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Limit-representation of linear, even discontinuous, linear functionals in Hilbert spaces

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LIMIT-REPRESENTATION OF LINEAR, EVEN DISCONTINUOUS, LINEAR FUNCTIONALS IN HILBERT SPACES

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1. - Usually only continuous linear functionals are considered in the literature, but there are discontinuous functionals in Hilbert space and they also deserve a study. To have an example of a discontinous linear functional defined everywhere in the Hilbert space $\boldsymbol{H}$, let us remind that a necessary and sufficient condition for the continuity of a linear functional $f(x)$ defined on $\boldsymbol{H}$ is its Hilbert-boundedness:

(1) $$|f(x)| \leq M \cdot \|x\|, \qquad \text{where } M > 0,$$

where (1) aught to hold for all x, and where $\|x\|$ is the norm of x.

Hence, to have a discontinuous $f(x)$, it suffices to assure its non-boundedness on the unit sphere $\|x\| \leq 1$.

Let H be a Hamel-basis [1]) in $\boldsymbol{H}$, composed of vectors a_α with norm ≤ 1.

If $x \in \boldsymbol{H}$, $x \neq \vec{0}$, there exists a unique representation

(2) $$x = \lambda_1 a_{\alpha_1} + \dots + \lambda_n a_{\alpha_n} .$$

with number coefficients $\lambda_1, \dots, \lambda_n$, all $\neq 0$, and where $a_{\alpha_i} \in H$ with mutually differing a_{α_i}.

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1) See [9]. It is a set H of independent vectors such that for every vector $x \neq \vec{0}$ there exists a finite number of vectors of H in terms of which x can be linearly expressed.

Let us attach to each a_α a number p_α such that their whole set be unbounded. Put, with reference to (2),

$$f(x) \underset{df}{=} \lambda_1 p_{\alpha_1} + \dots + \lambda_n p_{\alpha_n} \quad \text{for } x \neq \overrightarrow{0} \text{ and } f(\overrightarrow{0}) \underset{df}{=} 0.$$

The function thus defined is a linear discontinuous functional on $\boldsymbol{H}$.

2. - The following theorem is known [2]:

If $f(x)$, with domain $\boldsymbol{H}$, is a linear borelian functional, [i.e. for every open set E the counterimage $f^{-1}(E)$ is borelian], then $f(x)$ is continuous. Hence, if a linear functional is borelian and discontinuous, it cannot be defined on the whole space $\boldsymbol{H}$, but only on some linear sub-variety of $\boldsymbol{H}$. This variety must be even a set of the 1° cathegory [2].

An example of a borelian linear discontinuous functional may be constructed by a device of M. H. Stone [3]. Let $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$ be a complete orthonormal system of vectors in $\boldsymbol{H}$. Put

$$f(x) \underset{df}{=} a_1 + \dots + a_n$$

whenever

$$x = a_1\varphi_1 + \dots + a_n\varphi_n, \quad (n = 1, 2, \dots).$$

We easily prove that $f(x)$, which is defined on the linear variety L spanned by $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$, is discontinuous on L.

This example shows that there is some interest in linear discontinuous functionals whose domain of existence is any linear subvariety of $\boldsymbol{H}$.

In the general case they cannot be represented as ordinary limits of continuous linear functionals, but, if we apply stream limits, i.e. Moore-Smith generalized limits, we obtain the result that for every linear functional $f(x)$, continuous or not, defined on a linear subvariety L of $\boldsymbol{H}$, there exists a stream-sequence [2] $\{f_\alpha(x)\}$ of linear continuous functionals

[2]) A *stream-ordering* [4] (E. H. Moore & Smith « directed set ») is a *partial ordering* [5], [i. e. (|) $\boldsymbol{R} \neq \emptyset$; aRa is equivalent to $a \in$ (|) $\boldsymbol{R}$; aRb, bRc imply aRc; $a = b$ is equivalent to aRb, bRa] such that for

on $\boldsymbol{H}$ such that

$$f_\alpha(x) \to f(x) \quad \text{on} \quad L,$$

and that the sequence does not tend to any limit for $x \,\overline{\in}\, L$. To prove this we use a method whose priority belongs to L. Alaoglu [1]. Though our aim is Hilbert space, we like to lead our arguments so as to cover more general cases, because on this way the essential elements will be better visible.

3. - Let W be a linear space [3]), V its linear subvariety [4]), and $f(x) \neq 0$ a linear functional with $(| f = V$ [5]).

Let V be infinite dimensional, and have an infinite *deficiency* in W, (i.e. there exists in W an infinite dimensional linear subvariety which is independent of V).

Suppose W is provided with a topology (T) having the following property: if M is a finite dimensional linear subvariety in W, then there exists a linear subvariety N complementary to M in W, such that for every linear functional $\varphi(x)$ on M there exists [6]) a linear (T)-continuous functional

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every $a \in (\,)\,R$, $b \in (|)\,R$ there exists c with aRc and bRc. [$(|\ R$ is the set of all x such that there exists y with xRy; $|)\ R$ is the set of all y such that there exists x with xRy; $(|)\ R = (|\ R \cup |)\ R$].

A *stream sequence* [5] of numbers $\{ M_d \}$ corresponding to a stream ordering R is a number valued function defined for all $d \in (|)\ R$, i.e. $(|\ M = (|)\ R$. We say: R— stream sequence.

We say that $M_d \underset{R}{\to} N$, if for every $\varepsilon > 0$ there existe $d_0 \in (|)\ R$ such that $| M_d - N | < \varepsilon$ whenever $d_0 R d$. Since $\{ M_d \}$ cannot tend to two different stream-limits, we may write $\lim\limits_R M_d = N$.

3) A *linear space* is an Abelian group with real (or complex) multipliers, where $\lambda x \neq \vec{0}$ whenever $\lambda \neq 0$, $x \neq \vec{0}$.

4) A *linear subvariety of* W is a non empty set V such that, if $x_1, x_2 \in V$, then $\lambda_1 x_1 + \lambda_2 x_2 \in V$.

5) A linear functional $f(x)$ is defined by the condition

$$f(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

6) There are linear topological spaces where every linear continuous functional is identically equal 0, (see [6], [7]). Conditions are given in [8].

$\Phi(x)$ in W such that $\Phi(x) = 0$ wenever $x \in N$ and $\Phi(x) = \varphi(x)$ whenever $x \in M$.

Under these circumstances we shall prove that there exists a stream — ordering R and a corresponding R — stream sequence $\{F_d(x)\}$ of linear (T)-continuous functionals with $\mathbb{C}|F = W$ for $d \in \mathbb{C}|\mathbb{D}\ R$, such that

$$\lim_R F_d(x) = f(x) \text{ for all } x \in V$$

$$\lim_R F_d(x) \text{ does not exist whenever } x \bar{\in} V.$$

4. - Proof. Since $f(x) \neq 0$, we can choose y such that

$$f(y) = 1. \tag{1}$$

Let Z be the subvariety of V composed of all vectors x for which $f(x) = 0$. The variety Z has the deficiency $= 1$ with respect to V, and y is independent of V.

Let H_Z be a Hamel basis in Z, and consider the Hamel basis in W composed of H_Z, y and of a remaining class H_K of vectors. H_K is an infinite set, because V has an infinite deficiency in W. The set (H_K, y) is, of course, a Hamel basis of V. Let H_Z and H_K be well ordered. Denote vectors of H_Z by a a_α, and those of H_K by b_β, where α, β range over the field of some transfinite well-orderings of ordinals.

5. - We shall consider ordered couples (A, B) where A is a finite non empty set of vectors belonging to Z and B a finite non empty set of vectors belonging to K.

Let $S(\beta, A, B)$ be a number-valued function defined for all b_β, A, B; this function will be determined later on.

For any couple (A, B) we define in W a linear, (T)-continuous functional in the following way:

By hypothesis there exists a linear variety $W_{A,B}$, complementary to $\{A, B\}$ [7]) in W, such that for every linear func-

[7]) $\{A, B\}$ means the linear variety spanned by the vectors of A and those of B.

tional $\varphi(x)$ on $\{A, B\}$ there exists a linear (T)-continuous functional $\Phi(x)$ on W with $\Phi(x) = \varphi(x)$ for $x \in \{A, B\}$, and $\Phi(x) = 0$ for $x \in W_{A,B}$. Let $x \in W$; there is a unique representation

$$(2) \qquad x = x' + x''$$

where $x' \in \{A, B, y\}$, $x'' \in W_{A,B}$.

Let $A = \{a_{\alpha_1}, \ldots, a_{\alpha_s}\}$, $\quad B = \{b_{\beta_1}, \ldots, b_{\beta_t}\}$.

Since the vectors $a_{\alpha_1}, \ldots, a_{\alpha_s}, y, b_{\beta_1}, \ldots, b_{\beta_t}$ are independent they make up, in $\{A, B, y\}$, a system of coordinates. Hence x' has a unique representation

$$x' = x_1\overrightarrow{a_{\alpha_1}} + \ldots + x_s\overrightarrow{a_{\alpha_s}} + p\overrightarrow{y} + y_1\overrightarrow{b_{\beta_1}} + \cdots + y_t\overrightarrow{b_{\beta_t}}$$

where $x_1, \ldots, x_s, p, y_1, \ldots, y_t$ are numbers.

Put

$$\varphi_{A,B}(x') \underset{df}{=} p + y_1 s(A, B, \beta_1) + \ldots + y_t s(A, B, \beta_t).$$

This is a linear functional in $\{A, B, y\}$.

Now there exists, in W, a (T)-continuous functional $F_{A,B}(x)$ such that

$$F_{A,B}(x'') = 0 \quad \text{for} \quad x'' \in W_{A,B},$$

$$F_{A,B}(x') = \varphi_{A,B}(x') \quad \text{for} \quad x \in \{A, B, y\}.$$

$F_{A,B}(x)$ is even unique.

From (2) it follows

$F_{A,B}(x) = \varphi_{A,B}(x')$, provided that $x = x' + x''$, where $x' \in \{A, B, y\}$, $x'' \in W_{A,B}$.

6. - Define an ordering relation R for the couples (A, B):

$$(A_1, B_1)R(A_2, B_2) \quad \text{will mean} \quad A_1 \subseteq A_2, \; B_1 \subseteq B_2.$$

This is a stream-ordering, hence $\{F_{A,B}(x)\}$ is an R — stream sequence of linear (T)-continuous functionals on W. We shall prove that, whatever $S(A, B, \beta)$ may be, we have

$$\lim_{R} F_{A,B}(x) = f(x) \quad \text{for every } x \in V.$$

Let $x' \in V$. There exists a unique representation of x' in terms of the Hamel-basis (H_Z, y):

$$x' = \lambda_1 a_{\alpha_1'} + \dots + \lambda_\gamma a_{\alpha'_\gamma} + py .$$

Take $\varepsilon > 0$, and choose A_0 such that

$$a_{\alpha_1'}, \dots, a_{\alpha'_\gamma} \in A_0 .$$

Choose any H_0. Let $(A_0, B_0) R (A, B)$. If the put $x' = x_1' + x_1''$ where $x_1' \in \{A, B, y\}$, $x_1'' \in W_{A,B}$, we have

$$x_1'' = 0 \quad , \quad x_1' = x' ;$$

hence

$$F_{A,B}(x') = \varphi_{A,B}(x') = p .$$

On the other hand we have

$$f(x') = f(\lambda_1 a_{\alpha_1'} + \dots + \lambda_\gamma a_{\alpha'_\gamma}) + pf(y) = p$$

hence

$$| F_{A,B}(x') - f(x') | = 0 < \varepsilon \text{ for every } (A, B)$$

with

$$(A_0, B_0) R (A, B) .$$

Thus

$$\lim_R F_{A,B}(x) = f(x) \quad \text{for every} \quad x \in V .$$

7. - To assure that $F_{A,B}(x)$ do not converge whenever $x \,\overline{\in}\, V$, we shall make a suitable choice for $S(A, B, \beta)$.

If $b_\beta \,\overline{\in}\, B$, put

$$S(A, B, \beta) \underset{df}{=} 0 . \tag{3}$$

Let $b_\beta \in B$, and let

$$B = (b_{\beta_1} \dots, b_{\beta_t}) \tag{4}$$

where $\beta_1 < \dots < \beta_t$. Denote by $\nu(B, \beta)$ the number of those β_i in (4), for which $\beta \leq \beta_i$. Denote by $\mu(A)$ the number of vectors in A. Define

$$S(A, B, \beta) \underset{df}{=} \mu(A)^{\nu(B, \beta)}$$

8. - Let $x \bar{\in} V$. We have a unique representation of x in terms of vectors of the Hamel-basis (H_Z, y, H_K).

$$x = x_1 a_{\alpha_1} + \dots + x_s a_{\alpha_s} + py + y_1 b_{\beta_1} + \dots + y_t b_{\beta_t}.$$

where $t \geq 1$, $s \geq 0$, $y_1 \neq 0, \dots, y_t \neq 0$.

Let $\beta_1 < \dots < \beta_t$.

Choose B so as to have

$$b_{\beta_1}, \dots, b_{\beta_t} \in B,$$

and choose A such that

$$a_{\alpha_1}, \dots, a_{\alpha_s} \in A.$$

We have

$$\nu(B, \beta_1) > \nu(B, \beta_2) > \dots > \nu(B, \beta_t) \geq 1,$$

and

$$\varphi_{A,B}(x) = p + y_1 \mu(A)^{\nu(B,\beta_1)} + \dots + y_t \mu(A)^{\nu(B,\beta_t)}.$$

For a given x this formula holds for any A such that

$$a_{\alpha_1}, \dots, a_{\alpha_s} \in A$$

and for any B such that $b_{\beta_1}, \dots, b_{\beta_t} \in B$.

Suppose

$$\lim_{R} F_{A,B}(x) = q.$$

There exists (A_0, B_0) such that for every (A, B) with $(A_0, B_0) \, R \, (A, B)$ we have

$$| F_{A,B}(x) - q | < 1.$$

Hence

$$| F_{A,B}(x) | < q + 1. \tag{5}$$

Take A such that

$$A_0 \subseteq A, \; a_{\alpha_1}, \dots, a_{\alpha_s} \in A, \; \mu(A) = n, \qquad (n = 1, 2, \dots)$$

and B such that

$$B_0 \subseteq B, \; b_{\beta_1}, \dots, b_{\beta_t} \in B.$$

We have

$$F_{A,B}(x) = \varphi_{A,B}(x) = p + y_1 n^{\nu(B,\beta_1)} + \ldots + y_t n^{\nu(B,\beta_t)},$$

because $x \in \{A, y, B\}$.

We get

$$F_{A,B}(x) \geq | y_1 n^{\nu(B,\beta_1)} + \ldots + y_t n^{\nu(B,\beta_t)} | - | p |$$

$$\geq n^{\nu(B,\beta_1)} \left\{ | y_1 | - \frac{| y_2 | + \ldots + | y_t |}{n} \right\} - | p | .$$

Hence for

$$n > \frac{| y_2 | + \ldots + | y_t |}{2 \, | y_1 |}$$

we get

$$| F_{A,B}(x) | \geq n \frac{| y_1 |}{2} - | p |$$

and, by (5),

$$n \frac{| y_1 |}{2} - | p | < q + 1 \quad \text{for} \quad n \rightarrow \infty ,$$

which is impossible.

Thus we have proved that $\lim_{R} F_{A,B}(x)$ does not exist for $x \,\overline{\in}\, V$,

9. - The theorem proved settles the problem in the case where V is infinite dimensional and has an infinite deficiency in W. Now, if V is finite dimensional, the problem is easy, and in the case V has an at most finite deficiency, in W, we may plunge W into a wider Hilbert space W' so, as to obtain the required infinite deficiency for V. If W is finite dimensional, the problem is trivial.

Thus we can state the following

Theorem. - If

1. $\boldsymbol{H}$ is a Hilbert space (which may be neither separable nor complete),
2. V a linear subvariety in W,
3. $f(x)$ a linear functional defined on V, then there

exists a stream-ordering R and a corresponding stream-sequence $\{ f_\alpha(x) \}$ of linear continuous functional in H such that

1) $\lim_{R} f_\alpha(x) = f(x)$ whenever $x \in V$;

2) $\{ f_\alpha(x) \}$ does not tend to any limit, if $x \in V$.

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