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ON RELATIONS OF CURVATURE TENSORS OVER SEN'S SYSTEM OF AFFINE CONNECTIONS

Nota (*) *di* HRISHIKES SEN (*a Calcutta*)

In this paper an attempt has been made to generalise in a certain way some properties of curvature tensor in Riemannian geometry. In course of his investigations on parallel displacements of vectors in a Riemannian space, Prof. R. N. Sen (Sen, 1950 a) has obtained a system of affine connections which behaves in a very interesting manner.

His system may briefly be described as follows: Consider in a Riemannian space with the fundamental tensor g_{ij} an arbitrary parallel displacement of a contravariant vector ξ^t defined by

$$d\xi^t + \Gamma^t_{ij}\xi^i dx^j = 0. \tag{1.1}$$

Denoting the covariant derivative with respect to the parallelism (1.1) by a comma followed by indices, consider, along with (1.1) two other parallelisms given by

$$d\xi^t + (\Gamma^t_{ij} + g^{lt}g_{il,\,j})\xi^i dx^j = 0, \qquad d\xi^t + \Gamma^t_{ji}\xi^i dx^j = 0$$

which are respectively called the associate and conjugate of (1.1).

Denoting the coefficients of the three affine connections by

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the notations

$$(1.2) \qquad a = \Gamma^t_{ij}, \qquad a^* = \Gamma^t_{ij} + g^{lt} g_{il,j}, \qquad a' = \Gamma^t_{ji},$$

it is seen that the coefficients have the involutory property

$$a^{**} = a'' = a.$$

Parallelism (1.1) is called self-associate if $a = a^*$ and self-conjugate if $a = a'$. Further if we put

$$a_1 = a, \quad a_2 = a^*, \quad a_3 = a^{*\prime}, \quad a_4 = a^{*\prime *}, \ldots,$$

$$\alpha = g^{lt} g_{il,j}, \quad \alpha_c = g^{lt} g_{jl,i}, \quad \gamma = g^{lt} g_{ij,l},$$

$$\beta = g^{lt} g_{im}(\Gamma^m_{lj} - \Gamma^m_{jl}), \quad \beta_c = g^{lt} g_{jm}(\Gamma^m_{li} - \Gamma^m_{il}),$$

we obtain the following cyclic sequence of 12 terms (if they are all distinct):

$$(1.3) \quad \begin{cases} a_1 = a, \quad a_2 = a + \alpha, \quad a_3 = a' + \alpha_c, \quad a_4 = a + \alpha + \beta - \gamma, \\ a_5 = a' + \alpha_c + \beta_c - \gamma, \quad a_6 = a + \alpha + \alpha_c + \beta + \beta_c - \gamma, \\ a_7 = a' + \alpha + \alpha_c + \beta + \beta_c - \gamma, \quad a_8 = a' + \alpha_c + \beta + \beta_c - \gamma, \\ a_9 = a + \alpha + \beta + \beta_c - \gamma, \qquad a_{10} = a' + \alpha_c + \beta_c, \\ a_{11} = a + \alpha + \beta, \qquad a_{12} = a'. \end{cases}$$

The sequence (1.3) is then used to construct the following coefficients of affine connections

$$(1.4) \quad \left(\frac{1}{2}(a_p + a_q)\right)^* = \frac{1}{2}(a_p^* + a_q^*), \quad \left(\frac{1}{2}(a_p + a_q)\right)' = \frac{1}{2}(a_p' + a_q'),$$

where a_p, a_q belong to the sequence (1.3). Sen's system of affine connections generated by a is finally formed to consist of all affine connections which are generated by repeated applications of $*$, $/$ on the set (1.4). In this system, the Christoffel symbols (which define the Levi-Civita parallelism) are given by

$$(1.5) \qquad \left\{ {t \atop ij} \right\} = \frac{1}{2}(a_p + a_{p+6}), \qquad p = 1, 2, \ldots .$$

Levi-Civita parallelism is the only parallelism in Sen's system which is both self-associate and self-conjugate.

In a subsequent paper Sen (Sen, 1950 b) has obtained some fundamental relations connecting curvature tensors formed by the coefficients of affine connections of the system. We state the following result from his work as we shall need it frequently in our discussion.

Let Γ^t_{ij} and L^t_{ij} correspond to two arbitrary affine connections and let

$$T^t_{ij} = \Gamma^t_{ij} - L^t_{ij}, \qquad \Delta^t_{ij} = \frac{1}{2}(\Gamma^t_{ij} + L^t_{ij}).$$

If Γ^t_{ijk}, L^t_{ijk} and Δ^t_{ijk} be the curvature tensors formed with Γ^t_{ij}, L^t_{ij} and Δ^t_{ij} respectively, then

$$\Delta^t_{ijk} - \frac{1}{2}(\Gamma^t_{ijk} + L^t_{ijk}) = \frac{1}{4}(T^t_{sk}T^s_{ij} - T^t_{sj}T^s_{ik}). \tag{1.6}$$

Consequently if $a = \Gamma^t_{ij}$, $b = L^t_{ij}$, $c = \nabla^t_{ij}$, $d = \Omega^t_{ij}$ correspond to any four affine connections such that

$$|a - b| = |c - d|,$$

then

$$C(a) + C(b) - C(c) - C(d) = 2\left[C\left(\frac{1}{2}(a+b)\right) - C\left(\frac{1}{2}(c+d)\right)\right], \tag{1.7}$$

where

$$C(a) = \Gamma^t_{ijk} = \frac{\partial\Gamma^t_{ik}}{\partial x^j} - \frac{\partial\Gamma^t_{ij}}{\partial x^k} + \Gamma^t_{hj}\Gamma^h_{ik} - \Gamma^t_{hk}\Gamma^h_{ij}. \tag{1.8}$$

Similarly for the other curvature tensors in (1.7). As an aplication of (1.7), Sen has shown that since, by (1.5), $\frac{1}{2}(a_p + a_{p+6}) = \frac{1}{2}(a_q + a_{q+6})$, it follows immediately that

$$C(a_p) + C(a_q) - C(a_{p+6}) - C(a_{q+6}) = \tag{1.9}$$

$$= 2\left\{C\left(\frac{1}{2}(a_p + a_q)\right) - C\left(\frac{1}{2}(a_{p+6} + a_{q+6})\right)\right\}.$$

Now we know that the fully covariant Riemannian curvature tensor R_{hijk} satisfies the following relations with regard to indices:

$$R_{hijk} + R_{hikj} = 0, \tag{1.10}$$

$$R_{hijk} + R_{hjki} + R_{hkij} = 0, \tag{1.11}$$

$$R_{hijk} + R_{ihjk} = 0, \tag{1.12}$$

$$R_{hijk} - R_{jkhi} = 0, \tag{1.13}$$

$$R^t_{ijk;l} + R^t_{ikl;j} + R^t_{ilj;k} = 0, \tag{1.14}$$

and

$$R_{hijk;l} + R_{hikl;j} + R_{hilj;k} = 0,$$

where semi-colon denotes covariant derivative with respect to Levi-Civita parallelism. For any arbitrary affine connection $a = \Gamma^t_{ij}$, write

$$g_{ht}\Gamma^t_{ijk} = \Gamma_{hijk} \tag{1.15}$$

It is then obvious from (1.8) that Γ_{hijk} satisfies the property of the indices (1.10). It is also obvious that if a is self-conjugate, then Γ_{hijk} satisfies the cyclical property (1.11). Further, writing $g_{ht}C(a^*) = \Gamma^*_{hijk}$, Sen (Sen, 1950 b) has shown that

$$\Gamma_{hijk} + \Gamma^*_{ihjk} = 0. \tag{1.16}$$

Accordingly, if a is self-associate, Γ_{hijk} satisfies the property (1.12).

The relation (1.16) may be considered to be a generalisation of (1.12).

There are 6 such formulae over Sen's sequence (1.3), namely those formed with respect to the pairs (a_1, a_2), (a_3, a_4), ..., (a_{11}, a_{12}).

In this paper we propose to generalise the relations (1.11), (1.13) and the Bianchi identities (1.14) over Sen's system of affine connections.

2. - The relations (1.10) to (1.13) are not all independent. In this section we take up analogous consideration with regard to an arbitrary affine connection. Put

$$
(2.1)\left\{\begin{aligned} {}_1A_{hijk} &\equiv \Gamma_{hijk}+\Gamma_{hjki}+\Gamma_{hkij}, & {}_2A_{hijk} &\equiv \Gamma_{hijk}+\Gamma_{jikh}+\Gamma_{kihj}, \\ B_{hijk} &\equiv \Gamma_{hijk}+\Gamma_{ihjk}, & D_{hijk} &\equiv \Gamma_{hijk}-\Gamma_{jkhi}. \end{aligned}\right.
$$

It is easily seen that ${}_1A_{hijk}$ is skew in the 3 pairs of suffixes i, j, k. ${}_2A_{hijk}$ is skew in pairs of suffixes h, j, k. B_{hijk} is symmetric in h, i but skew in j, k. And $D_{hijk}+D_{jkhi}=0$.

We establish the following properties:

1) If ${}_1A_{hijk}-{}_1A_{jkhi}+{}_1A_{khij}-{}_1A_{ijkh}=0$ and $B_{hijk}=0$ then $D_{hijk}=0$.

This is easily seen by writing out in terms of Γ_{hijk} and simplifying.

2) If ${}_2A_{hijk}-{}_2A_{kjih}+{}_2A_{ikhj}-{}_2A_{jhki}=0$ and $B_{hijk}=0$, then $D_{hijk}=0$.

Proof is the same as above.

3) If $D_{hijk}=0$, then $B_{hijk}=0$, ${}_1A_{hijk}={}_1A_{jkhi}={}_1A_{ihkj}=$ $={}_1A_{kjih}$ and ${}_2A_{hijk}={}_2A_{jkhi}={}_2A_{ihkj}={}_2A_{kjih}$.

For, we have

$$\Gamma_{hijk}=\Gamma_{jkhi}=-\Gamma_{jkih}=-\Gamma_{ihjk}, \quad i.e., \quad B_{hijk}=0.$$

Again, since $B_{hijk}=0$, we have

$$\Gamma_{hijk}=\Gamma_{jkhi}, \quad \Gamma_{hjki}=\Gamma_{jhik}, \quad \Gamma_{hkij}=\Gamma_{jikh}.$$

Adding, we get ${}_1A_{hijk}={}_1A_{jkhi}$. Similarly for the others.

4) ${}_1A_{hijk}={}_1A_{jkhi}$ if and only if ${}_1A_{hijk}+{}_1A_{ihjk}=0$.

For

$${}_1A_{hijk}=-{}_1A_{hjik}={}_1A_{jhik}=-{}_1A_{jhki}={}_1A_{jkhi}.$$

Conversely,

$${}_1A_{hijk}=-{}_1A_{hikj}=-{}_1A_{kjhi}={}_1A_{kjih}=-{}_1A_{ihjk}.$$

This property remains unchanged if ${}_1A_{hijk}$ is replaced by ${}_2A_{hijk}$.

5) If $B_{hijk} = 0$, then ${}_1A_{hijk} + {}_2A_{ihjk} = 0$.
This is easily proved.

6) Let Γ^t_{ij} be self-conjugate. Then from (1.16), by taking the sum of cyclic permutations of i, j, k and remembering that ${}_1A_{hijk} = 0$, we have ${}_2A^*_{ihjk} = 0$. Thus it is seen that if Γ^t_{ij} is self-conjugate, then

$$ {}_2A^*_{hijk} = 0, \tag{2.2} $$

but ${}_2A_{hijk}$ and ${}_1A^*_{hijk}$ are not necessarily zero.

3. - In this and the next section we generalise the relations (1.11) and (1.13) over Sen's system of affine connections. For this purpose we adopt the following notation. If a_p and a_q are any two terms of the sequence (1.3), put

$$ C(a_p) = {}^{a_p}\Gamma^t_{ijk}\,, \qquad C\left(\frac{1}{2}(a_p + a_q)\right) = {}^{\frac{1}{2}(a_p + a_q)}\Gamma^t_{ijk}\,. \tag{3.1} $$

Now let us suppose that a_p and a_q are conjugate to one another.

Then $\frac{1}{2}(a_p + a_q)$ and $\frac{1}{2}(a_{p+6} + a_{q+6})$ are self-conjugate. Hence using notations (1.15) and (2.1), we get

$$ {}^{\frac{1}{2}(a_p + a_q)}{}_1A_{hijk} = 0, \qquad {}^{\frac{1}{2}(a_{p+6} + a_{q+6})}{}_1A_{hijk} = 0. $$

Accordingly, from (1.9) we arrive at the following result:

$$ {}^{a_p}{}_1A_{hijk} + {}^{a_q}{}_1A_{hijk} = {}^{a_{p+6}}{}_1A_{hijk} + {}^{a_{q+6}}{}_1A_{hijk}\,. \tag{3.2} $$

In the sequence (1.3), the pairs $(a_1, a_{12}), (a_2, a_3), (a_4, a_5)$ are conjugate. Therefore the relations (3.2) over Sen's system give 3 formulae

$$ \begin{cases} {}^{a_1}_{\ 1}A_{hijk} + {}^{a_{12}}_{\ 1}A_{hijk} - {}^{a_7}_{\ 1}A_{hijk} - {}^{a_6}_{\ 1}A_{hijk} = 0, \\ {}^{a_2}_{\ 1}A_{hijk} + {}^{a_3}_{\ 1}A_{hijk} - {}^{a_8}_{\ 1}A_{hijk} - {}^{a_9}_{\ 1}A_{hijk} = 0, \\ {}^{a_4}_{\ 1}A_{hijk} + {}^{a_5}_{\ 1}A_{hijk} - {}^{a_{10}}_{\ 1}A_{hijk} - {}^{a_{11}}_{\ 1}A_{hijk} = 0. \end{cases} \tag{3.3} $$

Again, by taking the cyclic permutations of i, j, k of (1.16) and adding and using notation (2.1) we get

$$ {}_1A_{hijk} + {}_2A^{*}_{ihjk} = 0. $$

Therefore

$$ {}^{a_p}_{1}A_{hijk} = - {}^{a_p^{*}}_{2}A_{ihjk} \, . $$

Accordingly (3.3) reduces to

$$ (3.4) \quad \begin{cases} {}^{a_2}_{2}A_{hijk} + {}^{a_{11}}_{2}A_{hijk} - {}^{a_8}_{1}A_{hijk} - {}^{a_5}_{2}A_{hijk} = 0, \\ {}^{a_1}_{2}A_{hijk} + {}^{a_4}_{2}A_{hijk} - {}^{a_7}_{2}A_{hijk} - {}^{a_{10}}_{2}A_{hijk} = 0, \\ {}^{a_3}_{2}A_{hijk} + {}^{a_6}_{2}A_{hijk} - {}^{a_9}_{2}A_{hijk} - {}^{a_{12}}_{2}A_{hijk} = 0. \end{cases} $$

The equations (3.3) or (3.4) give the required generalisation of (1.11).

4. - Let us put $u = \left\{ {t \atop ij} \right\}$, the Christoffel symbol. Then by (1.5)

$$ u = \frac{1}{2}(a_p + a_{p+6}) = \frac{1}{2}(a_q + a_{q+6}). $$

Now

$$ a_{p+6} - a_p = a_q - a_p + (a_{p+6} - a_q) $$

and

$$ a_q - a_{q+6} = a_q - a_p - (a_{q+6} - a_p) $$

Also

$$ a_{p+6} - a_q = a_{q+6} - a_p \, . $$

Put

$$ a_q - a_p = V^t_{ij} , \quad a_{p+6} - a_q = \Lambda^t_{ij} = a_{q+6} - a_p \, . $$

Then

$$ a_{p+6} - a_p = V^t_{ij} + \Lambda^t_{ij} , \qquad a_q - a_{q+6} = V^t_{ij} - \Lambda^t_{ij} \, . $$

Therefore, using notation (1.8), we have, by (1.6)

$$(4.1)\qquad C(u)=C\left(\frac{1}{2}(a_{p+6}+a_p)\right)=\frac{1}{2}\{C(a_{p+6})+C(a_p)\}$$
$$+\frac{1}{4}\left\{(V^t_{sk}+\Lambda^t_{sk})(V^s_{ij}+\Lambda^s_{ij})-(V^t_{sj}+\Lambda^t_{sj})(V^s_{ik}+\Lambda^s_{ik})\right\},$$

$$(4.2)\qquad C(u)=C\left(\frac{1}{2}(a_{q+6}+a_q)\right)=\frac{1}{2}\{C(a_{q+6})+C(a_q)\}$$
$$+\frac{1}{4}\left\{(V^t_{sk}-\Lambda^t_{sk})(V^s_{ij}-\Lambda^s_{ij})-(V^t_{sj}-\Lambda^t_{sj})(V^s_{ik}-\Lambda^s_{ik})\right\}.$$

Adding (4.1) and (4.2),

$$(4.3)\qquad 2C(u)=\frac{1}{2}\{C(a_p)+C(a_q)+C(a_{p+6})+C(a_{q+6})\}$$
$$+\frac{1}{2}\left\{V^t_{sk}V^s_{ij}+\Lambda^t_{sk}\Lambda^s_{ij}-V^t_{sj}V^s_{ik}-\Lambda^t_{sj}\Lambda^s_{ik}\right\}.$$

Again by (1.6),

$$2C\left(\frac{1}{2}(a_p+a_q)\right)-\{C(a_p)+C(a_q)\}=\frac{1}{2}(V^t_{sk}V^s_{ij}-V^t_{sj}V^s_{ik}),$$
$$2C\left(\frac{1}{2}(a_{p+6}+a_q)\right)-\{C(a_{p+6})+C(a_q)\}=\frac{1}{2}(\Lambda^t_{sk}\Lambda^s_{ij}-\Lambda^t_{sj}\Lambda^s_{ik}).$$

Therefore (4.3) reduces to

$$2C(u)-\frac{1}{2}\{C(a_p)+C(a_q)+C(a_{p+6})+C(a_{q+6})\}-$$
$$-2C\left(\frac{1}{2}(a_p+a_q)\right)+\{C(a_p)+C(a_q)\}$$
$$=2C\left(\frac{1}{2}(a_{p+6}+a_q)\right)-\{C(a_{p+6})+C(a_q)\}.$$

Or,

$$(4.4)\qquad 2C(u)-2C\left(\frac{1}{2}(a_p+a_q)\right)+\frac{1}{2}C(a_p)+\frac{3}{2}C(a_q)+\frac{1}{2}C(a_{p+6})$$
$$-\frac{1}{2}C(a_{q+6})-2C\left(\frac{1}{2}(a_{p+6}+a_q)\right)=0.$$

Also, interchanging p and q,

$$(4.5)\qquad 2C(u)-2C\left(\frac{1}{2}(a_p+a_q)\right)+\frac{1}{2}C(a_q)+\frac{3}{2}C(a_p)+\frac{1}{2}C(a_{q+6}) - \frac{1}{2}C(a_{p+6})-2C\frac{1}{2}(a_{q+6}+a_p)\Big)=0.$$

Taking half the sum of (4.4) and (4.5),

$$(4.6)\qquad 2C(u)-2C\left(\frac{1}{2}(a_p+a_q)\right)+C(a_p)+C(a_q)- -\left\{C\left(\frac{1}{2}(a_{p+6}+a_q)\right)+C\left(\frac{1}{2}(a_{q+6}+a_p)\right)\right\}=0.$$

Now $C(u)=R^t_{ijk}$ is the Riemannian curvature tensor which satisfies $R_{hijk}-R_{jkhi}=0$. Therefore, using notation (3.1), we derive from (4.6) the following relation

$$(4.7)\qquad ({}^{a_p}\Gamma_{hijk}-{}^{a_p}\Gamma_{jkhi})+({}^{a_q}\Gamma_{hijk}-{}^{a_q}\Gamma_{jkhi})- -2({}^{\frac{1}{2}(a_p+a_q)}\Gamma_{hijk}-{}^{\frac{1}{2}(a_p+a_q)}\Gamma_{jkhi}) =({}^{\frac{1}{2}(a_p+a_{q+6})}\Gamma_{hijk}-{}^{\frac{1}{2}(a_p+a_{q+6})}\Gamma_{jkhi})+ +({}^{\frac{1}{2}(a_q+a_{p+6})}\Gamma_{hijk}-{}^{\frac{1}{2}(a_q+a_{p+6})}\Gamma_{jkhi}).$$

The equation (4.7) may be considered as the required generalisation of (1.13).

5. - In this section we proceed to generalise Bianchi's identities (1.14) over Sen's system of affine connections.

As before let Γ^t_{ij} be the coefficients of an arbitrary affine connection and let Γ^t_{ijk} be the curvature tensor formed with it. Denote the covariant derivative with respect to Γ^t_i by a comma and put $\Gamma^t_{ij}-\Gamma^t_{ji}=V^t_{ij}$. Then by straightforward calculation we derive

$$(5.1)\qquad \Gamma^t_{ijk,l}+\Gamma^t_{ikl,j}+\Gamma^t_{ilj,k}=\Gamma^t_{isl}V^s_{jk}+\Gamma^t_{isj}V^s_{kl}+\Gamma^t_{isk}V^s_{lj}.$$

In particular when Γ^t_{ij} is symmetric, $V^t_{ij}=0$ and hence

(5.1) reduces to

$$\Gamma^t_{ijk,l} + \Gamma^t_{ikl,j} + \Gamma^t_{ilj,k} = 0, \tag{5.2}$$

which is the identity of Bianchi for a symmetric connection.

Multiplying (5.1) by g_{ht} and summing with respect to t, we get

$$\begin{aligned}\Gamma_{hijk,l} + \Gamma_{hikl,j} + \Gamma_{hilj,k} = \Gamma_{hisl} V^s_{jk} + \Gamma_{hisj} V^s_{kl} + \Gamma_{hisk} V^s_{lj} \\ + g_{ht,l}\Gamma^t_{ijk} + g_{ht,j}\Gamma^t_{ikl} + g_{ht,k}\Gamma^t_{ilj}.\end{aligned} \tag{5.3}$$

As we have now to consider the equations (5.1) and (5.3) over Sen's system, it seems convenient to use the following notations.

With reference to the sequence (1.3), put

$$a_p = {}^{a_p}\Gamma^t_{ij} \quad , \quad {}^{a_p}\Gamma^t_{ij} - {}^{a_p}\Gamma^t_{ji} = {}^{a_p}V^t_{ij}. \tag{5.4}$$

It is immediately seen from (1.3) that

$$\begin{cases} {}^{a_p'}V^t_{ij} = {}^{a_{p+6}}V^t_{ij} = -{}^{a_p}V^t_{ij}. \\ {}^{\frac{1}{2}(a_p+a_{p+5})}V^t_{ij} = {}^{a_p}V^t_{ij} \quad \text{if } p \text{ is odd,} \quad {}^{\frac{1}{2}(a_p+a_{p+7})}V^t_{ij} = {}^{a_p}V^t_{ij} \end{cases} \tag{5.5}$$

if p is even.

Also, we adopt the notation (3.1) for curvature tensors.

Further, denoting covariant derivative with respect to a_p by a solidus, put

$$\{{}^{a_p}\Gamma^t_{ijkl}\} \equiv {}^{a_p}\Gamma^t_{ijk|l} + {}^{a_p}\Gamma^t_{ikl|j} + {}^{a_p}\Gamma^t_{ilj|k}. \tag{5.6}$$

Thus, since $\Gamma^t_{ij} = a = a_1 = {}^{a_1}\Gamma^t_{ij}$, equations (5,1) and (5.3) may be written as

$$\{{}^{a_1}\Gamma^t_{ijkl}\} = {}^{a_1}\Gamma^t_{isl}\,{}^{a_1}V^s_{jk} + {}^{a_1}\Gamma^t_{isj}\,{}^{a_1}V^s_{kl} + {}^{a_1}\Gamma^t_{isk}\,{}^{a_1}V^s_{lj}$$

and

$$\begin{aligned}\{{}^{a_1}\Gamma_{hikl}\} = {}^{a_1}\Gamma_{hisl}\,{}^{a_1}V^s_{jk} + {}^{a_1}\Gamma_{hisj}\,{}^{a_1}V^s_{kl} + {}^{a_1}\Gamma_{hisk}\,{}^{a_1}V^s_{lj} \\ + g_{ht,l}\,{}^{a_1}\Gamma^t_{ijk} + g_{ht,j}\,{}^{a_1}\Gamma^t_{ikl} + g_{ht,k}\,{}^{a_1}\Gamma^t_{ilj}.\end{aligned}$$

Now using these notations and applying (5.5) and (1.9), we get from (5.1)

$$(5.7)\qquad \{{}^{a_1}\Gamma^t_{ijkl}\} + \{{}^{a_{12}}\Gamma^t_{ijkl}\} + \{{}^{a_7}\Gamma^t_{ijkl}\} + \{{}^{a_6}\Gamma^t_{ijkl}\}$$

$$= ({}^{a_1}\Gamma^t_{isl} - {}^{a_{12}}\Gamma^t_{isl} - {}^{a_7}\Gamma^t_{isl} + {}^{a_6}\Gamma^t_{isl})\,{}^{a_1}V^s_{jk}$$

$$+ ({}^{a_1}\Gamma^t_{isj} - {}^{a_{12}}\Gamma^t_{isj} - {}^{a_7}\Gamma^t_{isj} + {}^{a_6}\Gamma^t_{isj})\,{}^{a_1}V^s_{kl}$$

$$+ ({}^{a_1}\Gamma^t_{isk} - {}^{a_{12}}\Gamma^t_{isk} - {}^{a_7}\Gamma^t_{isk} + {}^{a_6}\Gamma^t_{isk})\,{}^{a_1}V^s_{lj}$$

$$= 2[{}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{isl} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{isl}]\,{}^{a_1}V^s_{jk}$$

$$+ 2[{}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{isj} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{isj}]\,{}^{a_1}V^s_{kl}$$

$$+ 2[{}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{isk} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{isk}]\,{}^{a_1}V^s_{lj}.$$

But

$$\{{}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{ijkl}\} = {}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{isl}\,{}^{\frac{1}{2}(a_1+a_6)}V^s_{jk}$$

$$+ {}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{isj}\,{}^{\frac{1}{2}(a_1+a_6)}V^s_{kl} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{isk}\,{}^{\frac{1}{2}(a_1+a_6)}V^s_{lj}.$$

Similarly for $\{{}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{ijkl}\}$.
Therefore, applying (5.5)

$$(5.8)\qquad \{{}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{ijkl}\} + \{{}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{ijkl}\} =$$

$$[{}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{isl} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{isl}]\,{}^{a_1}V^s_{jk} +$$

$$+ [{}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{isj} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{isj}]\,{}^{a_1}V^t_{kl} +$$

$$+ [{}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{isk} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{isk}]\,{}^{a_1}V^s_{lj}.$$

Hence, from (5.7) and (5.8), we have finally

$$\{{}^{a_1}\Gamma^t_{ijkl}\} + \{{}^{a_{12}}\Gamma^t_{ijkl}\} + \{{}^{a_7}\Gamma^t_{ijkl}\} + \{{}^{a_6}\Gamma^t_{ijkl}\}$$

$$= 2[\{{}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{ijkl}\} + \{{}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{ijkl}\}].$$

Thus we have

$$
(5.9)\quad \begin{cases}
\{{}^{a_1}\Gamma^t_{ijkl}\} + \{{}^{a_{12}}\Gamma^t_{ijkl}\} + \{{}^{a_7}\Gamma^t_{ijkl}\} + \{{}^{a_6}\Gamma^t_{ijkl}\} \\
\qquad - 2[\{{}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{ijkl}\} + \{{}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{ijkl}\}] = 0, \\
\{{}^{a_2}\Gamma^t_{ijkl}\} + \{{}^{a_3}\Gamma^t_{ijkl}\} + \{{}^{a_8}\Gamma^t_{ijkl}\} + \{{}^{a_9}\Gamma^t_{ijkl}\} \\
\qquad - 2[\{{}^{\frac{1}{2}(a_2+a_9)}\Gamma^t_{ijkl}\} + \{{}^{\frac{1}{2}(a_3+a_8)}\Gamma^t_{ijkl}\}] = 0, \\
\{{}^{a_4}\Gamma^t_{ijkl}\} + \{{}^{a_5}\Gamma^t_{ijkl}\} + \{{}^{a_{10}}\Gamma^t_{ijkl}\} + \{{}^{a_{11}}\Gamma^t_{ijkl}\} \\
\qquad - 2[\{{}^{\frac{1}{2}(a_4+a_{11})}\Gamma^t_{ijkl}\} + \{{}^{\frac{1}{2}(a_5+a_{10})}\Gamma^t_{ijkl}\}] = 0.
\end{cases}
$$

The three equations (5.9) constitute the required generalisation of the first of the identities (1.14). We now proceed to generalise the other part of (1.14). In order to do so we require the following property which is obtained from (1.3):

Writing V^t_{ij} for ${}^{a_1}V^t_{ij}$, covariant derivative (with adopted notation) of g_{ij}

$$
(5.10)\quad \begin{cases}
\text{with respect to } a_1,\ a_8 \text{ is } g_{ij,k}, \\
\quad \text{«} \qquad a_3,\ a_{10} \text{ is } g_{ij,k} - g_{ik,j} - g_{jk,i} - g_{is}V^s_{kj} - g_{js}V^s_{ki}, \\
\quad \text{«} \qquad a_5,\ a_{12} \text{ is } g_{ij,k} - g_{is}V^s_{kj} - g_{js}V^s_{ki}, \\
\quad \text{«} \qquad a_p \text{ and } a_{p+6} \text{ are negative of one another.}
\end{cases}
$$

Now multiplying first of the equations (5.9) by g_{ht}, summing for t and applying (5.10), we get

$$
\begin{aligned}
(5.11)\quad & \{{}^{a_1}\Gamma_{hijkl}\} + \{{}^{a_{12}}\Gamma_{hijkl}\} + \{{}^{a_7}\Gamma_{hijkl}\} + \{{}^{a_6}\Gamma_{hijkl}\} \\
& \qquad - 2[\{{}^{\frac{1}{2}(a_1+a_6)}\Gamma_{hijkl}\} + \{{}^{\frac{1}{2}(a_7+a_{12})}\Gamma_{hijkl}\}] \\
& = g_{ht,l}({}^{a_1}\Gamma^t_{ijk} + {}^{a_{12}}\Gamma^t_{ijk} - {}^{a_7}\Gamma^t_{ijk} - {}^{a_6}\Gamma^t_{ijk}) \\
& + g_{ht,j}({}^{a_1}\Gamma^t_{ikl} + {}^{a_{12}}\Gamma^t_{ikl} - {}^{a_7}\Gamma^t_{ikl} - {}^{a_6}\Gamma^t_{ikl}) \\
& + g_{ht,k}({}^{a_1}\Gamma^t_{ilj} + {}^{a_{12}}\Gamma^t_{ilj} - {}^{a_7}\Gamma^t_{ilj} - {}^{a_6}\Gamma^t_{ilj}) \\
& + (g_{hs}V^s_{tl} + g_{ts}V^s_{hl})({}^{a_{12}}\Gamma^t_{ijk} - {}^{a_6}\Gamma^t_{ijk} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{ijk} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{ijk}) \\
& + (g_{hs}V^s_{tj} + g_{ts}V^s_{hj})({}^{a_{12}}\Gamma^t_{ikl} - {}^{a_6}\Gamma^t_{ikl} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{ikl} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{ikl}) \\
& + (g_{hs}V^s_{tk} + g_{ts}V^s_{hk})({}^{a_{12}}\Gamma^t_{ilj} - {}^{a_6}\Gamma^t_{ilj} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{ilj} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{ilj}).
\end{aligned}
$$

By (1.9), we have

$$ {}^{a_1}\Gamma^t_{ijk} + {}^{a_{12}}\Gamma^t_{ijk} - {}^{a_7}\Gamma^t_{ijk} - {}^{a_6}\Gamma^t_{ijk} = 2[{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{ijk} - {}^{\frac{1}{2}(a_6+a_7)}\Gamma^t_{ijk}], $$

$$ {}^{a_{12}}\Gamma^t_{ijk} - {}^{a_6}\Gamma^t_{ijk} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{ijk} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{ijk} $$

$$ = {}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{ijk} - {}^{\frac{1}{2}(a_6+a_7)}\Gamma^t_{ijk}). $$

Therefore the right-hand side of (5.11) becomes

$$ (5.12) \qquad (2g_{ht,l} + g_{hs}V^s_{tl} + g_{ts}V^s_{hl})({}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{ijk} - {}^{\frac{1}{2}(a_6+a_7)}\Gamma^t_{ijk} $$

$$ + (2g_{ht,j} + g_{hs}V^s_{tj} + g_{ts}V^s_{hj})({}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{ikl} - {}^{\frac{1}{2}(a_6+a_7)}\Gamma^t_{ikl}) $$

$$ + (2g_{ht,k} + g_{hs}V^s_{tk} + g_{ts}V^s_{hk})({}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{ilj} - {}^{\frac{1}{2}(a_6+a_7)}\Gamma^t_{ilj}). $$

Again, since $\frac{1}{2}(a_1 + a_{12})$ and $\frac{1}{2}(a_6 + a_7)$ are symmetric, we have from (5.3) and (5.10)

$$ \{{}^{\frac{1}{2}(a_1+a_{12})}\Gamma_{hijkl}\} = \left[g_{ht,l} + \frac{1}{2}(g_{hs}V^s_{tl} + g_{ts}V^s_{hl})\right]{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{ijk} $$

$$ + \left[g_{ht,j} + \frac{1}{2}(g_{hs}V^s_{tj} + g_{ts}V^s_{hj})\right]{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{ikl} $$

$$ + \left[g_{ht,k} + \frac{1}{2}(g_{hs}V^s_{tk} + g_{ts}V^s_{hk})\right]{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{ilj}. $$

Similarly for

$$ \{{}^{\frac{1}{2}(a_6+a_7)}\Gamma_{hijkl}\}. $$

Therefore,

$$ (5.13) \qquad 2[\{{}^{\frac{1}{2}(a_1+a_{12})}\Gamma_{hijkl}\} + \{{}^{\frac{1}{2}(a_6+a_7)}\Gamma_{hijkl}\}] $$

$$ = (2g_{ht,l} + g_{hs}V^s_{tl} + g_{ts}V^s_{hl})[{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{ijk} - {}^{\frac{1}{2}(a_6+a_7)}\Gamma^t_{ijk}] $$

$$ + (2g_{ht,j} + g_{hs}V^s_{tj} + g_{ts}V^s_{hj})[{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{ikl} - {}^{\frac{1}{2}(a_6+a_7)}\Gamma^t_{ikl}] $$

$$ + (2g_{ht,k} + g_{hs}V^s_{tk} + g_{ts}V^s_{hk})[{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{ilj} - {}^{\frac{1}{2}(a_6+a_7)}\Gamma^t_{ilj}]. $$

Hence, from (5.12) and (5.13) we see that (5.11) finally reduces to

$$\{{}^{a_1}\Gamma_{hijkl}\} + \{{}^{a_{12}}\Gamma_{hijkl}\} + \{{}^{a_7}\Gamma_{hijkl}\} + \{{}^{a_6}\Gamma_{hijkl}\}$$
$$-2[\{{}^{\frac{1}{2}(a_1+a_6)}\Gamma_{hijkl}\} + \{{}^{\frac{1}{2}(a_7+a_{12})}\Gamma_{hijkl}\}$$
$$+\{{}^{\frac{1}{2}(a_1+a_{12})}\Gamma_{hijkl}\} + \{{}^{\frac{1}{2}(a_6+a_7)}\Gamma_{hijkl}\}] = 0.$$

Thus we have

$$(5.14)\quad \left\{\begin{aligned}
&\{{}^{a_1}\Gamma_{hijkl}\} + \{{}^{a_{12}}\Gamma_{hijkl}\} + \{{}^{a_7}\Gamma_{hijkl}\} + \{{}^{a_6}\Gamma_{hijkl}\}\\
&\quad -2[\{{}^{\frac{1}{2}(a_1+a_6)}\Gamma_{hijkl}\} + \{{}^{\frac{1}{2}(a_7+a_{12})}\Gamma_{hijkl}\}\\
&\quad +\{{}^{\frac{1}{2}(a_1+a_{12})}\Gamma_{hijkl}\} + \{{}^{\frac{1}{2}(o_6+a_7)}\Gamma_{hijkl}\}] = 0,\\
&\{{}^{a_2}\Gamma_{hijkl}\} + \{{}^{a_3}\Gamma_{hijkl}\} + \{{}^{a_8}\Gamma_{hijkl}\} + \{{}^{a_9}\Gamma_{hijkl}\}\\
&\quad -2[\{{}^{\frac{1}{2}(a_2+a_9)}\Gamma_{hijkl}\} + \{{}^{\frac{1}{2}(a_3+a_8)}\Gamma_{hijkl}\}\\
&\quad +\{{}^{\frac{1}{2}(a_2+a_3)}\Gamma_{hijkl}\} + \{{}^{\frac{1}{2}(a_8+a_9)}\Gamma_{hijkl}\}] = 0,\\
&\{{}^{a_4}\Gamma_{hijkl}\} + \{{}^{a_5}\Gamma_{hijkl}\} + \{{}^{a_{10}}\Gamma_{hijkl}\} + \{{}^{a_{11}}\Gamma_{hijkl}\}\\
&\quad -2[\{{}^{\frac{1}{2}(a_4+a_{11})}\Gamma_{hijkl}\} + \{{}^{\frac{1}{2}(a_5+a_{10})}\Gamma_{hijkl}\}\\
&\quad +\{{}^{\frac{1}{2}(a_4+a_5)}\Gamma_{hijkl}\} + \{{}^{\frac{1}{2}(a_{10}+a_{11})}\Gamma_{hijkl}\}] = 0.
\end{aligned}\right.$$

The three equations (5.14) constitute the generalisation of the second of the identities (1.14) over Sen's system of affine connections.

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