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A theorem on infinite sequences of finitely additive real valued measures

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A THEOREM ON INFINITE SEQUENCES OF FINITELY ADDITIVE REAL VALUED MEASURES [1]).

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Introduction.

The known Helly's theorem (1912) on sequences of functions $f_n(x)$ of bounded variation is not only important in the spectral theory of infinite matrices but it has also another aspect of importance. Indeed, this theorem, as stated at the end of Section 2 of this paper, can be given the form of a theorem on finitely additive measures. To see this, let us remark that to a non decreasing function $f(x)$ in $0 \leq x \leq 1$ with $f(0) = 0$ can be attached a finitely additive measure $\mu(F) \geq 0$ defined on finite unions F of halfopen subintervals $0 \leq a < x \leq b \leq 1$.

These sets F make up a finitely additive Boolean lattice (B_1). Now, if we have a bounded sequence $\mu_n(F) \geq 0$ of finitely additive measures on (B_1), the Helly's theorem states that a subsequence $\mu_{k(n)}(F)$ can be extracted therefrom, converging on every F, to a similar measure.

We shall prove that this theorem can be extended to a

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finite or even to the denumerable cartesian product of replicas of (B_1). To do this is the aim of this paper.

First we shall prove a theorem, analogous to the Helly's one, for functions of a finite number of variables.

The ordinary increment $f(x_2) - f(x_1)$ of the function of a single variable, and which is the brickstone for the notion of the variation of a function, will be replaced in the case of several variables, by the Vitali-increment (defined in Section 2.), so we shall consider functions of bounded Vitali-variation.

The main device of the proof is the same as in the case of a single variable, but cannot be used without a range of quite delicate preparations, stated in several lemmas, so we think that a rather detailed exposition of the whole proof is needed. To avoid unnecessary complications we shall confine our detailed proof to the case of two variables only. The general case can be treated by induction, by using quite analogous argument. Having the theorem on measure settled for the the case of finite dimensions, the passage to that of infinite denumerable number of dimensions does not offer any difficulties. We do not know whether the theorem is true for non denumerable dimensions, but we like to conjecture that the answer should be negative.

2. Notations and terminology.

The unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ will be denoted by $Q = \overline{Q}$, and the set of all its interior points by Q^0. A function $f(x, y)$ will be termed *increasing* if $x_1 \leq x_2$, $y_1 \leq y_2$ imply $f(x_1, y_1) \leq f(x_2, y_2)$. Thus the constant function is an increasing one.

The same term will be used for functions of a single variable. We shall write $(x_1, y_1) \leq (x_2, y_2)$, if and only if $x_1 \leq x_2$, $y_1, \leq y_2$, and $(x_1, y_1) < (x_2, y_2)$, if and only if $x_1 < x_2$, $y_1 < y_2$.

The sets $x > a$, $y > b$; $x < a$, $y > b$; $x < a$, $y < b$; $x > a$, $y < b$ will be termed *quarters at* (a, b), the I, II, III and IV quarter respectively. The I and III quarters will be termed *main quarters*.

Let $\{A_n\}$ be an at most denumerable set of lines parallel to the y-axis, and $\{B_m\}$ an at most denumerable set of lines parallel to the x-axis.

The point-set union $\Sigma_n A_n + \Sigma_m B_m$ will be termed *grate*. The empty set is a grate. The rectangles we shall deal with will always be supposed to have their sides parallel to the axes of the system of coordinates.

The closed rectangle whose two vertices are (a, b), (c, d) where $0 \leq a < c \leq 1$, $0 \leq b < d \leq 1$ will be denoted by $(a, b; c, d)$. By a *partition of a rectangle* R we shall understand any finite set of rectangles R_p whose union is R and where any two of which have no interior point in common.

For union of sets we shall use the symbol $+$, Σ, and for intersection the dot and Π. The signs $\subseteq$, $\supseteq$ mean inclusion of sets; $\subset$, $\supset$ denote strict inclusion (equality not permitted). A superscript dash and $\langle\ \rangle$ will denote closure.

Let $f(x, y)$ be a function defined on Q, an let $R = \langle a, b; c, d \rangle \subseteq Q$ be the set $a \leq x \leq c$, $b \leq y \leq d$.

By the *Vitali-increment* of f on this generalized rectangle R we shall understand the number $V(f(x, y); R) = f(a, b) + + f(c, d) - f(a, d) - f(c, b)$.

The function f is said to be of *bounded Vitali-variation* in Q if there exists $M > 0$ such that for every partition $R_1, \ldots, R_n$ of Q, $|V(f; R_1)| + |V(f; R_2)| + \ldots + |V(f; R_n)| \leq M$.

We shall rely on Helly-s theorem which is this:

If 1) $f_n(x)$, $(n, = 1, 2, \ldots)$ are increasing functions on $(0 \leq x \leq 1$, and such that 2) $f_n(0) = 0$, and 3) $f_n(x) \leq K (n = 1, 2, \ldots)$, then there exists an increasing sequence $k(n)$ of indices such that

$$f(x) = \lim f_{k(n)}(x) \qquad \ldots (1)$$

exists. Such a function $f(x)$ is also increasing, $f(0) = 0$ and $f(x) \leq K$. This theorem is a particular case of the analogous theorem, where 2) is dropped and 3) replaced by $|f_n(x)| \leq M$; in the thesis $f(0) = 0$ and $f(x) \leq K$ are replaced by $|f(x)| \leq M$.

A slightly more general theorem for a sequence of functions of bounded variation is known. It is equivalent to the above theorems.

3. The Helly theorem for functions of two variables.

1. DEFINITION. Let E be an everywhere dense set of points in Q^0, and $f(p) = f(x, y)$ a function defined on E. Let $(x_0, y_0) \in Q^0$. We say that $f(p)$ *has a limit at* (x_0, y_0) *in the* I *quarter, if*

$$\lim f(x_0 + a_n, y_0 + b_n) \mid n \to \infty$$

exists for all sequences $(x_0 + a_n, y_0 + b_n) \in E$ where $a_n > 0$, $b_n > 0$, $a_n \to 0$, $b_n \to 0$.

In this case the limit is the same for all these sequences. We shall write

$$\lim \text{(I)} f(p) \mid p \in E, \ p \to (x_0, y_0).$$

Analogously we define the limits

$$\lim (J) f(p) \mid p \in E, \ p \to (x_0, y_0).$$

where J stands for II, III or IV.

2. DEFINITION. We say that a set A of points is a Γ-*set if* there exists a grate G, such that $A \subseteq G$.

We see that if A is a Γ-set, so is every subset of A. If $A_1, A_2. \dots, A_n, \dots$ are Γ-sets, so is their union $\Sigma_n A_n$. The empty set is a Γ-set.

3. LEMMA. If

1. E is everywhere dense in Q^0, $E \subseteq Q^0$,
2. $f(p) = f(x, y)$ is a bounded function defined for all $p \in E$,
3. for every point $(x_0, y_0) \in Q^0$ all four quarter limits exist,
4. we put
   $g(x, y) = \lim \text{(I)}\ f(p) \mid p \in E, \ p \to (x, y), \ (x, y) \in Q^0,$

5. F is the set of all discontinuity points of $g(x, y)$ in Q^0,

then F is a Γ-set.

3.2. Since F is not a Γ-set, there exists a closed square Q' $m(x, y)$ the supremum and the infimum of the function g at (x, y) respectively, i. e. the supremum and the infimum of all lim $g(z_n)$ for $z_n \rightarrow (x, y)$, $z_n \in Q^0$. The numbres $M(x, y)$, $m(x, y)$ exist for f is bounded.

z is a discontinuity point of g if and only if $m(z) < M(x, y)$.

3.2. Since F is not a Γ-set, there exists a closed square Q' such that $Q' \subset Q^0$ and $F' = F \cdot Q'$ is not a F-set. Take such a square Q'. Let $r < s$ be rational numbers and put

$$F_{r,s} = \{ z \mid m(z) \leq r \leq s \leq M(z) \} \cdot F'.$$

Since $F' = \Sigma_{r<s} F_{r,s}$ where the union is taken for all rationals r, s, and since the set of all couples $r < s$ is denumerable, there exist. r, s, for which $F_{r,s}$ is not a Γ-set.

3.3. Take such a couple $r < s$ and partition Q' into four equal closed squares Q_1, Q_2, Q_3, Q_4. At least one of the sets

$$F_{r,s} \cdot Q_1 \quad , \quad F_{r,s} \cdot Q_2 \quad , \quad F_{r,s} \cdot Q_3 \quad , \quad F_{r,s} \cdot Q_4$$

must not be a Γ-set, say $F_{r,s} \cdot Q_i$.

Partition Q_i again and repeat the above argument. We obtain an infinite sequence of closed squares $R_1 \supset R_2 \supset \ldots$ shrinking to a point (x_0, y_0), and where $R_n \cdot F_{r,s}$ is not a Γ-set whatever n may be.

We have $(x_0, y_0) \in Q' \subset Q^0$.

3.4. Let us partition R_n into at most four rectangles $R_n(1)$, $R_n(2)$, $R_n(3)$, $R_n(4)$ by means of the lines $x = x_0$ and $y = y_0$. Whatever the case may be there exist a sequence $i_1, i_2, \ldots,$ of number 1, 2, 3, 4, such that

$$R_n(i_n) \cdot F_{r,s}, \qquad (n = 1, 2, 3, \ldots),$$

is not a Γ-set. Hence there exists in $(R_{k(n)}(i))^0$ a point $z_n \in F_{r,s}$.

3.5. Consider the sequence $z_1, z_2, \ldots$, which surely tends to (x_0, y_0).

We have

$$m(z_n) \leq r < s \leq M(z_n), \qquad (n = 1, 2, \ldots).$$

Put $z_n = (x_n, y_n)$ and choose $a_n > 0$ so that the closed square

$$S_n = \langle x_n - a_n, \quad x_n + a_n; \quad y_n - a_n, \quad y_n + a_n \rangle \subseteq (R_{k(n)}(i))^0.$$

There exist $u_n, v_n \in s_n$ such that

$$| m(z_n) - g(u_n) | \leq \varepsilon, \qquad | M(z_n) - g(v_n) | \leq \varepsilon,$$

where $\varepsilon = \dfrac{s - r}{6} > 0$.

Choose squares U_n, V_n around u_n, v_n respectively and contained in $(R_{k(n)}(i))^0$ and find points $p_n, q_n \in E$ such that

$$| g(u_n) - f(p_n) | \leq \varepsilon, \; | g(v_n) - f(q_n) | \leq \varepsilon.$$

We obtain the inequalities

$$f(p_n) \leq m(z_n) + 2\varepsilon \leq r + 2\varepsilon \leq s - 2\varepsilon \lesseqgtr M(z_n) - 2\varepsilon \leq f(q_n),$$

which give

$$f(q_n) - f(p_n) \geq \frac{s - r}{3}.$$

Now this inequality is impossible, since $\lim f(q_n) = \lim f(p_n)$, because, by hypothesis, all four quarter limits exist and q_n p_n are lying in the same quarter, and both tend to (x_0, y_0).

The contradiction thus obtained proves the lemma.

4. We recollect that:

Given two points (x, y) and (x', y'), the symbols $(x, y) \leq \leq (x', y')$, $(x, y) < (x', y')$ mean that $x \leq x'$, $y \leq y'$ and $x < x'$, $y < x'$ respectively.

5. We shall say that a function $f(p)$ defined on a set P in Q is increasing on P if and only if $p \leq p'$, $p \in P$, $p' \in P$ imply $f(p) \leq f(p')$.

6. Lemma. If

1. P is everywhere dense in Q,
2. $F(p)$ is an increasing function on P,
3. $z_0 \in Q^0$,

then the main quarter-limits

$$F\,(\mathrm{I})\,(z_0) = \lim\ (\mathrm{I})\ F(p) \mid p \to z_0,\ p \in P,$$

$$F\,(\mathrm{III})\,(z_0) = \lim\ (\mathrm{III})\ F(p) \mid p \to z_0,\ p \in P$$

both exist.

Proof. 6.1. First we see that if $p_1 \geq p_2 \geq \dots \geq p_n \geq \dots > z_0$, with $\lim p_n = z_0$, $p_n \in P$, then $\lim F(p_n)$ exists.

Indeed, the sequence $\{F(p_n)\}$ is bounded, because there exists $p \in P$ with $p < z_0 < p_n$.

Analogously, if $q_1 \leq q_2 \leq \dots \leq q_n \leq \dots < z_0$ with $\lim q_n = z_0$, them $\lim F(q_n)$ exists.

6.2. Having this, we shall prove that if $\{p_n\}$ is any sequence with $p_n > z_0$, $p_n \to z_0$, then $\overline{\lim}\, F(p_n)$ and $\underline{\lim}\, F(p_n)$ exist.

There exists $p \in P$ with $p < z_0$; therefore $F(p) \leq F(p_n)$ ... (1) for all n. There exists $a > 0$ such that, if we put $z_0 = (x_0, y_0)$, we have $(x_0 + a,\ y_0 + a) \in Q^0$. There exists N such that, if $n > N$, we have $p_n \in (x_0, y_0;\ x_0 + a,\ y_0 + a)$. Now we can find $p' \in P$ such that $p' \in (x_0 + a, 1;\ y_0 + a, 1)$ for P is everywhere dense.

Consequently, for $n \geq N$ we have $p_n \leq p'$ and then $F(p_n) \leq F(p')$ ... (2).

From (1) and (2) the statement follows.

Put $\lambda = \underline{\lim}\ F(p_n)$, $\mu = \overline{\lim}\ F(p_n)$.

6.3. Choose partial sequences $\{a_n\}$, $\{b_n\}$ out of $\{p_n\}$ such that

$$\lambda = \lim F(a_n),\ \mu = \lim F(b_n).$$

Since $a_n \to z_0$, $b_n \to z_0$ and since they are in the I quarter, it follows that we can select indices tending to infinity n_1, n_2, ... ; m_1, m_2, .., such that

$$a_{n_1} > b_{m_1} > a_{n_2} > b_{m_2} > \dots \to z_0.$$

Since $F(a_{n_1})$, $F(b_{m_1})$, $F(a_{n_2})$, $F(b_{m_2})$, ... tends to a limit, we get $\lambda = \mu$.

6.4. Now take two sequences $p'_n > z_0$, $p''_n > z_0$ both tending to z_0 and where $p'_n \in P$, $p''_n \in P$. Since $p'_1, p''_1, p'_2, p''_2, \ldots \to z_0$, the sequence $F(p'_1)$, $F(p''_1)$, $F(p'_2)$, $F(p''_2)$,... also tends to a limit. This limit equals both the limits $\lim F(p'_n)$ and $\lim F(p''_n)$. It follows that lim (I) $F(p)$ exists. In an analogous way we prove that lim (III) $F(p)$ exists too.

Obviously lim (III) $F(p) \leq$ lim (I) $F(p)$ which follows from the existence of sequences

$$q_1 < q_2 < \ldots < z_0 < \ldots < p_2 < p_1 \text{ with } q_n \to z_0,\ p_n \to z_0.$$

7. The following *lemma* has an easy proof:

If 1. $f_n(p)$ are increasing functions defined on the set P,

2. P is everywhere dense in Q,

3. $\rho(p) = \lim f_n(p)$ exists for all $p \in P$,

then

$\rho(p)$ is an increasing function on P.

8. LEMMA.

If 1. $f(z)$ *is* defined on $\bar{Q}$,

2. $f(0, y) = f(x, 0) = 0$ for all x and y,

3. all Vitali-increments of $f(z)$ in $\bar{Q}$ are non negative,

then

$f(z)$ is increasing in $\bar{Q}$.

PROOF. Let $(x', y') \leq (x'', y'')$. We have, in general, $f(x, y) = V \langle 0, x; 0, y \rangle$, where V denotes the Vitali increment.

Put $R_1 = \langle 0, 0; x', y' \rangle$, $R_2 = \langle x', 0; x'', y' \rangle$, $R_3 = \langle 0, y'; x'', y'' \rangle$.

We have

$$f(x'', y'') = V(R_1 + R_2 + R_3) = V(R_3) + V(R_1 + R_2) \geq V(R_1 + R_2) = V(R_1) + V(R_2) \geq V(R_1) = f(x', y'),$$

which concludes the proof.

9. LEMMA. If

1. P is a dense set in Q,
2. $f_n(z)$ are functions in $\overline{Q}$ with
$f_n(0, y) = f_n(x, 0) = 0$ for all x and y,
3. $f_n(x, y)$ has non negative Vitali increments only,
4. $\rho(p) = \lim f_n(p)$ for all $p \in P$,
5. $| f_n(z) | \leq K$, $(n = 1, 2, ...)$,

then all four quarter-limits lim (J) $\rho(p) \,|\, p \rightarrow z_9$ exist for every $z_9 \in Q^0$.

PROOF. 9.1. By Lemma 8 all $f_n(z)$ are increasing functions on $\overline{Q}$; hence, by Lemma 7, $\rho(p)$ is an increasing function on P. Hence, by Lemma 6, there exist the main quarter limits

$$\lim \text{ (I) } \rho(p), \lim \text{ (III) } \rho(p) \text{ at all } z_0 \in Q^0.$$

It remains to prove the existence of two other quarter limits.

The proof will be based essentially on Helly's theorem for functions of a single variable, (and in analogous way, in a general cas of n-variables, the proof relies on the Helly's theorem for (n-1) variables).

9.2. Place a nem system of coordinates at the point $0' = (0, 1)$ with the x'-axis concordantly parallel to the x-axis, and with y'-axis directed oppositely to the y-axis. The gauge unit will be the same.

Let $z \in Q$, let a, b be its (x, y)-coordnates. Its (x', y')-coordinates are a, $1 - b$. If a', b' are the (x', y')-coordinates of z, then $(a', 1 - b')$ are its (x, y)-coordinates.

A couple (a, b) of numbers with $0 \leq a \leq 1$, $0 \leq b \leq 1$ denotes two points in $\overline{Q}$, one denoted by (a, b), if we refer them to the (x, y)-system, and the point denoted by (a, b'), if we refer them to the (x', y')-system.

9.3. Take $z = (a, b)' = (a, 1 - b) \in \overline{Q}$.

Put

(1) $\quad g_n(z) = g_n(a, b)' = g_n(a, 1 - b) = V_{x, y} \langle f_n ; 0 , 1 - b ; a, 1 \rangle$

for all $z \in \overline{Q}$.

We have

$$g_n(x,\ 0)' = g_n(0,\ y)' = 0$$

$$(2) \qquad g_n(a,\ b)' = f_n(a,\ 1) - f_n(a,\ 1-b)$$

for

$$f_n(0,\ 1-b) = f_n(0,\ 1) = 0.$$

Let us calculate the $(x',\ y')$ - Vitali increment of the function $g_n(z)$ on a rectangle R. A simple computation gives:

$$V_{x',y'}(g_n,\ R) = V_{x',y'}(f_n,\ R) \geq 0.$$

Since $g_n(z)$ is a function with non-negative $(x',\ y')$-Vitali increment in $\overline{Q}$ and vanishing on the x' and y' axes, the Lemma 8 gives that $g_n(z)$ is a $(x',\ y')$-increasing function in $\overline{Q}$. (3)

We have for every $z = (x,\ y)'$

$$(4) \quad g_n(z) = g(x,\ y)' = f_n(x,\ 1) - f_n(x,\ 1-y) = f_n(x,\ 1) - f_n(z).$$

9.4. Notice that hyp. 2 and 3 implies, on account of Lemma 8, that $f_n(z)$ is (x, y)-increasing in $\overline{Q}$; hence the function $f_n(x, 1)$ of the single variable x, defined on $0 \leq x \leq 1$ is increasing too.

In addition to that we have $f_n(0,\ 1) = 0$ and $f_n(x,\ 1) \leq \leq f_n(1,\ 1) \leq K$, so we are in the conditions of the Helly theorem for a single variable.

Hence a subsequence $\{ f_{k(n)}(x,\ 1) \}$ can be extracted from $\{ f_n(x, 1) \}$, converging everywhere on $\langle 0,\ 1 \rangle$ to an increasing function $\varphi(x)$ on $\langle 0,\ 1 \rangle$,

$$(5) \quad \varphi(x) = \lim f_{k(n)}(x,\ 1), \ 0 \leq x \leq 1, \text{ with } \varphi(x) \leq K,\ \varphi(0) = 0.$$

9.5. Consider the sequence $\{ g_{k(n)}(p) \}$ for $p \in P$, $(n = 1, 2, \ldots)$. Put $p = (a,\ b)' = (a,\ 1-b)$. From (4) we have

$$g_{k(n)}(p) = f_{k(n)}(a,\ 1) - f_{k(n)}(p).$$

By (5) and hyp. 4 we obtain

$$(6) \qquad t(p) = \lim g_{k(n)}(p) = \varphi(a) - \rho(p),$$

and

$$(7) \qquad |\ g_{k(n)}(p)\ | \leq |\ f_{k(n)}(a,\ 1)\ | + |\ f_{k(n)}(p)\ | \leq 2K.$$

9.6. Recapitulating, we see that $\{ g_{k(n)}(p) \}$ satisfies the following conditions: These functions are defined on the everywhere dense set P; $g_{k(n)}(0, y)' = g_{k(n)}(x, 0)'$; $g_{k(n)}$ has a non-negative (x', y')-Vitali increment; $t(p) = \lim g_{k(n)}(p)$ on P; $| g_{k(n)}(p) | \leq 2K$.

Thus we can apply to this sequence the result already obtained in 9.1. i. e.

$\lim(\mathrm{I}')\ t(p)$ and $\lim(\mathrm{III}')\ t(p)$ exist for $p \to z_0$ for all $z_0 \in Q^0$.

Having this, take $z_0 \in Q^0$ and a sequence $\{ p_n \} \to P$ such that $\lim(\mathrm{II})\ p_n = y_0$.

Let

$$p_n = (a_n, b_n) = (a_n, 1 - b_n)', \ z_0 = (x_0, y_0) = (x_0, 1 - y_0)'.$$

We have

$$a_n \to x_0, \ a_n < x_0, \ b_n \to y_0, \ b_n > y_0,$$

hence

$$(a_n, 1 - b_n)' <_{x', y'} (x_0, 1 - y_0)'.$$

It follows, by (6), $t(p_n) = \varphi(a_n) - \rho(p_n)$, and since $\varphi(x)$ is monotonous, it follows that $\lim \rho(p_n)$ exists.

Thus we have proved that for every p_n tending to z_0 in the II — (x, y) — quarter, $\lim \rho(p_n)$ exists. By an already used argument, in 6.4, we prove that this limit does not depend on the choice of $\{ p_n \}$, so

$$\lim\ (\mathrm{II})\ \rho(p) \mid p \to z_0 \text{ exists.}$$

In an analogous way we prove that

$\lim\ (\mathrm{IV})\ \rho(p) \mid p \to z_0$ also exists. The Lemma 9 is proved.

10. Definition. Suppose that the hypotheses of Lemma 9 are satisled. We delne the functions $\bar{\sigma}(z)$, $\underline{\sigma}(z)$ for $z \in Q^0$ by putting

$$\bar{\sigma}(z) = \lim\ (\mathrm{I})\ \rho(p) \mid p \to z, \ p \in P,$$

$$\underline{\sigma}(z) = \lim\ (\mathrm{III})\ \rho(p) \mid p \to z, \ p \in P.$$

We have $\underline{\sigma}(z) \leq \bar{\sigma}(z)$.

11. All preparations being ready, we can apply the argument taken from Wintner's book (1929).

We give it for commodity of the reader in this and in the next subsection. First we prove that if $z_0 = (x_0, y_0) \in Q^0$ is a continuity point of $\bar{\sigma}(z)$, then $\bar{\sigma}(z_0) = \underline{\sigma}(z_0)$.

PROOF. We have

$$\underline{\sigma}(z_0) \leq \bar{\sigma}(z_0).$$

Suppose that

$$\delta = \bar{\sigma}(z_0) - \underline{\sigma}(z_0) > 0 . \tag{1}$$

By hypothesis $\bar{\sigma}$ is continuous at z_0. Hence we can find (x_1, y_1) with $x_1 < x_0$, $y_1 < y_0$, such that

$$\bar{\sigma}(x_0, y_0) < \bar{\sigma}(x_1, y_1) + \frac{\delta}{2} .$$

Hence, by (I)

$$\underline{\sigma}(x_0, y_0) + \frac{\delta}{2} < \bar{\sigma}(x_1, y_1). \tag{2}$$

Put

$$a = \frac{x_0 - x_1}{3} > 0 \quad , \quad b = \frac{y_0 - y_1}{3} > 0,$$

and chose points p', p'' where

$$p' \in (x_0 - a, x_0; y_0 - b, y_0) \cdot P$$

$$p'' \in (x_1, x_1 + a; y_1, y_1 + b) \cdot P, \text{ (open rectangles).}$$

We have $p'' < p'$, and

$$\rho(p') \leq \underline{\sigma}(x_0, y_0), \ \bar{\sigma}(x_1, y_1) \leq \rho(p''),$$

Hence, from (2), we obtain

$$\rho(p') + \frac{\delta}{2} < \rho(p''),$$

which is impossible, since ρ is increasing on P.

The assertion is proved.

12. We shall prove an inequality under the hypothesis of Lemma 9.

Let $z = (x, y) \in Q^0$.

Take four positive numbers a, b, c, d such that

$$(x - a,\ y - b) \in P, \qquad (x + c,\ y + d) \in P.$$

We have

$$\rho(x - a,\ y - b) = \lim f_n(x - a,\ y - b) \leq \liminf f_n(x, y) \leq$$
$$\leq \limsup f_n(x, y) \leq \lim f_n(x + c,\ y + d) = \rho(x + c,\ x + d).$$

Let $a \to 0$, $b \to 0$, $c \to 0$, $d \to 0$, We get

$$\underline{\sigma}(z) \leq \liminf f_n(x, y) \leq \limsup f_n(x, y) \leq \bar{\sigma}(z).$$

REMARK. The above inequality shows, on account of 11, that if z_0 is a continuity of $\bar{\sigma}(z)$, them $\lim f_n(z_0)$ exists.

13. Suppose once more that the hypotheses of Lemma 9 are satisfied. Since, by Lemma 9, $\rho(z)$ possesses everywhere in Q^0 all four quarter limits, and since $\bar{\sigma}(z) = \lim$ (I) $\rho(p)$, $(p \in P$, $p \to z)$, therefore, by Lemma 3, the set of all discontinuity-points of $\bar{\sigma}(z)$ is a Γ-set. The remark in 12. allows to deduce that $\lim f_n(z_0)$ exists for all $z_0 \in \bar{Q}$ excepting perhaps for points of a grate G.

Let G be composed of lines A_1, A_2, ... parallel to the x-axis and of lines B_1, B_2, ... parallel to the y-axis.

On A_1 the functions $f_n(z)$ can be considered as functions of a single variable. Since $f_n(z)$ is on A_1 monotonic and the sequece $\{ f_n(z) \}$ is bounded, we can extract a subsequence $\{ f_{l(n)}(z) \}$ converging everywhere on $A_1 \cdot Q$.

From $\{ f_{l(n)} \}$ we analogously extract another subseèuence $\{ f_{lk(n)} \}$ which converges everywhere on B_1.

We apply the same argument respectively to A_2, yielding the indices $lkl_1(n)$ and so on.

If we use the indices

$$l(1),\ lk(1),\ lkl_1(1),\ lkl_1k_1(1), \ldots$$

we obtain a subsequence of $f_n(z)$ converging everywhere in $\bar{Q}$.

The function $f(z) = \lim f_n(z)$ is, by Lemma 7, an increasing function on $\overline{Q}$. We also have $f(x, 0) = f(0, y) = 0$ for all x and y with $0 \leq x \leq 1$, $0 \leq y \leq 1$.

Thus we have proved the following lemma:

LEMMA. Under the hypotheses of Lemma 9, there exists a subsequence $\{ f_{s(n)}(z) \}$ of $\{ f_n(z) \}$ converging everywhere in $\overline{Q}$.

If we put $f(x, y) = \lim f_{s(n)}(z)$, we have

1) $f(x, y)$ has a non negative Vitali-increment,
2) $f(x, 0) = f(0, y) = 0$ for all x and y,
3) $| f(x, y) | \leq K$.

14. Now we can prove the Helly theorem for two dimensions.

TEOREM. If

1. Q is the closed square $0 \leq x \leq 1$, $0 \leq y \leq 1$,
2. $f_n(x, y)$ are functions in Q with non negative Vitali-increments,
3. $| f_n(x, y) | \leq K$, $(n = 1, 2, ...)$,
4. $f_n(x, 0) = f_n(0, y) = 0$ for all x and y,

then there evists a subsequence $\{ f_{k(n)}(x, y) \}$ of $\{ f_n \}$ converging everywhere in Q to a function $f(x, y)$.

Such a function $f(x, y)$ has the properties

1) $f(x, y)$ has a non negative Vitali increment,
2) $| f(x, y) | \leq K$.

PROOF. We choose in Q a denumerable everywhere dense set P:

$$z_1, z_2 \cdot \dots, z_m, \dots$$

Since $\{ f_n(z_1) \}$ is bounded, we can find an increasing sequence $\{ n' \}$ of indices such that $f_{n'}(z_1)$ converges. From $\{ n' \}$ we extract another sequence $\{ n'' \}$ with converging $f_{n''}(z_2)$. We repeat this argument. The sequence

$$f_1(z_m),\ f_{1'}(z_m),\ f_{1''}(z_m), \dots$$

converges on the whole P. If we apply the Lemma 13., we obtain the theorem.

15. The theorem 14. holds true if we replace the condition 4. by the following more general

4'. Each $f_n(x, \theta)$ and each $f_n(0, y)$ are increasing functions.

Indeed, let us remark that the Vitali-increment of a function of two variables $g(x, y)$ on R will not change if we add to $g(x, y)$ any function $p(x)$ or $q(y)$ of a single variable.

Having this, put $g_n(x, y) = f_n(x, y) - f_n(x, 0) - f_n(0, y) + f_n(0, 0)$.

For any rectangle R we have

$$V(g_n, R) = V(f_n, R) \geq 0.$$

In addition to that $g_n(0, y) = g_n(x, 0) = 0$ and $|g_n(x, y)| \leq 2K$.

Since we are in the conditions of Lemma 13, we can find an increasing sequence $k(n)$ of indices such that $g(x, y) = \lim g_{k(n)}(x, y)$ exists.

Since $\{ f_{k(n)}(x, 0) \}$ is a bounded sequence of increasing functions, the one-dimensional Helly theorem yields a subsequence $\{ kk_1(n) \}$ of $\{ k(n) \}$ with converging

$$\{ f_{kk_1(n)}(x, 0) \}.$$

A further extraction of a subsequence gives a converging sequence

$$f_{kk_1k_2(n)}(0, y),$$

and finally a converging sequence $\{ f_{kk_1k_2k_3}(0, 0 \}$.

This proves the theorem.

1. Remark. The square Q can be replaced by any rectangle with sides parallel to the axes of the systm of coordinates.

4. Functions of several variables.

The Vitali increment constitutes a generalization of the increment $f(b) - f(a)$ of a function $f(x)$ of the single variable x. We have, for $x' \leq x''$, $y' \leq y''$,

$$V(f(x, y); x', y'; x'', y'') =$$
$$= V(f(x, y''); x'; x'') - V(f(x, y'); x'; x''),$$

if we put, in general, for $a < b$, $V(g(x); a; b) = g(b) - g(a)$.

This remark yields the recurrent definition of the Vitali-increment for $f(x_1, \dots, x_{n+1})$:

$$V(f(x_1, \dots, x_n, x_{n+1});\ x_1', \dots, x_{n+1}';\ x_1'', \dots, x_{n+1}'') =$$
$$= V(f(x_1, \dots, x_n, x_{n+1}'');\ x_1', \dots, x_n';\ x_1'', \dots, x_n'') -$$
$$- V(f(x_1, \dots, x_n, x_{n+1}');\ x_1', \dots, x_n';\ x_1'', \dots, x_n''),$$

where the symbol $x_1', \dots, x_n';\ x_1'', \dots, x_n''$ means the n-dimensional box with sides parallel to the system of coordinates and with two opposite vertices $(x_1', \dots, x_n')$, $(x_1'', \dots, x_n'')$.

Of course we supose that $x_1' \leq x_1'', \dots, x_n' \leq x_n''$.

If the function $f(x_1, \dots, x_n)$ has thoroughly non negative Vitali-increments on every box, and if

$$f(0, x_2, \dots, x_n) = \dots = f(x_1, \dots, 0, \dots, x_n) = \dots =$$
$$= f(x_1, \dots, x_n, 0) = 0.$$

then the function is increasing i. e. if $x_1' \leq x_1'', \dots, x_n' \leq x_n''$, then $f(x_1', \dots, x_n') \leq f(x_1'', \dots, x_n'')$.

2. The method applied in Section 3 can be used succesfully to the n-dimensional case. By induction we can prove the n-dimensional Helly theorem.

THEOREM. If

1. Q is the closed n-dimensional box

$$0 \leq x_1 \leq 1, \dots, 0 \leq x_n \leq 1,$$

2. $f_p(x_1, \dots, x_n)$, $(p = 1, 2, \dots)$, are functions in Q with non negative all Vitali increments,
3. $f_p(0, x_2, \dots, x_n) = f_p(x_1, \dots, x_{n-1}, 0) = 0$ for all $x_1, \dots, x_n, \dots$
4. $|f_p| \leq K$,

then there exists a subsequence $\{f_{k(p)}\}$ of $\{f_p\}$ converging everywhere to a function $f(x_1, \dots, x_n)$. Such a function has non negative all Vitali increments, and $|f| \leq K$.

We also get a similar theorem where the condition 3. is replaced by the condition 3′ stating that $f_p(x_1, \dots, x_n)$ will also have a non negative Vitali increments if we replace any k variables $(k = 1, \dots, n-1)$ by zero.

We suppose that the theorem, under hypotheses 1, 2, 3', 4, is true for functions with k variables where $k \leq n$ and we prove the theorem under hypotheses 1, 2, 3, 4 for $k = n + 1$. The theorem with hypostheses 1, 2, 3', 4 for $k = n + 1$ will follow.

The inductive proof requires the same steps as in the case of two variables.

In the case of n-dimensions, by a grate we shall understand any at most denumerable union of $(n - 1)$ — planes parallel to the system of coordinates, and Γ-set is defined as any subset of a grate.

Quarters are replaced, in a case of three dimensions, by octants, and we do analogously for n-dimensions.

5. Functions of bounded Vitali variation, measure.

1. To avoid unessential complications of n-dimensions we shall confine oursevels to two dimensions. The general case can be treated similarly.

Let $g(x, y)$ be defined in $\bar{Q}$, have non negative Vitali increments, be bounded, and let $g(0, y) = g(x, 0) = 0$ for all x and y.

We shall consider halfopen « rectangles »

$H = H(x_1, x_2; y_1, y_2)$ defined as the sets of (x, y) for which $x_1 < x \leq x_2$, $y_1 < y \leq y_2$.

$H \neq 0$ if and only if $x_1 < x_2$, $y_1 < y_2$.

We define for all $H \subseteq Q$ where $H \neq 0$, the set-function

$$\mu(H) = V(g(x, y); x_1, y_1; x_2, y_2) \text{ and } \mu(0) = 0.$$

If we partition H into a finite number of halfopen rectangles H_p disjoint with one another, we see that

$$\mu(H) = \Sigma_p \mu(H_p).$$

This allows to extend the function $\mu(H)$ to a set function $\mu(F)$ defined for all sets F which are finite unions of halfopen rectangles. We call these sets *figures*.

If $F = \Sigma_q H_q$ where the rectangles are disjoint, we define $\mu(F)$ as $\Sigma_q \mu(H_q)$.

The function $\mu(f)$ is finitely additice i. e. if $F_1 \cdot F_2 = 0$, then $\mu(F_1 + F_2) = \mu(F_1) + \mu(F_2)$. It is also non negative.

The figures F, if ordered by the inclusion relation of point-sets, will generate a Boolean finitely additive lattice (B_2) with 0 as null element and $H(0, 0; 1, 1)$ as unit element.

Indeed, the complement of a figure is a figure too. and the sum of two figures is also a figure.

We see that $\mu(F)$ is a finitely additive, finite an non negative measure on (B_2).

2. The Boolean lattice (B_2) can be conceived as the cartesian product $(B_1) \times (B_1)$ of two Boolean lattices (B_1) and (B_1), where B_1 is composed of finite unions of halfopen subintervals of $0 \leq x \leq 1$.

3. Now suppose that we have defined on (B_2) any non negative, finite, finitely additive measure $v(F)$.

This measure constitutes an extension of the function $f(x, y) = vH(0, 0; x, y)$, for $0 \leq x \leq 1$, $0 \leq y \leq 1$.

We see that $f(x, 0) = f(0, y) = 0$ for all x and y, and that f is bounded and has non negative Vitali increments.

Indeed

$$V(f(x, y); x_1, x_2; y_1, y_2) = vH(x_1, x_2; y_1, y_2). \tag{1}$$

Thus between the functions $g(x, y)$ of two variables and satisfying the conditions stated in subsection 1 of the present section, and the non negative, finite, finitely additive measures on (B_2) there exists a one-to-one correspondence, which will be denoted by C.

4. The above remarks yield the

THEOREM. If

1. $\mu_n(F) \geq 0$ are finitely additive measures on (B_2), (see 2.),
2. $\mu_n(F) \leq K$, $(n = 1, 2, \ldots)$,

then there exists an increasing sequence $k(n)$ of indices such that

$$\mu(F) = \lim \mu_{k(n)}(F)$$

for all figures $F \in B_2$. The function $\mu(F)$ is also a finitely additive, non negative measure on (B_2) with $\mu(F) \leq K$.

Proof. Let $g_n(x, y) C \mu_n(F)$, and apply to $\{g_n\}$ the theorem 14 of Section 4.

Let $g(x, y) = \lim g_{k(n)}$ for all $(x, y) \in \overline{Q}$.

Since $g_n(x, y) \leq K$, we also have $g(x, y) \leq K$ and $g(x, y)$ has non negative Vitali increments.

Now

$$\mu_n H(x_1, x_2; y_1, y_2) = g_n(x_1, y_1) + g_n(x_2, y_2) - g_n(x_1, y_2) - g_n(x_2, y_1).$$

It follows that

$$\mu H(x_1, x_2; y_1, y_2) = \lim \mu_{k(n)} H(x_1, x_2; y_1, y_2)$$

exists for $x_1 < x_2$; $y_1 < y_2$.

This equality can be extended to figures, so we get $\mu(F) = \lim \mu_{k(n)}(F)$. The proof is easy to conclude.

5. Let $g(x, y)$ be a function of bounded Vitali variation in Q, and such that $g(x, 0) = g(0, y) = 0$ for all x and y,

if we take, like in 1., $H = H(x_1, x_2; y_1, y_2)$, and define $\mu(H) = V(g(x, y); x_1, y_1; x_2, y_2)$.

The function $\mu(H)$ can be extended to figures in (B_2), yielding a real-valued, finite, finitely additive measure on (B_2), and conversely to every such real valued measure there corresponds, in a unique way a bounded function $g(x, y)$ of bounded Vitali variation in Q and with $g(x, 0) = g(0, y) = 0$.

Let $\{\mu_n(F)\}$ be a sequence of real valued, bounded, finitely additive measures on (B_2), and let $|\mu_n(F)| \leq K$, $(n = 1, 2, ...)$.

It is known that $\mu_n(F)$ can be represented as the difference of two non negative measures

$$\mu_n(F) = \mu_n'(F) - \mu_n''(F),$$

where

$$|\mu_n'(F)| \leq K, \quad |\mu_n''(F)| \leq K.$$

If we apply the theorem 4. and the method of selecting sequences, we obtain the

THEOREM. If

1. $\mu_n(F)$ are finitely additive real valued measures on (B_p), where p is a natural number,
2. $|\mu_n(F)| \leq K$, $(n = 1, 2, \ldots)$,

then there exists an increasing sequence $k(n)$ of indices such that

$$\mu(F) = \lim \mu_{k(n)}(F)$$

exists. Such a function $\mu(F)$ is also a real valued, finitely additive measure on (B_p) with

$$|\mu(F)| \leq K.$$

6. This theorem is equivalent to the following general Helly theorem for functions $f_n(x_1, \ldots, x_p)$ of bounded Vitali variation:

THEOREM. If

1. $f_n(x_1, \ldots, x_p)$ are functions of bounded Vitali variation in a box Q,
2. if we put any number k of variables $(k = 1, \ldots, p-1)$ equal 0, in f_n, we obtain also a function of bounded Vitali variation,
3. $|f_n(x_1, \ldots, x_p)| \leq M$, $(n = 1, 2, \ldots)$,
4. if the Vitali variation of f_n are $\leq N$, then there exists a subseqence $f_{s(n)}(x_1, \ldots, x_p)$ converging everywhere in Q toward a function of bounded Vitali variation, satisfying the analogous conditions 1, 2, 3, 4.

7. A theorem analogous to Theor. 5. is true for an infinite sequence of measures on the cartesian product of a denumerable number of replicas of (B_1):

$$(B_\omega) = \Pi^\times (B_1).$$

Let us recall the definition of this product. In the space of all infinite sequences $x = \{x_1, x_2, \ldots, x_n, \ldots\}$ of real numbers with $0 \leq x_n \leq 1$, we consider boxes H, defined as sets of all x for which $0 \leq a_n < x_n \leq b_n \leq 1$, $(n = 1, 2, \ldots)$, where all $a_n = 0$ and all $b_n = 1$ excepting for an at most *finite* number of indices.

The empty set and finite unions of boxes constitute the elements of (B_ω).

If these sets F, called figures, are ordered by the relation of inclusion of sets, they make up a Boolean lattice which is finitely additive.

Let $\mu_p(F) \geq 0$ be a finitely additive measure on (B_ω), $p = 1, 2 \ldots$ and suppose that $|\mu_p(F)| \leq K$ for $p = 1, 2, \ldots$

If we fix $m \geq 1$ and consider only those figures which are finite sums of boxes each of which having $a_n = 0, b_n = 1$ for $n > m$, we obtain a Boolean lattice (A_m) on which the measure $\mu_p(F)$ satisfies the conditions of Theor. 5. So a subsequence $\mu_{k(p)}(F)$ can be extracted in such a way that $\lim \mu_{k(p)}(F)$ exists for all those figures.

If we take progressively $m = 1, 2, \ldots$ and use the principle og extracting suitable subsequences, we prove the assertion. Thus the following theorem is true:

THEOREM. If

1. (B_ω) is the cartesian product of an infinite denumerable number of replicas of (B_1),
2. $\mu_p(F)$ is a real valued finitely additive measure on (B_ω),
3. $|\mu_p(F)| \leq K$ for $p = 1, 2, \ldots$

then a subsequence $\{\mu_{k(p)}(F)\}$ can be extracted out of $\mu_p(F)$ such that the limit

$$\mu(F) = \lim \mu_{k(p)}(F)$$

exists for all $F \in B_\omega$, and $\mu(F)$ is finitely additive on (B_ω) with $|\mu(F)| \leq K$.

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