1. Introduction. - The present note generalises some familiar results of De Franchis and Comessatti concerning irregular multiple planes. In the first place, a classical theorem of De Franchis [4, 5] states that, on any double plane of irregularity \( q > 0 \), the branch curve is reducible, consisting of a number of curves belonging to a pencil; it follows from this that any surface \( V_2 \) which is a simple model of the double plane must contain a pencil, of genus \( q \), of curves.

The theorem in question is established by computing the simple integrals of the first kind attached to \( V_2 \). Actually, it is the second of the above results which is significant, for it means that \( V_2 \) cannot possess a proper model \( V_2^* \) on its Picard-Severi variety \( V_q \) (see [9]). Thus we may conclude that the existence on \( V_2 \) of a rational involution \( I_2 \) of order 2 implies the non-existence of \( V_2^* \) and hence, by a theorem of Severi [10], that \( V_2 \) contains a pencil of genus \( q \). From this the result concerning the branch curve can be deduced.

Now it appears that the De Franchis theorem is merely a special case of a proposition about superficially irregular algebraic varieties \( V_r \) of any dimension \( r \geq 2 \) which carry superficially regular involutions \( I_z \). Denoting by \( g_k(I_z) \) the number of linearly independent differential forms of the first kind and of degree \( k (k = 1, 2, ..., r) \) attached to the image variety of \( I_z \), we show that:

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If a variety $V_r(r \geq 2)$ of superficial irregularity $q > 0$ carries an involution $I_2$ such that $g_1(I_2) = g_2(I_2) = 0$, then $V_r$ must contain a congruence of subvarieties (of some dimension $\geq 1$), the congruence having superficial irregularity $q$.

Moreover, we readily see that the coincidence locus of $I_2$ always belongs to the congruence in question.

We show further that this result is itself a special case of the following theorem:

If a variety $V_r(r \geq 2)$ of superficial irregularity $q > 0$ carries an involution $I_2$ such that $g_1(I_2) = 0$, while for any even value of $k \leq r$, $g_k(I_2) < \binom{q}{k}$, then $V_r$ must contain an irregular congruence (of superficial irregularity $\leq q$).

The limitation $q > r$ is required for the method of proof; the restriction is, however, inessential, since whenever $q < r$, $V_r$ must contain some congruence of superficial irregularity $q$ ([10]).

We then indicate how the same methods can be applied to the case where $V_r$ carries a superficially regular cyclic involution $I_m$ of any order $m \geq 3$. Here, however, the results are less precise, since — in constrast with the case $m = 2$ — there are now various types of associated involution of order $m$ on the Picardian $V_q$ of $V_r$. Moreover, unlike the case $m = 2$, we have now always to deal with singular transformations of $V_q$, and these necessarily give rise to problems of existence. Examples of irregular cyclic planes quoted by Comessatti [2] demonstrate that general results analogous to those obtained in the case $m = 2$ cannot be established.

Finally we remark that the previous considerations may be extended to the case where the involution $I_m$ is non cyclic, provided that the associated involution on $V_q$ is generable by a (finite) group of automorphisms of $V_q$; the results then obtained are exactly similar to those mentioned above.

2. Generalities. - Consider a non-singular algebraic variety $V_r(r \geq 2)$ having superficial irregularity $q \geq r$; in all that follows the case $q < r$ can be set aside since we know that $V_r$ will then contain a congruence of superficial irregu-
Assuming that $V_r$ does not contain such a congruence we may obtain for $V_r$ a model (simple or multiple) of dimension $r$ on the second Picardian or Picard-Severi variety $V_q$ constructed with the period matrix associated with the linearly independent simple integrals $u_i (i = 1, 2, \ldots, q)$ of the first kind attached to $V_r$ (see [1, 9, 13]). Denoting by $x$ a point current on $V_r$, we write $u_i(x) = u_i$, and take $u_i$ for coordinates on $V_q$; then the locus of the corresponding point $(u)$ on $V_q$ is an irreducible algebraic variety $V_r^*$. This will be a simple model of $V_r$ if and only if the congruences

\begin{equation}
 u_i(x) \equiv u_i(y) \quad \text{(mod. periods)}
\end{equation}

where $x, y$ are points of $V_r$, in general admit a single solution. If instead for arbitrary $x$, the equations (1) admit $v (> 1)$ solutions, it follows that $V_r^*$ is a $v$-ple model of $V_r$: to a point of $V_r^*$ there then corresponds a set of $v$ distinct points on $V_r$, belonging to the fundamental involution $I_r$. In either case we shall assume that $V_r^*$ is non-singular; such a hypothesis may possibly be restrictive.

We observe that a necessary and sufficient condition for the existence of $V_r^*$ on $V_q$ is that $V_r$ should not contain any congruence of superficial irregularity $q$ ([10]).

Suppose now that $V_r$ carries an involution $I$ of order 2; this generates an automorphism between points $P, P'$ of $V_r$, under which all the integrals $u_i$ must be invariant. Hence we have a transformation from $u_i(P)$ to $u'_i(P')$ of the form

\begin{equation}
 u'_i = \sum_{j=1}^{q} \lambda_{ij} u_j + \mu_i \quad (i = 1, 2, \ldots, q)
\end{equation}

where $\lambda_{ij}, \mu_i$ are constants.

Evidently the transformation (2), applied to $V_r^*$, is subordinate to a transformation of the entire variety $V_q$ which generates an involution $J$, likewise of order 2, on $V_q$. From the theory of Picard varieties it is known ([7]) that $J$ can be represented by the canonical form

\begin{align}
 u_i &= u_i + a_i \quad (i = 1, 2, \ldots, p; p \geq 0) \\
 u_j &= -u_j \quad (j = p + 1, p + 2, \ldots, q)
\end{align}
where the $a_i$ are constants (possibly zero) and where $p$ is the superficial irregularity of $J$. We note that if $V_\varphi$ has general moduli, there are just two possibilities: either $p = 0$ or $p = q$.

In all other cases we have a singular transformation of $V_\varphi$ which can exist only for particular values of the moduli of $V_\varphi$ ([3]).

It is clear, by comparison of equations (1) and (3), that if $\nu > 1$, $I$ cannot belong to the fundamental involution $I_\nu$.

In the case $\nu = 1$, the sets of $I$ are in birational correspondence with the sets of an involution $I^*$ of order 2 on $V_\varphi^*$, which is subordinate to $J$. When $\nu > 1$, we have instead that $I$ is mapped on a $\nu$-fold involution $I^*$ (likewise of order 2) on $V_\varphi^*$, which is subordinate to $J$; this follows by comparing equations (1) and (3). In particular, when $q = r$, $V_\varphi^*$ coincides with $V_\varphi$ and $I^*$ with $J$.

3. On the characters $g_k$, $q_k$. - We denote by $g_k(k = 1, 2, \ldots, r)$ the number of linearly independent differential forms of the first kind and of degree $k$ which are attached to $V_r$.

Here $g_r$ is the geometric genus $P_g(V_r)$, while $g_1$ is the superficial irregularity $q$ of $V_r$. The arithmetic genus is then given by the Severi-Kodaira relation ([11])

\[
P_a = g_r - g_{r-1} + \ldots + (-1)^{r-1} g_1.
\]  

Defining the $r$-dimensional irregularity $q_r$ as the difference $P_a - P_a$, we then introduce the set of $k$-dimensional irregularities $q_k(k = 2, 3, \ldots, r - 1)$ by taking appropriate linear sections of $V_r$ and applying (4) to each in turn. We thus obtain the relations ([11, 13]):

\[
\begin{align*}
g_k &= q_k + q_{k+1} \\
g_1 &= q_2.
\end{align*}
\]

We say that $V_r$ is completely regular if and only if $q_k = 0$ ($k = 2, 3, \ldots, r - 1$). Clearly a necessary and sufficient condition for the complete regularity of $V_r$ is $g_s = 0$ ($s = 1, 2, \ldots, r - 1$).
Suppose now that $V_r$ is mapped on a multiple non-singular variety $V_r'$; in that case we have the inequalities

$$g_k(V_r) \geq g_k(V_r') \quad (k = 1, 2, \ldots, r).$$

It follows from (5) and (6) that, if $V_r$ is completely regular, then so also is $V_r'$. For, if for some $s (1 \leq s \leq r - 1)$ we had $g_s(V_r') > 0$, then we should have $g_s(V_r) > 0$, whence $V_r$ could not be completely regular.

One last preliminary remark: suppose that $V_r$ contains a congruence $\Gamma$ of some positive superficial irregularity ($\leq q_2$); then $\Gamma$ will be mapped by a congruence $\Gamma'$ of subvarieties on $V_r'$. Now, in the case where $V_r'$ is superficially regular, $\Gamma'$ will perforce be superficially regular: this means that $\Gamma'$ cannot correspond birationally, element for element, to $\Gamma$. Applying this result to the case we have to consider, let $V_r'$ denote a birational image of the involution $I$ on $V_r$; if $I$ is superficially regular, we deduce that to a member of $\Gamma'$ there will correspond two members of $\Gamma$, in general distinct. Moreover, the coincidence locus of $I$ must belong to $\Gamma$, and the branch locus on $V_r'$ must belong to $\Gamma'$.

4. On the Wirtinger involution. - Returning to the Picard variety $V_q$, we consider the case where the involution $J$ is superficially regular; the involution, represented by equations (32), then has for image a generalised Wirtinger variety (in the case $r = 2$, a generalised Kummer surface) which we shall denote by $W_q$.

Now every differential form of the first kind and of degree $k$ attached to $W_q$ must arise from an analogous form attached to $V_q$; and it is known ([12]) that every such form is given by an expression of the type $du_1 du_2 \ldots du_k$. Evidently this furnishes a corresponding differential form on $W_q$ if and only if it is invariant under the transformation (32). We thus

(1) The name of Wirtinger variety is usually restricted to the case where $V_q$ has all its divisors equal to unity; we may call this the ordinary Wirtinger variety (for $r = 2$, the ordinary Kummer surface).
obtain the results

\begin{equation}
\begin{aligned}
g_k(W_q) &= \binom{q}{k} \\
g_k(W_q) &= 0
\end{aligned}
\tag{7}
\end{equation}

In these formulae, \(k\) takes the values 1, 2, ..., \(r\). It now follows from (4) that the arithmetic genus \(P_a\) of \(W_q\) is given by

\[ P_a(W_q) = (-1)^q(2^q - 1). \]

This result was obtained by Gröbner [3] for the ordinary Wirtinger variety by computing the Hilbert characteristic function for the manifold in question and then applying the Severi postulation formula.

5. First applications. - With the notation of n. 2, suppose that the involution \(I\) carried by \(V_r\) has superficial irregularity \(p(0 < p \leq q)\); this means that precisely \(p\) linearly independent differential forms of the first kind and first degree — say \(du_1, du_2, \ldots, du_p\) — take the same values at corresponding points \(P, P'\) of \(I\). The equations (2) for \(I\) assume the form (3).

If \(V_q\) has general moduli, and thus admits only ordinary transformations, we must have \(p = q\). If instead \(p < q\), we have a singular transformation; evidently the involution \(J\) is now pseudo-Abelian of type \(p\) ([8]). Hence \(V_r^*\) must contain a superficially irregular congruence, and thus so also must \(V_r\). Whence the result: If \(I\) has superficial irregularity \(p(0 < p < q)\), \(V_r\) must contain a superficially irregular congruence.

Suppose next that \(I\) is superficially regular and — as usual — that \(V_r\) does not contain any congruence of superficial irregularity \(q(= g_1)\); in this case the model \(V_r^*\) on \(V_q\) certainly exists and if \(q = r\), coincides with \(V_q\). The equations (1) now take the form (3), so that \(J\) is a generalised Wirtinger (or Kummer) involution, whose characters \(g_k(J)\) are given by (7); in particular, then, we have \(g_2(J) = \binom{q}{2}\).
Now the number $g_2(I^*)$ will equal $g_2(J)$ provided that $V_r^*$ does not contain a superficially irregular congruence, for in that case none of the differential forms $du_i du_j$ can vanish identically on $V_r^*$ ([12]). In any event, since we know that $V_r^*$ certainly does not contain any congruence of superficial irregularity $q$ (n. 2), not all the integrals $u_i$ attached to $V_r^*$ can be functions of one integral alone ([12]), and therefore we must have $g_2(I^*) > 0$; hence, whatever the value of $\nu$, it follows that $g_2(I) > 0$, by equations (6). Thus

If $r$ carries an involution $I$ of the second order such that $g_1(I) = 0$, $g_2(I) = 0$, then $V_r$ must contain a congruence of superficial irregularity $q$.

As remarked in n. 3, the members of the congruence are conjugate in $I$, and the coincidence locus of $I$ belongs to the congruence.

In the case where $r \geq 3$, it follows from (5) that, if the characters $g_1$ and $g_2$ are both zero, then $q_2$ and $q_3$ are also zero, and vice-versa. Thus

If $V_r(r \geq 3)$ carries an involution $I$ of the second order which is bidimensionally and also tridimensionally regular, then $V_r$ must contain a congruence of superficial irregularity $q$.

In particular, then, if $I$ is unirational or birational, $V_r$ must contain a congruence of superficial irregularity $q$.

6. The double space $S_r$. - Consider first the case $r = 2$; suppose that $V_2$ contains a rational involution $I$, which means that $V_2$ can be mapped on a double plane $S_2$ of irregularity $q$. By the previous theorem, $V_2$ must contain an irrational pencil, of genus $q$, and the coincidence locus of $I$ must consist of curves belonging to the pencil. Hence the branch curve consists of a number curves of curves belonging to a pencil in $S_2$, and the general curve of this pencil maps a pair of curves of $V_2$; this result is due to De Franchis [4].

Next, let $r = 3$; then the double planes in the corresponding double space $S_3$ are «generic» surfaces, having irregularity $q$. Hence, by the previous result, the branch surface in $S_3$ consists of a number of surfaces of a pencil, from which
it follows that $V_r$ must contain a pencil, of genus $q$, of surfaces.

Proceeding by induction, we thus obtain the result: Every double space $S_r, (r \geq 2)$ of superficial irregularity $q > 0$ contains a pencil, of genus $q$, of hypersurfaces; and the branch locus in $S_r$ consists of a number of primals belonging to a pencil.

It is clear that the image of the pencil on $V_r$ is a hyperelliptic curve, since to a member of the (linear) pencil in $S_r$ which maps it there corresponds a pair of hypersurfaces, in general distinct. This type of double $S_r$ has been studied by Gallarati [6], who has calculated the invariants $g_k(V_r)$ in the case where the base of the pencil in $S_r$ is irreducible and non-singular.

7. Extension of previous results.

I. - Let $q = r$; in this case, if $v = 1$, the involution $I$ is coincident with the generalised Wirtinger involution $J$, and its characters $g_k(I)$ are given by (7). If $v > 1$, we have $g_k(I) \geq g_k(J)$. It follows that, in order that the model $V_r^* (\equiv V_q)$ should exist, the inequalities $g_k(I) \geq \binom{q}{k}$ must be satisfied for every even value of $k$. Hence,

If, when $q = r$, the variety $V_r$ carries a superficially regular involution $I$ of the second order such that, for any even value of $k$, $g_k(I) < \binom{k}{q}$, then $V_r$ must contain a congruence of superficial irregularity $q$.

II. - In the case where $q > r$, we can obtain a result which is more general than that of n. 5. Previously we have allowed $V_r$ to contain some irregular congruence (necessarily of superficial irregularity $< q$). Suppose now that $V_r$ contains no superficially irregular congruence whatever; this entails that $V_r^*$ also can contain no such congruence. On this hypothesis the differential forms of the first kind of any degree $k \leq r$ attached to $J$ must give rise to precisely the same number
of differential forms of the first kind and of like degree attached to $I^*$. We thus have, for every even value of $k \leq r$,

$$g_k(I^*) = \binom{q}{k}, \quad \text{whence} \quad g_k(I) \geq \binom{q}{k}.$$

Therefore, if $V_r$ carries a superficially regular involution $I$ of the second order such that, for any even value of $k \leq r$,

$$g_k(I) \leq \binom{q}{k},$$

then $V_r$ must contain an irregular congruence (of superficial irregularity $\leq q$).

As remarked, in n. 3, the members of this congruence must be conjugate in $I$, so that the coincidence locus of $I$ belongs to the congruence in question.

8. Notes and examples. - We add a few comments upon the preceding results.

In the first place we remark that the conditions of n. 5 are not necessary in order that $V_r$ should contain a congruence of superficial irregularity $q$. Thus, consider a product variety $V_r = V_t \times V_{r-t}$ ($t \geq 1$), where $V_t$ is the simple model of a double space $S_t$; in particular, when $t = 1$, $V_t$ is a hyperelliptic curve. In this case $V_r$ carries an involution $I$ which is mapped by the product $S_t \times V_{r-t}$; hence, if we assume that $g_t(V_{r-t}) = 0$, we shall have $g_t(I) = 0$. Now in this case, $g_2(I) = g_2(V_{r-t})$, from which it follows that the character $g_2(I)$ can have any non-negative value whatever. Evidently the variety $V_r$ contains a congruence of varieties $V_{r-t}$, which is mapped by the points of $V_t$, and which has maximum superficial irregularity $g_1(V_t) = q$.

Returning to the general case we observe that, from the correspondence between $V_r$ and $I$ we have (n. 3), for every $k(1 \leq k \leq r)$, $g_k(I) \leq g_k(V_r)$. For the particular double spaces $S_t$ considered by Gallarati [6], we have $g_k(V_r) = 0$ ($k = 2, 3, \ldots, r - 1$). This suggests an interesting problem: what are the most general conditions of validity for this last result?

In the second place, since for any birational involution $I$ on $V_r$ we have $g_k(I) = 0$ (all $k$), it follows that, in the pre-
vious example, \( g_k(I) = g_k(V_r) \) \((k = 2, 3, \ldots, r - 1)\). This suggests another problem: under what conditions can we assert that this set of relations will hold? An analogous question can of course be raised for any involution, superficially regular or not, carried by a given variety \( V_r \); but the answer is unknown even in the relatively simple case just considered, at any rate for a variety \( V_r \) of general character.

A certain amount is, however, known concerning involutions on a Picard variety \( V_q \) ([9]). Thus, for an involution \( I \) of any order on \( V_q \), the sole condition \( g_i(I) = q \) ensures that the image of \( I \) should also be a Picard variety. But the effect of other analogous conditions on the nature of \( I \) has not yet been investigated. The cyclic involutions — to which we now turn — on \( V_q \) have been studied by Lefschetz [7].

9. The general cyclic involution \( I_m (m \geq 3) \). - Consider next the case where \( V_r \) carries a cyclic involution \( I_m \) of any order \( m \geq 3 \); such an involution is generated by an automorphism of \( V_r \) to which the remarks made in n. 2 apply. We have now a system of equations analogous to (3), which are of the form

\[
\begin{align*}
\{ & u'_i = u_i + a_i \quad (i = 1, 2, \ldots, p; \ p \geq 0) \\
& u'_j = \varepsilon_j u_j \quad (j = p + 1, p + 2, \ldots, q)
\end{align*}
\]

(8)

where \( p \) is the superficial irregularity of the associated involution \( J \) on the Picard - Severi variety \( V_q \), which certainly exists provided \( V_r \) contains no congruence of superficial irregularity \( q \); and where \( \varepsilon_j \) denotes an \( m \) th root of unity, other than unity itself ([7]).

Precisely as in n. 5 we see that: if \( I_m \) has superficial irregularity \( p (0 < p < q) \), then \( V_r \) must contain a superficially irregular congruence. Supposing instead that \( g_i(I_m) = 0 \), we have \( g_i(I^*_{m}) = 0 \), in which case \( p = 0 \) in equations (8).

On this hypothesis, we may proceed to calculate the characters \( g_k(J) \), for \( k = 1, 2, \ldots, r \). To begin with, we have \( g_i(J) = 0 \). Next, \( g_2(J) \) is equal to the number of products
where are different numbers \( k \neq 1 \) occurring in (8), which are equal to unity. And similarly for the remaining characters \( g_k(J) \).

An essential difference between the present case and the preceding is that, while for \( m = 2, p = 0 \), we have an ordinary transformation of \( V_q \), for \( m > 2, p = 0 \), we always have a singular transformation. Such transformations can exist only on varieties \( V_q \) with particular moduli ([3]); and in every case which is a priori possible it must be shown that the corresponding \( V_q \) can effectively be constructed. Moreover, since there is now a number of different involutions \( J \) for any given value of \( m \), the results are necessarily less precise. We have the following analogue of the previous theorems:

If \( V_r \) carries a cyclic involution \( I_m(m \geq 3) \) such that \( g_k(I_m) = 0 \) \( (\forall k) \), then either there exists an associated involution \( J \) on \( V_q \) for which \( g_k(J) = 0 \) \( (\forall k) \), or else \( V_r \) contains a superficially irregular congruence.

The proof is exactly as before. It should be noted that, in the case where the above-mentioned involution \( J \) actually exists, no general conclusion can be drawn. Thus Comessatti [2], in his classification of the irregular cyclic triple planes \( (r = 2, m = 3) \) has shown that all such surfaces contain irrational pencils, though not necessarily of genus \( q \); this had been previously noticed by Bagnera and De Franchis in their study of the hyperelliptic surfaces. Comessatti also quotes an example of an irregular quintuple plane \( (r = 2, m = 5) \) which contains no irrational pencil whatever.

In conclusion, we point out that the previous methods will apply also to the case where, instead of the cyclic involution \( J \), we have on \( V_q \) any superficially regular involution \( J_m(m \geq 3) \), provided always that it is generable by a group \( \mathcal{G} \) (of order \( m \)) of automorphisms of \( V_q \). It is known ([9]) that a sufficient, but not a necessary, condition for \( J_m \) to be so generable is that the image variety of \( J_m \) should have some positive plurigenus. When the group \( \mathcal{G} \) exists, it may be represented analytically by a number of sets of equations such as (8); in that case the characters \( g_k(J_m) \) may be calculated from these equations, and we may then deduce results similar to the preceding.
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