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On certain definite integrals involving Legendre's polynomials

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ON CERTAIN DEFINITE INTEGRALS INVOLVING LEGENDRE'S POLYNOMIALS

Nota (*) di S. K. CHATTERJEE (*a Calcutta*)

1. - The object of the present note is to evaluate certain definite integrals involving Legendre's Polynomials. Using the symbol $\{u(x)\}_r$ for $\dfrac{d^r u(x)}{dx^r}$, we write

$$(1.1) \qquad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n = \frac{1}{2^n n!} \{ (x^2 - 1)^n \}_n .$$

2. - Let m, n, r and s be four positive integers such that $1 \leqq m \leqq n$ and $1 \leqq r \leqq m$, $1 \leqq s \leqq n$.

Then

$$(2.1) \qquad \int_{-1}^{1} \frac{d^r P_m(x)}{dx^r} \cdot \frac{d^s P_n(x)}{dx^s} dx$$

$$= \frac{1}{2^{m+n} m!\, n!} \int_{-1}^{1} \{ (x^2 - 1)^m \}_{m+r} \{ (x^2 - 1)^n \}_{n+s} dx .$$

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3. - CASE I. - If $2 \leqq r+s \leqq m$ we have

$$(3.1) \qquad \int_{-1}^{1} \{(x^2-1)^m\}_{m+r} \{(x^2-1)^n\}_{n+s}\, dx$$

$$= \sum_{t=0}^{s-1} (-)^t \left[\{(x^2-1)^m\}_{m+r+t} \{(x^2-1)^n\}_{n+s-t-1} \right]_{-1}^{1}$$

$$+ (-)^s \int_{-1}^{1} \{(x^2-1)^m\}_{m+r+s} \{(x^2-1)^n\}_n\, dx .$$

The last integral vanishes on account of the orthogonal property of Legendre's Polynomials. If however $r+s \geqq m+1$, the process comes to an end earlier.

CASE II. - Let $m+1 \leqq r+s \leqq 2m$. Writing $r+s = m+\delta$ we have $1 \leqq \delta \leqq m$. In this case

$$(3.2) \qquad \int_{-1}^{1} \{(x^2-1)^m\}_{m+r} \{(x^2-1)^n\}_{n+s}\, dx$$

$$= \sum_{t=0}^{s-\delta} (-)^t \left[\{(x^2-1)^m\}_{m+r+t} \{(x^2-1)^n\}_{n+s-t-1} \right]_{-1}^{1} .$$

CASE III. - Let $r+s \geqq 2m+1$, where $r \leqq m < s \leqq n$ writing $r+\Delta = m$, we have $0 \leqq \Delta \leqq m-1$.

Here, we have

$$(3.3) \qquad \int_{-1}^{1} \{(x^2-1)^m\}_{m+r} \{(x^2-1)^n\}_{n+s}\, dx$$

$$= \sum_{t=0}^{\Delta} (-)^t \left[\{(x^2-1)^m\}_{m+r+t} \{(x^2-1)^n\}_{n+s-t-1} \right]_{-1}^{1} .$$

Now, it may be easily shewn that

$$(3.4) \qquad \left[\{(x^2-1)^m\}_{m+r+t} \{(x^2-1)^n\}_{n+s-t-1} \right]_{-1}^{1}$$

$$= 2^{m+n-r-s+1}\, m!\, n!\, (r+t)!\, (s-t-1)!$$

$$\binom{m+r+t}{r+t} \binom{m}{r+t} \binom{n+s-t-1}{s-t-1} \binom{n}{s-t-1}$$

provided $r+t \leqq m$ and $s-t-1 \geqq 0$, which is certainly true in the present cases.

4. - It follows, therefore, in case I

$$\int_{-1}^{1} \frac{d^rP_m(x)}{dx^r}\frac{d^sP_n(x)}{dx^s}\,dx \tag{4.1}$$

$$=K\sum_{t=0}^{s-1}(-)^t\left[(r+t)!\,(s-t-1)!\binom{m+r+t}{r+t}\binom{m}{r+t}\binom{n+s-t-1}{s-t-1}\binom{n}{s-t-1}\right]$$

where

$$K=\frac{1}{2^{r+s-1}}\,[1+(-)^{m+n-r-s}].$$

In case II

$$\int_{-1}^{1} \frac{d^rP_m(x)}{dx^r}\frac{d^sP_n(x)}{dx^s}\,dx \tag{4.2}$$

$$=K\sum_{t=0}^{s-\delta}(-)^t\left[(r+t)!\,(s-t-1)!\binom{m+r+t}{r+t}\binom{m}{r+t}\binom{n+s-t-1}{s-t-1}\binom{n}{s-t-1}\right]$$

And in case III

$$\int_{-1}^{1} \frac{d^rP_m(x)}{dx^r}\frac{d^sP_n(x)}{dx^s}\,dx \tag{4.3}$$

$$=K\sum_{t=0}^{\Delta}(-)^t\left[(r+t)!\,(s-t-1)!\binom{m+r+t}{r+t}\binom{m}{r+t}\binom{n+s-t-1}{s-t-1}\binom{n}{s-t-1}\right]$$

If $r=s=1$ and $m \geqq 2$, case I gives the well known result [1] (Clare, 1898)

$$\int_{-1}^{1} \frac{dP_m(x)}{dx}\frac{dP_n(x)}{dx}\,dx=\frac{1+(-)^{m+n}}{2}\,m(m+1). \tag{4.4}$$

From the above it is also easy to verify that [2] (Math.

Trip, 1897)

(4.5) $$\int_{-1}^{1} \frac{d^2P_m(x)}{dx^2}\,\frac{d^2P_n(x)}{dx^2}\,dx$$

$$= \frac{(m-1)m(m+1)(m+2)}{48}\,\{\,3n(n+1)-m(m+1)+6\,\}$$
$$\times\,\{\,1+(-)^{m+n}\,\}.$$

5. - We next evaluate the integral $\int_{-1}^{1} \prod_{r=1}^{k} \frac{dP_{n_r}(r)\,dx}{dx}$ where $n_1, n_2, \ldots, n_k$ are k *positive integers such that*

$1 \leqq n_1 \leqq n_2 \ldots \leqq n_{k-1} \leqq n_k$ and $n_1+n_2+\ldots n_{k-1} < n_k + k$.

Now,

$$\int_{-1}^{1} \prod_{r=1}^{k} \frac{dP_{n_r}(x)}{dx}\,dx = L\int_{-1}^{1} \prod_{r=1}^{k} \{\,(x^2-1)^{n_r}\,\}_{n_r+1}\,dx$$

where

$$L = \frac{1}{2^{\sum_{r=1}^{k} n_r} \prod_{r=1}^{k} n_r\,!}.$$

Again

(5.1) $$\int_{-1}^{1} \prod_{r=1}^{k} \{\,(x^2-1)^{n_r}\,\}_{n_r+1}\,dx$$

$$= [\{(x^2-1)^{n_1}\}_{n_1+1}\{(x^2-1)^{n_2}\}_{n_2+1}\ldots\{(x^2-1)^{n_{k-1}}\}_{n_{k-1}+1}\{(x^2-1)^{n_k}\}_{n_k}]_{-1}^{1}$$

$$-\int_{-1}^{1}\{(x^2-1)^{n_1}\}_{n_1+2}\{(x^2-1)^{n_2}\}_{n_2+1}\ldots\{(x^2-1)^{n_{k-1}}\}_{n_{k-1}+1}\{x^2-1)^{n_k}\}_{n_k}\,dx$$

$$-\int_{-1}^{1}\{(x^2-1)^{n_1}\}_{n_1+1}(x^2-1)^{n_2}\}_{n_2+2}\ldots\{(x^2-1)^{n_{k-1}}\}_{n_{k-1}+1}\{(x^2-1)^{n_k}\}_{n_k}\,dx$$

. . . .

$$-\int_{-1}^{1}\{(x^2-1)^{n_1}\}_{n_1+1}\{(x^2-1)^{n_2}\}_{n_2+1}\ldots\{(x^2-1)^{n_{k-1}}\}_{n_{k-1}+2}\{(x^2-1)^{n_k}\}_{n_k}\,dx$$

Each of the $k-1$ integrals vanishes if $n_1+n_2+\dots+$ $+n_{k-1}<n_k+k$.

Also

$$(5.2) \qquad [\{(x^2-1)^{n_1}\}_{n_1+1}\{(x^2-1)^{n_2}\}_{n_2+1}\dots$$
$$\dots\{(x^2-1)^{n_{k-1}}\}_{n_{k-1}+1}\{(x^2-1)^{n_k}\}_{n_k}]^{1}_{-1}$$
$$=2^{\sum_{r=1}^{k} n_r-(k-1)}[1-(-)^{\sum_{r=1}^{k} n_r-(k-1)}]\cdot F$$

where

$$F=(n_1+1)!\,(n_2+1)!\,\dots\,(n_{k-1}+1)!\,n_k!\,n_1n_2\,\dots\,n_{k-1}.$$

So, under the conditions stated above

$$(5.3) \qquad \int_{-1}^{1}\prod_{r=1}^{k}\frac{dP_{n_r}(x)}{dx}\,dx=\frac{[1-(-)^{\sum_{r=1}^{k} n_r-(k-1)}]}{2^{k-1}}\prod_{r=1}^{k-1}n_r(n_r+1).$$

If $k=2$, we again get the result [1].

$$(5.4) \qquad \int_{-1}^{1}\frac{dP_m(x)}{dx}\frac{dP_n(x)}{dx}\,dx=\frac{1+(-)^{m+n}}{2}\,m(m+1)$$

where $m=n_1$ and $n=n_2$.

The results of this paper are embodied in formulae (4,1), (4.2), (4.3) and 5.3).

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REFERENCES

[1] WHITTAKER E. T. and WATSON G. N. - *Modern Analysis*. 1952, p. 309.

[2] WHITTAKER E. T. and WATSON G. N. - *Modern Analysis*. 1952, p. 309.