

RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

S. K. CHATTERJEE

**On certain definite integrals involving
Legendre's polynomials**

Rendiconti del Seminario Matematico della Università di Padova,
tome 27 (1957), p. 144-148

http://www.numdam.org/item?id=RSMUP_1957__27__144_0

© Rendiconti del Seminario Matematico della Università di Padova, 1957, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

ON CERTAIN DEFINITE INTEGRALS INVOLVING LEGENDRE'S POLYNOMIALS

Nota ()* di S. K. CHATTERJEE (*a Calcutta*)

1. - The object of the present note is to evaluate certain definite integrals involving Legendre's Polynomials. Using the symbol $\{u(x)\}_r$ for $\frac{d^r u(x)}{dx^r}$, we write

$$(1.1) \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n = \frac{1}{2^n n!} \{(x^2 - 1)^n\}_n.$$

2. - Let m, n, r and s be four positive integers such that $1 \leq m \leq n$ and $1 \leq r \leq m, 1 \leq s \leq n$.

Then

$$(2.1) \quad \int_{-1}^1 \frac{d^r P_m(x)}{dx^r} \cdot \frac{d^s P_n(x)}{dx^s} dx \\ = \frac{1}{2^{m+n} m! n!} \int_{-1}^1 \{(x^2 - 1)^m\}_{m+r} \{(x^2 - 1)^n\}_{n+s} dx.$$

(*) Pervenuta in Redazione il 24 ottobre 1956.

Indirizzo dell'A.: Department of Pure Mathematics, University, Calcutta (India).

3. - CASE I. - If $2 \leq r + s \leq m$ we have

$$\begin{aligned}
 (3.1) \quad & \int_{-1}^1 \{ (x^2 - 1)^m \}_{m+r} \{ (x^2 - 1)^n \}_{n+s} dx \\
 &= \sum_{t=0}^{s-1} (-)^t [\{ (x^2 - 1)^m \}_{m+r+t} \{ (x^2 - 1)^n \}_{n+s-t-1}]_{-1}^1 \\
 &+ (-)^s \int_{-1}^1 \{ (x^2 - 1)^m \}_{m+r+s} \{ (x^2 - 1)^n \}_n dx.
 \end{aligned}$$

The last integral vanishes on account of the orthogonal property of Legendre's Polynomials. If however $r + s \geq m + 1$, the process comes to an end earlier.

CASE II. - Let $m + 1 \leq r + s \leq 2m$. Writing $r + s = m + \delta$ we have $1 \leq \delta \leq m$. In this case

$$\begin{aligned}
 (3.2) \quad & \int_{-1}^1 \{ (x^2 - 1)^m \}_{m+r} \{ (x^2 - 1)^n \}_{n+s} dx \\
 &= \sum_{t=0}^{s-\delta} (-)^t [\{ (x^2 - 1)^m \}_{m+r+t} \{ (x^2 - 1)^n \}_{n+s-t-1}]_{-1}^1.
 \end{aligned}$$

CASE III. - Let $r + s \geq 2m + 1$, where $r \leq m < s \leq n$ writing $r + \Delta = m$, we have $0 \leq \Delta \leq m - 1$.

Here, we have

$$\begin{aligned}
 (3.3) \quad & \int_{-1}^1 \{ (x^2 - 1)^m \}_{m+r} \{ (x^2 - 1)^n \}_{n+s} dx \\
 &= \sum_{t=0}^{\Delta} (-)^t [\{ (x^2 - 1)^m \}_{m+r+t} \{ (x^2 - 1)^n \}_{n+s-t-1}]_{-1}^1.
 \end{aligned}$$

Now, it may be easily shewn that

$$\begin{aligned}
 (3.4) \quad & [\{ (x^2 - 1)^m \}_{m+r+t} \{ (x^2 - 1)^n \}_{n+s-t-1}]_{-1}^1 \\
 &= 2^{m+n-r-s+1} m! n! (r+t)! (s-t-1)! \\
 & \binom{m+r+t}{r+t} \binom{m}{r+t} \binom{n+s-t-1}{s-t-1} \binom{n}{s-t-1}
 \end{aligned}$$

provided $r + t \leq m$ and $s - t - 1 \geq 0$, which is certainly true in the present cases.

4. - It follows, therefore, in case I

$$(4.1) \quad \int_{-1}^1 \frac{d^r P_m(x)}{dx^r} \frac{d^s P_n(x)}{dx^s} dx$$

$$= K \sum_{t=0}^{s-1} (-)^t \left[(r+t)! (s-t-1)! \binom{m+r+t}{r+t} \binom{m}{r+t} \binom{n+s-t-1}{s-t-1} \binom{n}{s-t-1} \right]$$

where

$$K = \frac{1}{2^{r+s-1}} [1 + (-)^{m+n-r-s}].$$

In case II

$$(4.2) \quad \int_{-1}^1 \frac{d^r P_m(x)}{dx^r} \frac{d^s P_n(x)}{dx^s} dx$$

$$= K \sum_{t=0}^{s-\delta} (-)^t \left[(r+t)! (s-t-1)! \binom{m+r+t}{r+t} \binom{m}{r+t} \binom{n+s-t-1}{s-t-1} \binom{n}{s-t-1} \right]$$

And in case III

$$(4.3) \quad \int_{-1}^1 \frac{d^r P_m(x)}{dx^r} \frac{d^s P_n(x)}{dx^s} dx$$

$$= K \sum_{t=0}^{\Delta} (-)^t \left[(r+t)! (s-t-1)! \binom{m+r+t}{r+t} \binom{m}{r+t} \binom{n+s-t-1}{s-t-1} \binom{n}{s-t-1} \right]$$

If $r = s = 1$ and $m \geq 2$, case I gives the well known result [1] (Clare, 1898)

$$(4.4) \quad \int_{-1}^1 \frac{dP_m(x)}{dx} \frac{dP_n(x)}{dx} dx = \frac{1 + (-)^{m+n}}{2} m(m+1).$$

From the above it is also easy to verify that [2] (Math.

Trip, 1897)

$$(4.5) \quad \int_{-1}^1 \frac{d^2 P_m(x)}{dx^2} \frac{d^2 P_n(x)}{dx^2} dx$$

$$= \frac{(m-1)m(m+1)(m+2)}{48} \{ 3n(n+1) - m(m+1) + 6 \}$$

$$\times \{ 1 + (-1)^{m+n} \}.$$

5. - We next evaluate the integral $\int_{-1}^1 \prod_{r=1}^k \frac{dP_{n_r}(x)}{dx}$ where n_1, n_2, \dots, n_k are k positive integers such that

$$1 \leq n_1 \leq n_2 \leq \dots \leq n_{k-1} \leq n_k \quad \text{and} \quad n_1 + n_2 + \dots + n_{k-1} < n_k + k.$$

Now,

$$\int_{-1}^1 \prod_{r=1}^k \frac{dP_{n_r}(x)}{dx} dx = L \int_{-1}^1 \prod_{r=1}^k \{ (x^2 - 1)^{n_r} \}_{n_r+1} dx$$

where

$$L = \frac{1}{2^{\sum_{r=1}^k n_r} \prod_{r=1}^k n_r!}.$$

Again

$$(5.1) \quad \int_{-1}^1 \prod_{r=1}^k \{ (x^2 - 1)^{n_r} \}_{n_r+1} dx$$

$$= [\{ (x^2 - 1)^{n_1} \}_{n_1+1} \{ (x^2 - 1)^{n_2} \}_{n_2+1} \dots \{ (x^2 - 1)^{n_{k-1}} \}_{n_{k-1}+1} \{ (x^2 - 1)^{n_k} \}_{n_k}]_{-1}^1$$

$$- \int_{-1}^1 \{ (x^2 - 1)^{n_1} \}_{n_1+2} \{ (x^2 - 1)^{n_2} \}_{n_2+1} \dots \{ (x^2 - 1)^{n_{k-1}} \}_{n_{k-1}+1} \{ (x^2 - 1)^{n_k} \}_{n_k} dx$$

$$- \int_{-1}^1 \{ (x^2 - 1)^{n_1} \}_{n_1+1} \{ (x^2 - 1)^{n_2} \}_{n_2+2} \dots \{ (x^2 - 1)^{n_{k-1}} \}_{n_{k-1}+1} \{ (x^2 - 1)^{n_k} \}_{n_k} dx$$

.

$$- \int_{-1}^1 \{ (x^2 - 1)^{n_1} \}_{n_1+1} \{ (x^2 - 1)^{n_2} \}_{n_2+1} \dots \{ (x^2 - 1)^{n_{k-1}} \}_{n_{k-1}+2} \{ (x^2 - 1)^{n_k} \}_{n_k} dx$$

Each of the $k-1$ integrals vanishes if $n_1 + n_2 + \dots + n_{k-1} < n_k + k$.

Also

$$(5.2) \quad \begin{aligned} & [\{ (x^2 - 1)^{n_1} \}_{n_1+1} \{ (x^2 - 1)^{n_2} \}_{n_2+1} \dots \\ & \dots \{ (x^2 - 1)^{n_{k-1}} \}_{n_{k-1}+1} \{ (x^2 - 1)^{n_k} \}_{n_k}]^{-1} \\ & = 2^{\sum_{r=1}^k n_r - (k-1)} [1 - (-1)^{\sum_{r=1}^k n_r - (k-1)}] \cdot F \end{aligned}$$

where

$$F = (n_1 + 1)! (n_2 + 1)! \dots (n_{k-1} + 1)! n_k! n_1 n_2 \dots n_{k-1}.$$

So, under the conditions stated above

$$(5.3) \quad \int_{-1}^1 \prod_{r=1}^k \frac{dP_{n_r}(x)}{dx} dx = \frac{[1 - (-1)^{\sum_{r=1}^k n_r - (k-1)}]}{2^{k-1}} \prod_{r=1}^{k-1} n_r (n_r + 1).$$

If $k=2$, we again get the result [1].

$$(5.4) \quad \int_{-1}^1 \frac{dP_m(x)}{dx} \frac{dP_n(x)}{dx} dx = \frac{1 + (-1)^{m+n}}{2} m(m+1)$$

where $m = n_1$ and $n = n_2$.

The results of this paper are embodied in formulae (4.1), (4.2), (4.3) and 5.3).

I am indebted to Dr. H. M. Sengupta for the kind interest he has taken in the work.

REFERENCES

- [1] WHITTAKER E. T. and WATSON G. N. - *Modern Analysis*. 1952, p. 309.
 [2] WHITTAKER E. T. and WATSON G. N. - *Modern Analysis*. 1952, p. 309.