Some theorems on recurrent and Ricci-recurrent spaces

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SOME THEOREMS ON RECURRENT AND RICCI-RECURRENT SPACES

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Introduction. - Ruse [4] and Walker [5] introduced the recurrent spaces for which the curvature tensor satisfies a relation of the form

\[ R_{\lambda t^j k} = R^t_{\lambda t^j k} \lambda_t \]

where \( \lambda_t \) is a non-zero vector and comma denotes covariant derivative. An \( n \)-space of this kind was denoted by \( K_n \). In a recent paper [3] Patterson considered a Riemannian space of more than two dimensions whose Ricci tensor satisfies

\[ R_{ij} = 0, \quad R_{ij, k} = R_{ij} \lambda_k \]

for some non-zero vector \( \lambda_k \). He called such a space Ricci-Recurrent and denoted an \( n \)-space of this kind by \( R_n \). Obviously every \( K_n \) is an \( R_n \) but the converse is not true. The question as to when an \( R_n \) can be a \( K_n \) has been considered in section 1 of this paper. In the remaining sections some necessary conditions for a space \( V_n \) (\( n > 2 \)) to be conformal to a recurrent and to a Ricci-recurrent space have been given.

1. - It is known that the conformal tensor \( C_{ijkl} \) of a Riemannian space \( V_n \) with metric tensor \( g_{ij} \) is given by

\[ C_{ijkl} = R_{ijkl} - \frac{1}{n-2} (R_{jk} g_{iu} - R_{jk} g_{iu} + R_{ik} g_{ju} - R_{ik} g_{ju}) \]

\[ + \frac{R}{(n-1)(n-2)} (g_{jk} g_{iu} - g_{jk} g_{iu}) \]

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Now let the $V_n$ be a Ricci-recurrent space $R_n$ specified by a non-zero vector $\lambda_i$. Then since

$$R_{ij,p} = R_{ij}\lambda_p \quad \text{and} \quad R_{i,p} = R\lambda_p$$

it follows from (1.1) that

$$C_{ijkl,p} = R_{ijkl,p} + (C_{ijkl} - R_{ijkl})\lambda_p$$

(1.2)

or

$$C_{ijkl,p} = C_{ijkl}\lambda_p = R_{ijkl,p} - R_{ijkl}\lambda_p.$$

Conversely, if (1.2) holds, multiplying both sides of (1.2) by $g^{ik}$ and summing for $j$ and $k$ we get

$$C_{ui,p} - C_{ui}\lambda_p = R_{ui,p} - R_{ui}\lambda_p$$

which reduces, in virtue of $C_{ij} = 0$, to

(1.3)

$$R_{ui,p} = R_{ui}\lambda_p.$$

From (1.2) and (1.3) we have therefore the following theorem:

**Theorem 1.** A necessary and sufficient condition that a $V_n$ be an $R_n$ is that (1.2) holds.

In particular, if the $V_n$ is conformal to a flat space or if $n = 3$ then $C_{ijkl} = 0$. In the first case it follows from (1.2) that the $R_n$ is a $K_n$. In the second case it follows that the $R_3$ is a $K_3$. Thus we have the following theorem:

**Theorem 2.** Every $R_n$ ($n > 3$) is a $K_n$ if it is conformal to a flat space and every $R_3$ is a $K_3$.

Again for an $R_n$ specified by a non-zero vector $\lambda_i$ we have

$$R_{ij,kl} - R_{ij,ik} = (\lambda_{k, l} - \lambda_{l, k})R_{ij}$$

or by Ricci's identity

(1.4)

$$R_{pj}R_{ik}^p + R_{ip}R_{jk}^p = (\lambda_{k, l} - \lambda_{l, k})R_{ij}.$$

If $\lambda_i$ be a gradient then $\lambda_{k, l} = \lambda_{l, k}$ and so the lefthand side of (1.4) is zero. Conversely if the lefthand side of (1.4) is zero then $\lambda_i$ is a gradient. From this we have the following theorem:
THEOREM 3. A necessary and sufficient condition that the vector of recurrence of an \( R_n \) be a gradient is that at every point of the \( R_n \)

\[ R_{pj}R^p_{kl} + R_{ip}R^p_{jk} = 0. \]

Since the vector of recurrence of every \( K_n \) is a gradient it follows in virtue of Theorem 2 that the relation (1.5) holds at every point of an \( R_n \) \( (n > 3) \) which is conformal to a flat space and at every point of an \( R_n \).

2. - In this section we deduce a necessary condition that a space \( V_n \) \( (n > 3) \) with metric tensor \( g_{ij} \) be conformal to a recurrent space.

Let \( \tilde{V}_n \) be a space with metric tensor

\[ g_{ij} = e^{2\sigma} \bar{g}_{ij} \]  

Then since \( \tilde{C}_{ijk} = C_{ijk} \) we have

\[ C_{hijk} = e^{2\sigma} C_{hijk} \]

Now suppose \( \tilde{V}_n \) is a recurrent space specified by a non-zero vector \( \lambda_i \) and let a comma and a semi-colon denote covariant derivatives with respect to \( g_{ij} \) and \( \bar{g}_{ij} \) respectively. Then from (1.1) we get

\[ \tilde{C}_{hijk;j} = \tilde{C}_{hijk} \lambda_i \]

Writing \( \sigma_i = \frac{\partial \sigma}{\partial x_i} \), \( \sigma^j = g^{ij} \sigma_i \), we obtain from (2.2)

\[ e^{2\sigma} C_{hijk} \lambda_i = e^{2\sigma} C_{hijk;j} + 2 e^{2\sigma} C_{hijk} \sigma_i \]

(2.3)

or \( C_{hijk} \lambda_i = 2 C_{hijk} \sigma_i + C_{hijk;j} \)

But

\[ C_{hijk;j} = C_{hijk;i} - 4 C_{hijk} \sigma_i - (C_{lijk} \sigma_h + C_{hlijk} \sigma_i + C_{hhiik} \sigma_j + C_{hijl} \sigma_k) \]

\[ + \sigma_i (g_{hi} C_{sejk} + g_{ij} C_{hijk} + g_{jl} C_{hihk} + g_{kl} C_{hijh}). \]

Therefore, since \( C_{hijk} \) possesses the properties of the indi-
ces of $R_{hijk}$, the equation (2.3) reduces to

\begin{align}
C_{hijk}\lambda_l &= C_{hijk,1} - 2C_{hijk}\sigma_i \\
&- (C_{ljhk}\sigma_h + C_{lkhi}\sigma_j + C_{ljik}\sigma_k) \\
&+ \sigma^i(g_{kl}C_{nijk} + g_{lj}C_{ehkj} + g_{jl}C_{skhi} + g_{kl}C_{efsjk}).
\end{align}

We may therefore state the following theorem:

**Theorem 4.** A necessary condition, given in terms of the conformal tensor, that a $V_n$ be conformal to a $K_n$ is that the equation (2.4) holds, provided that the $V_n$ is not conformal to a flat space.

Further, substituting from (2.1) in the last two lines of (2.4) and simplifying we obtain the condition (2.4) as

\begin{align}
C_{hijk}\lambda_l &= C_{hijk,1} - 2C_{hijk}\sigma_i \\
&- [(R_{ljhk}\sigma_j + R_{lkhj}\sigma_l + R_{lkhi}\sigma_j + R_{ljik}\sigma_k) \\
&+ \frac{1}{n-2} \left\{ (g_{ik}R_{ij} - g_{ij}R_{ik})\sigma_h + (g_{kj}R_{ik} - g_{ik}R_{kj})\sigma_l \\
&+ (g_{ik}R_{lk} - g_{lk}R_{ik})\sigma_j + (g_{kj}R_{ik} - g_{ik}R_{kj})\sigma_k \right\}] \\
&+ \sigma^i [(h_{kl}R_{sk} + g_{kl}R_{sk}) + g_{lj}(g_{ij}R_{sk} - g_{ik}R_{sj}) \\
&+ g_{jl}(g_{ik}R_{sh} - g_{hk}R_{si}) + g_{kl}(g_{hi}R_{si} - g_{ij}R_{sh})].
\end{align}

If now the vector $\sigma_i$ be supposed to be parallel in the space $V_n$ and therefore $\sigma^R_{R_{sk}} = 0 = \sigma^R_{R_{sk}}$, then the last three lines in the above equation vanish. On the other hand if the space $V_n$ be supposed to be a recurrent space specified by $\sigma_i$ then the second line of the above equation reduces to $-2R_{hijk}\sigma_l$.

3. - In this section we deduce a necessary condition that a $V_n$ be conformal to a Ricci-recurrent space.

With reference to the conformal relation (2.1) we have the following equations

\begin{align}
e^{-2\sigma}R_{hijk} &= R_{hijk} + g_{hk}(\sigma_i,j - \sigma_i\sigma_j) + g_{ij}(\sigma_k,h - \sigma_k\sigma_h) \\
&- h_j(\sigma_i,k - \sigma_i\sigma_k) - g_{ik}(\sigma_j,h - \sigma_j\sigma_h) + (g_{hk}g_{ij} - g_{kj}g_{hi})\Delta_i\sigma
\end{align}
whence

\begin{align}
(3.2) \quad \sigma_{i,j} - \sigma_{i,j} &= \frac{1}{n-2} (\bar{R}_{ij} - R_{ij}) \\
&- \frac{1}{2(n-1)(n-2)} (g_{ij}\bar{R} - g_{ij}R) - \frac{1}{2} g_{ij}\Delta_1 \sigma.
\end{align}

Now let \( \tilde{V}_n \) with metric tensor \( g_{ij} \) be supposed to be a Ricci-recurrent space. Differentiating (3.2) covariantly with respect to \( g_{ij} \) and substituting

\begin{align}
(3.3) \quad \bar{R}_{ij,k} &= \bar{R}_{ij}\lambda_k + 2\bar{R}_{ij}\sigma_k + \bar{R}_{jk}\sigma_i + \bar{R}_{ik}\sigma_j - g^{sm}\sigma_m(\bar{R}_{sj}g_{ik} + \bar{R}_{sk}g_{jk} \\
g_{ij,k} &= 2g_{ij}\sigma_k, \quad \bar{R}_k = R\lambda_k
\end{align}

we get

\begin{align}
(3.4) \quad \sigma_{i,j,k} - (\sigma_{i,j,k} + \sigma_{j,i,k}) &= \frac{1}{n-2} \left\{ \bar{R}_{ij}\lambda_k + 2\bar{R}_{ij}\sigma_k + \bar{R}_{jk}\sigma_i + \bar{R}_{ik}\sigma_j - g^{sm}\sigma_m(\bar{R}_{sj}g_{ik} + \bar{R}_{sk}g_{jk}) - R_{ij,k} \right\} \\
&- \frac{g_{ij}}{2(n-1)(n-2)} \left\{ e^{\sigma R}(\lambda_k + 2\sigma_k) - R_k \right\} - \frac{1}{2} g_{ij}\Delta_1 \sigma, k.
\end{align}

But from (3.2)

\begin{align}
\sigma_{i,j,k} + \sigma_{j,i,k} - 2\sigma_{i,j}\sigma_k &= \frac{1}{n-2} \left\{ \sigma_i(\bar{R}_{jk} - R_{jk}) + \sigma_j(\bar{R}_{ik} - R_{ik}) \right\} \\
&- \frac{1}{2(n-1)(n-2)} \left\{ \sigma_i(\bar{R}_{jk}\bar{R} - g_{jk}R) + \sigma_j(\bar{R}_{ik}\bar{R} - g_{ik}R) \right\} \\
&- \frac{1}{2} \Delta_1 \sigma(\sigma_{i,j} + \sigma_{i,k}).
\end{align}

Substituting in (3.4), we obtain

\begin{align}
(3.5) \quad \sigma_{i,j,k} - 2\sigma_{i,j}\sigma_k &= \frac{1}{n-2} \left\{ \bar{R}_{ij}\lambda_k + 2(\bar{R}_{ij}\sigma_k + \bar{R}_{jk}\sigma_i + \bar{R}_{ik}\sigma_j) \\
&- \sigma_m(\bar{R}_{j}^m g_{ik} + \bar{R}_i^m g_{jk}) - (R_{ij,k} + R_{jk}\sigma_i + R_{ik}\sigma_j) \right\}
\end{align}
This is therefore a necessary condition to be satisfied by the function $\sigma$ in order that a $V_n$ with metric tensor $g_{ij}$ be conformal to the Ricci-recurrent $V_n$ with metric tensor $\tilde{g}_{ij} = e^{2\sigma}g_{ij}$.

If $V_n$ is supposed to be an Einstein space we have simply to put $R_{ij, k} = R_{i, k} = 0$ and $R_{ij} = \frac{R}{n} g_{ij}$ in (3.5).

4. - The condition (3.5) which has been obtained by straightforward method is not simple enough. We look for a simpler condition. For this purpose we first obtain the condition of integrability of the equation (3.2).

Differentiating (3.2) covariantly with respect to $x^k$, interchanging the indices $j$ and $k$ and subtracting, the lefthand side becomes, after the operation,

$$
\sigma_{i, jk} - \sigma_{i, kj} + \sigma_{i, j} g_{k} - \sigma_{i, k} g_{j} = \sigma_{s} R_{ijk} + \sigma_{i, j} g_{k} - \sigma_{i, k} g_{j}.
$$

From (3.1) we obtain

$$
\sigma_{s} R_{ijk} = \sigma_{s} \tilde{R}_{ijk} - \tilde{\delta}_{k}^{s} (\sigma_{i, j} - \sigma_{i, j}) - g_{ij} g_{hk} (\sigma_{h, k} - \sigma_{h, k}) + g_{ik} g_{hs} (\sigma_{h, j} - \sigma_{h, j}) - (\tilde{\delta}_{k}^{s} g_{ij} - \tilde{\delta}_{k}^{s} g_{hk}) \Delta_{1} \sigma
$$

$$
= \sigma_{s} \tilde{R}_{ijk} - \sigma_{k} (\sigma_{i, j} - \sigma_{i, j}) - \frac{1}{2} g_{ij} (\Delta_{1} \sigma), k + g_{ij} \sigma_{k} \Delta_{1} \sigma
$$

$$
+ \sigma_{i} (\sigma_{k} - \sigma_{i} \sigma_{k}) + \frac{1}{2} g_{ij} (\Delta_{1} \sigma), j - g_{ik} \sigma_{j} \Delta_{1} \sigma - (\sigma_{k} g_{ij} - \sigma_{j} g_{ik}) \Delta_{1} \sigma
$$

$$
= \sigma \tilde{R}_{ijk} + \sigma_{i} \sigma_{k} - \sigma_{k} \sigma_{i} + \frac{1}{2} \{ g_{ij} (\Delta_{1} \sigma), j - g_{ij} (\Delta_{1} \sigma), k \}.
$$
Therefore after the operation we obtain from (3.2)

\begin{equation}
\sigma_s \bar{R}_{ijk} + \frac{1}{2} \left| g_{ik}(\Delta_1 \sigma), j - g_{ij}(\Delta_1 \sigma), k \right| \\
= \frac{1}{n-2} \left\{ (R_{ij,k} - \bar{R}_{ik,j}) - (R_{ij,k} - R_{ik,j}) \right\}
\end{equation}

\begin{align*}
&\quad - \frac{1}{2(n-1)(n-2)} \left\{ \bar{R}(g_{ij,k} - \bar{g}_{ik,j}) + (\bar{g}_{ij}\bar{R}_k - \bar{g}_{ik}\bar{R}_j) \\
&\quad - (g_{ij}R_{k} - g_{ik}R_{j}) \right\} - \frac{1}{2} \left| g_{ij}(\Delta_1 \sigma), i - g_{ik}(\Delta_1 \sigma), j \right|.
\end{align*}

Now apply (3.3). Then (4.1) reduces to

\begin{align*}
\sigma_s \bar{R}_{ijk} &= \frac{1}{n-2} \left[ \bar{R}_{ij,k} - \bar{R}_{ik,j} + \bar{R}_{ij}\sigma_k - \bar{R}_{ik}\sigma_j + \sigma g^{st}(g_{ij}\bar{R}_{sk} - \bar{g}_{ik}\bar{R}_{sj}) \\
&\quad - \frac{1}{2(n-1)} \left\{ 2R(g_{ij}\sigma_k - \bar{g}_{ik}\sigma_j) + g_{ij}R_k - \bar{g}_{ik}\bar{R}_j \right\} \\
&\quad - \frac{1}{n-2} \left\{ R_{ij,k} - R_{ik,j} - \frac{1}{2(n-1)} (g_{ij}R_k - g_{ik}R_j) \right\}.
\end{align*}

Writing as usual (Eisenhart, 1926, P. 91)

\begin{equation}
\begin{align*}
R_{ijk} &= R_{ij,k} - R_{ik,j} - \frac{1}{2(n-1)} (g_{ij} R_k - g_{ik} R_j) \\
\bar{R}_{ijk} &= \bar{R}_{ij,k} - \bar{R}_{ik,j} - \frac{1}{2(n-1)} (\bar{g}_{ij} \bar{R}_k - \bar{g}_{ik} \bar{R}_j)
\end{align*}
\end{equation}

the above equation reduces to

\begin{align*}
\sigma_s \bar{R}_{ijk} - \frac{1}{n-2} \left[ \bar{R}_{ij}\sigma_k - \bar{R}_{ik}\sigma_j + \sigma g^{st}(g_{ij}\bar{R}_{sk} - \bar{g}_{ik}\bar{R}_{sj}) \\
- \frac{R}{n-1} (\bar{g}_{ij}\sigma_k - \bar{g}_{ik}\sigma_j) \right] &= \frac{1}{n-2} \left[ \bar{R}_{ijk} - R_{ijk} \right].
\end{align*}
Now from the conformal tensor (1.1) we get

\[ \sigma_h C_{ijk} = \sigma_h \bar{C}_{ijk} = \sigma_h \bar{R}_{ijk} \]

\[ + \frac{1}{n-2} \left\{ \bar{R}_{ik} \sigma_j - \bar{R}_{ij} \sigma_k + \sigma_h \bar{g}^{hl} (\bar{g}_{lk} \bar{R}_{ij} - \bar{g}_{ij} \bar{R}_{lk}) \right\} \]

\[ + \frac{\bar{R}}{(n-1)(n-2)} (\bar{g}_{ij} \sigma_k - \bar{g}_{ik} \sigma_j). \]

Substituting this in the above equation we obtain finally

(4.3) \[ \sigma_h C_{ijk} = \frac{1}{n-2} (\bar{R}_{ijk} - R_{ijk}) \]

This is the condition of integrability of equation (3.2). This reduces to a known result given by Brinkmann [1] in the case when \( \bar{V}_n \) is an Einstein space.

Now suppose that \( \bar{V}_n \) is a Ricci-recurrent space specified by a vector \( \lambda_i \). Then from (4.2)

\[ \bar{R}_{ijk} = \bar{R}_{ij} \lambda_k - \bar{R}_{ik} \lambda_j + \frac{\bar{R}}{2(n-1)} (\bar{g}_{ik} \lambda_j - \bar{g}_{ij} \lambda_k). \]

Substituting in (4.3) we get

(4.4) \[ \sigma_h C_{ijk} + \frac{1}{n-2} R_{ijk} \]

\[ = \frac{1}{n-2} \left\{ \bar{R}_{ij} \lambda_k - \bar{R}_{ik} \lambda_j + \frac{\bar{R} e^{2\sigma}}{2(n-1)} (\bar{g}_{ik} \lambda_j - \bar{g}_{ij} \lambda_k) \right\}. \]

We may therefore state the following theorem:

**Theorem 5.** A necessary condition that a space \( V_n \) be conformal to a Ricci-recurrent \( \bar{V}_n \) is that there exists a function \( \sigma \) satisfying (4.4).

If \( V_n \) be an Einstein space, we have simply to put \( R_{ijk} = 0 \) in (4.4).
And if $V_n$ be a Ricci-recurrent space specified by a vector $\mu_i$ we have simply to put in (4.4)

$$R_{ijk} = R_{ij}^k \mu_k - R_{ik}^j \mu_j + \frac{R}{2(n-1)}(g_{ik}^j \mu_j - g_{ij}^k \mu_k).$$

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