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IMPROPERLY ABELIAN VARIETIES

Nota (*) *di* LEONARD ROTH (*a Londra*)

The present note continues the study of the pseudo-Abelian varieties initiated in (**7**). We begin by recalling that a *pseudo-Abelian variety of type* q $(1 \leq q \leq p-1)$ is any non-singular variety W_p which is invariant under a continuous group of ∞^q automorphisms whose trajectories form a *congruence* (∞^{p-q} system of index 1) of Picard varieties V_q; it is thus a natural generalization of the elliptic surfaces, and of the elliptic and hyperelliptic threefolds previously considered in (**8**) and (**9**). Just as the elliptic surfaces contain a subspecies, which we have elsewhere (**9**) called *improperly hyperelliptic,* and which map irregular involutions on a Picard surface (necessarily of particular moduli, containing pencils of elliptic curves), so the elliptic and hyperelliptic threefolds include subspecies, namely the *improperly Abelian threefolds,* the principal types of which have been determined in (**10**).

In this work the above results are generalized to the case $p > 3$. We first show that any Abelian variety W_p having some plurigenus greater than zero, which maps an involution of superficial irregularity $q (0 < q < p)$ on a Picard variety V_p, is pseudo-Abelian of type q; then, from the representation of W_p obtained in (**7**) we deduce that W_p is improperly Abelian, i.e. is representable parametrically by means of Abelian functions of genera less than p: more precisely, we show that the coordinates of the general point of W_p are expressible as algebraic functions of Abelian functions of

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genus q and other Abelian functions of genus $p-q$, and that, in many cases, the genera of the functions required for the representation may be lowered still further.

We also remark that the classification of the improperly Abelian varieties is a recursive process depending on the determination of all species of lower dimension which map involutions on Picard varieties that are generable by finite groups of automorphisms, including, it should be added, those species which have all their plurigenera equal to zero. It follows from this that the classical discussion (**2**, **4**) of the hyperelliptic surfaces, which excludes all the rational types, forms an inadequate basis for the treatment of the higher Abelian varieties.

The note concludes with a brief account of the *para-Abelian* varieties, which bear the same relation to the improperly Abelian varieties as do the paraelliptic surfaces (**10**) to the improperly elliptic. While these manifolds do not possess the group of automorphisms characteristic of the improperly Abelian varieties, they have certain affinities with the latter: thus, they are in general superficially irregular, and they are representable on multiple product varieties with a particularly simple kind of branch manifold; while their systems of canonical hypersurfaces are compounded of congruences of Picard varieties.

1. The classification of Abelian varieties. - We begin by outlining the process of classifying the Abelian varieties, on which our work depends, referring to (**2**), (**5**), (**10**) for details and illustrations of the method. Let V_p be a Picard variety (Abelian variety of genus p and rank 1) which we assume to be free from singularities and exceptional manifolds; then any Abelian variety W_p of genus p and rank $r>1$ may be regarded as the image of a simple involution I_n on V_p, so that the problem of classifying W_p is equivalent to that of classifying I_n. We call W_p properly or improperly Abelian, according as the variety cannot or can be represented parametrically by means of Abelian functions of genus lower than p. Thus V_p itself is properly Abelian (in fact we shall find that it is the only superficially irregular proper type); again, among the proper types, we have the Wirtinger variety (**16**): this is of rank 2, and maps an involution on V_p which is generated by transformations of the second kind. We then have the following results.

I. *The pure canonical and pluricanonical hypersurfaces*[1]) *of W_p, if effective, are all of order zero.*

For the canonical system of V_p, which is of order zero, is the transform of the canonical system of W_p, together with the coincidence hypersurface (if any) of I_n. It follows that the geometric genus P_g and the plurigenera P_i of W_p satisfy the inequalities $P_g \leq 1$, $P_i \leq 1$.

II. *The superficial irregularity q of W_p is at most equal to p.*

For a generic surface of W_p maps an involution on a surface of V_p which is likewise generic and hence of irregularity p, whence the result (**2**). (Actually, we shall see that $q = p$ if, and only if, W_p is a Picard variety).

III. *If I_n possesses ∞^{p-1} coincidences, then the geometric genus and plurigenera of W_p are all zero.*

By hypothesis, there is a coincidence hypersurface on V_p and hence a branch hypersurface B on W_p. Consider a general linear system $|C|$ of hypersurfaces on W; this maps an irreducible system $|C_1|$ on V_p, which belongs to I_n, and whose adjoint system $|C_1'|$ is equivalent to the sum of the transform of $|C'|$ and the coincidence hypersurface. Now since V_p has a canonical hypersurface of order zero, we have $C'_1 \equiv C_1$, from which it follows that $C \equiv C' + B$: that is to say, W_p possesses an *anticanonical system* (**11**, **12**). Hence, on W_p, the process of successive adjunction always terminates (**11**), so that W_p must have geometric genus and plurigenera zero.

The group-theoretic method of classifying the varieties W_p, which was first applied systematically to the case $p = 2$

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[1]) That is, varieties of dimension $p - 1$.

by Bagnera and De Franchis (**2**), is based on the above concepts, used in conjunction with the following theorem:

IV. *If* W_p *has some plurigenus greater than zero, then* I_n *can be generated by a finite group* $\mathcal{G}_n$ *of automorphisms of* V_p.

For, in the first place, it can be shown (**3**, **1**) that, on this hypothesis, I_n cannot possess a united point which is conjugate to a hypersurface; in the second place, it follows from III that I_n can have at most ∞^{p-2} coincidences. Hence, by the transcendental-topological argument due to Bagnera-De Franchis (**2**) and Andreotti (**1**), it may be shown that I_n is generable by a group $\mathcal{G}_n$ of automorphisms of V_p.

It may be noted that the conjecture that, if the involution I_n has at most ∞^{p-2} coincidences, it can be generated by a group $\mathcal{G}_n$, was first made by Lefschetz (**6**) without, however, the additional hypothesis concerning the plurigenera of W_p; that this hypothesis is essential may be shown by examples.

The converse theorems suggested by these results are not in general true. Thus, in the case $p = 2$, Bagnera-De Franchis have shown that a rational involution I_n may possess a finite (non-zero) number of coincidences, and have also indicated that an involution may be endowed with a coincidence curve and yet be generable by a finite group of automorphisms; while Scorza (**13**) has actually obtained all the involutions — necessarily rational — of this type. And, generally, it may be shown that there exist *anticanonical involutions* (i.e. such that W_p possesses an anticanonical system) endowed with ∞^{p-1} coincidences which are generable by finite groups of automorphisms.

In order, however, to obtain a complete classification of the varieties W_p within the limits of the group-theoretic method, it is evidently necessary, in view of III, to exclude those types for which $P_i = 0$. In the case $p = 2$, this amounts to no more than omitting rational and elliptic scrollar surfaces from the scheme; in the case $p > 2$, however, the precise extent of such a limitation is not known. Even so, in order to obtain all the members of this restricted class of varieties W_p, we shall see that it is still necessary to determine those types W_r $(r < p)$ with zero plurigenera which map involutions on V_r generable by finite groups of automorphisms.

2. The superficially irregular types. - Supposing that, with the customary notation, the general point of V_p has coordinates $(u_1, u_2, ..., u_p)$, let W_p be an Abelian variety which represents any (simple) involution I_n generable by a group $\mathcal{G}_n$; then it may be shown (**2**, **5**) that $\mathcal{G}_n$ itself can be generated by a finite set of substitutions of the form

$$u_i' = \sum_{j=1}^{p} a_{ij}u_j + + b_i \qquad (i = 1, 2, ..., p) \tag{1}$$

where a_{ij} and b_i are constants.

In the case where W_p has superficial irregularity $q > 0$, we may show further (**2**, **5**) that q of the above relations may be taken to be

$$u_i' = u_i + b_i \qquad (i = 1, 2, ..., q) \tag{2}$$

Lefschetz (**5**) has remarked that, by modifying suitably the period matrix of V_p, the remaining relations of the set (1) may be reduced to the form

$$u_j' = \varepsilon_j u_j + b_j \quad (j = q + 1, q + 2, ..., p) \tag{3}$$

The constants ε_j are called the multipliers of the substitution; and since the group generated by (2) and (3) is finite, they must be roots of unity, other than unity itself.

The first stage in the classification of the varieties W_p consists in determining the finite groups of collineations represented by (2) and (3); the second stage consists in showing that there exist period matrices for V_p of the requisite kinds. In this process an important part is played by the theorem:

V. *A necessary and sufficient condition that W_p should have geometric genus unity is that each substitution in $\mathcal{G}_n$ should have modulus unity.*

The necessity of the condition is obvious, for W_p must then possess a p-ple integral $\iint\ldots\int du_1 du_2 \ldots du_p$ of the first kind, from which it follows that the Jacobian $\partial(u'_1, u'_2, \ldots, u'_p)\partial/(u_1, u_2, \ldots, u_p)$ must be equal to unity. The converse result is likewise true (**2**).

It is an immediate consequence of (2) that I_n is invariant under a continuous group $\mathcal{G}$ of ∞^q transformations of the first kind; evidently $\mathcal{G}$ is completely transitive and permutable, so that its trajectories V_q, which are of course invariant under $\mathcal{G}$, must be Picard varieties (**5**); that is, W_p is a pseudo-Abelian variety of type q (**7**). Hence

VI. *Every Abelian variety W_p of superficial irregularity $q(0 < q < p)$ and with some plurigenus greater than zero is pseudo-Abelian of type q.*

We may show further that

VII. *Every Abelian variety W_p of superficial irregularity p is a Picard variety.*

This result is due to Severi (**15**); a shorter proof, of geometrical character, is as follows. Suppose, if possible, that W_p has geometric genus zero; then (Severi, **14**) W_p must contain a congruence of irregularity p, and consequently V_p must contain a congruence of irregularity p at least, in contradiction to the known fact that every congruence on V_p has irregularity less than p. Thus W_p has geometric genus unity and therefore possesses a pure canonical hypersurface of order zero (I), so that, by (**7**, § 2), it is a Picard variety.

We have seen that, when $q < p$, W_p contains a congruence $\{V_q\}$ of Picard varieties V_q. Now this congruence can arise only from a congruence of varieties $\overline{V}_q$ on V_p; from the general theory of Picard varieties we know that this congruence must be of Picard type and that its members are Picard varieties; a variety V_p containing such a congruence is said to be *special of type q*. We know also that V_p must contain a second Picard congruence of Picard varieties $\overline{V}_{p-q}$; evidently these will give rise to a congruence $\{V_{p-q}\}$ of varieties V_{p-q} on W_p. Hence

VIII. *Every Abelian variety W_p of superficial irregularity $q(0 < q < p)$ and with some plurigenus greater than zero is the image of an involution on a Picard variety V_p which is special of type q; in addition to the congruence $\{V_q\}$ of trajectories V_q, W_p contains a congruence $\{V_{p-q}\}$ of varieties V_{p-q}.*

In previous work (7) the existence of this second congruence $\{V_{p-q}\}$ on a pseudo-Abelian variety W_p was established on the assumption that the trajectories of $\mathfrak{G}$ had general moduli: here, however, such an assumption is not made.

3. Further properties of W_p. - Before proceeding it will be convenient to recall the properties of pseudo-Abelian varieties which will be required later. Let W_p be a pseudo-Abelian variety of type q whose congruence of trajectories V_q is $\{V_q\}$: the varieties V_{p-q} of the associated congruence $\{V_{p-q}\}$ on W_p are transforms of one another under the group $\mathfrak{G}$ and are thus birationally equivalent. They cut on the generic V_q an involution i_d, where the number $d = [V_q V_{p-q}]$ is called the determinant of W_p. This involutions is without coincidences, from which it follows incidentally that $\{V_{p-q}\}$ is necessarily an Abelian congruence of a restricted type. Again, while the generic trajectory V_q is irreducible, there exist in general reducible members of $\{V_q\}$ of the form $sV_{q,s} (\equiv V_q)$ where $V_{q,s}$ is itself a Picard variety; here the number s may *a priori* be any divisor of d, including d. The varieties $V_{q,s}$ generate a certain number of irreducible manifolds whose dimensions may vary from q to $p-1$.

In the case where $d = 1$, we may evidently map W_p on the product $V_q \times V_{p-q}$; to obtain a representation in the case where $d > 1$, we first construct the variety $W_p^* = \overline{V}_q^* \times \overline{V}_{p-q}^*$, where $\overline{V}_q^*$ and $\overline{V}_{p-q}^*$ are birationally equivalent to $\{V_{p-q}\}$ and $\{V_q\}$ respectively; then, making correspond the generic point of W_p^* to the set $(V_q V_{p-q})$ we have a mapping of W_p on the d-ple variety W_p^*. From this representation we deduce that the superficial irregularity q_2 of W_p satisfies the inequality

$q_2 \geq q_2' + q_2''$, where q_2' and q_2'' denote respectively the irregularities of $\{V_q\}$ and $\{V_{p-q}\}$.

It remains only to recall the nature of the branch variety of the representation. Now the sets $(V_q V_{p-p})$ define an involution I_d on W_p whose coincidence locus is generated by the varieties $V_{q,s}$, each counted $(s-1)$ times; then, corresponding to $V_{q,s}$, we have a variety $V^*_{q,s}$ of the congruence which maps $\{V_p\}$, constituting an $(s-1)$-ple element of the branch locus. Such varieties generate a number of irreducible manifolds which may have any dimension from q to $p-1$ inclusive; and any component of the branch locus with dimension less than $p-1$ will be fundamental in the correspondence between W_p and W^*.

4. The improperly Abelian varieties. - Suppose now that the variety W_p has some plurigenus greater than zero, and that it maps a simple involution I_n of superficial irregularity $q(0 < q < p)$ on V_p; then the congruence $\{\overline{V}_q\}$ on V_p is mapped, simply or multiply, on the congruence $\{V_q\}$ and, in the latter case, there will possibly be $(s-1)$-ple branch elements $V_{q,s}$. Hence $\{V_q\}$ is an Abelian congruence. Similarly, the congruence $\{\overline{V}_{p-q}\}$ is mapped, simply or multiply, on $\{V_{p-q}\}$, although here the representation is without branch elements (§ 3); thus, as already remarked, $\{V_{p-q}\}$ is an Abelian congruence. It follows that the coordinates of the generic point P^* of the variety W_p^* of § 3 are expressible as rational functions of the coordinates of two points, lying on Abelian varieties of genera q and $p-q$ respectively; hence the coördinates of the generic point P of W_p, which is mapped on the d-ple variety W_p^*, are expressible as algebraic functions of the coordinates of P^*. Thus

IX. *Every Abelian variety W_p of superficial irregularity $q(0 < q < p)$ and with some plurigenus greater than zero is representable parametrically by means of algebraic functions of Abelian functions of genus q and other Abelian functions of genus $p-q$.*

It is by virtue of this result that we may call W_p an improperly Abelian variety. In many cases, one or both of the congruences $\{V_q\}$ and $\{V_{p-q}\}$ are themselves improperly Abelian, and then the genera of the Abelian functions required for the parametric representation can be lowered further. The precise form of the representation will evidently depend on that of the mapping of W_p on W_p^*; the problem of determining this form has so far been solved completely only for $p=1$.

5. The classification of improperly Abelian varieties. It will now become clear that the determination of all possible species of variety W_p is a recursive process depending on the classification of the Abelian varieties $W_r (r < p)$ representing involutions which are generable by finite groups of automorphisms. In the case $p=2$, the solution of this problem is classical (**2**); for $p=3$, the procedure to be followed has been outlined in (**10**): it must be added that our present knowledge of the theory of surfaces does not suffice for a completely geometrical solution in this case.

As an illustration we consider the particularly interesting case where I_n is free from coincidences. We observe first that in this case W_p *must be superficially irregular*; for the equations (3) always admit solutions and, if W_p were superficially regular, the corresponding set of equations (2) would be absent. Next we remark that W_p *and all its submanifolds are free from exceptional varieties*; for any such variety contains rational curves and, since the correspondence between V_p and W_p is without branch points, any rational curve on W_p would arise from some rational curve on V_p: whereas, as is well known, such curves do not exist.

Suppose, in the first place, that $\overline{V}_q$ and $\overline{V}_{p-q}$ both have general moduli, so that $\{\overline{V}_q\}$ and $\{\overline{V}_{p-q}\}$ are general Picard congruences; then $\{V_{p-q}\}$ is a Picard congruence of Picard varieties. For $\{\overline{V}_{p-q}\}$ is mapped on $\{V_{p-q}\}$ without branch points, and also $\overline{V}_{p-q}$ is mapped on V_{p-q} without branch points.

Again, since $\{\overline{V}_p\}$ is general, the congruence $\{V_p\}$ is uniquely specified: thus, when $p-q=1$, it is a rational pencil; when $p-q=2$, it is a Kummer surface or its generalization; and, for $p-q>2$, it is mapped by a Wirtinger manifold (**16**), or its generalization, according as $\{V_q\}$, regarded as a Picard variety U_{p-q}, has divisors unity or at least one divisor greater than unity. With an obvious extension of the terminology we may say that $\{V_q\}$ is a generalized Wirtinger congruence. In any case it follows from the general theory that the number of varieties $V_{q,s}$ is finite.

If, in the second place, we allow either or both of $\overline{V}_q$ and $\overline{V}_{p-q}$ to have particular moduli, a great number of possibilities at once present themselves. Thus, supposing that $\overline{V}_{p-q}$ has particular moduli, all that can be asserted about V_{p-q} is that it maps $\overline{V}_{p-q}$ in a correspondence free from coincidences; hence, by the above opening remark, V_{p-q} is a superficially irregular Abelian variety. Again, if the congruence $\{\overline{V}_{p-q}\}$ has particular moduli, $\{V_{p-q}\}$ may be improperly Abelian of a type which maps an involution without coincidences. And $\{V_q\}$ may *a priori* be any Abelian congruence with ∞^i branch elements $V_{q,s}$ $(0 \leq i \leq p-q-1)$; if, in particular $i=p-q-1$, the congruence is of anticanonical type (§ 1).

The fact that involutions of anticanonical type will actually have to be considered is revealed by the case $p=3$, $q=1$ (**10**); here we may have a W_3 which contains an elliptic pencil $\{V_2\}$ of Picard surfaces, and a rational congruence $\{V_1\}$ of elliptic curves which possesses ∞^1 curves $V_{1,s}$, and cuts on each V_2 a (rational) involution endowed with a coincidence curve. As stated in § 1, the involutions of this type have been classified by Scorza.

Continuing the previous discussion, we next observe that the congruence $\{V_q\}$ cuts on each V_{p-q} an involution of order d, having for $(s-1)$-ple coincidences the points where V_{p-q} meets each variety $V_{q,s}$. Now, by a property of pseudo-Abelian varieties (**7**), this involution is generable by a finite group $\mathcal{G}_d$ of automorphisms of V_{p-q}: thus, in a geometrical treatment of the problem, we require to determine all possible groups $\mathcal{G}_d$ and their coincidence loci. When V_{p-q} is a Picard variety, this problem will already have been solved in the recursive process; when, however, V_{p-q} is Abelian, it may be reduced to the consideration of compound involutions [2]) on a Picard variety U_{p-q}.

In conclusion, then, we see that, *in order to obtain, by the group-theoretic method, a complete classification of the types W_p, it is necessary to determine all the types W_r $(r < p)$, even those with plurigenera zero, representing involutions which are generable by finite groups of automorphisms of V_r.*

6. Para-Abelian varieties. Consider the variety W_p constructed by analogy with an improperly Abelian variety in the following manner. Instead of an Abelian congruence $\{V_q\}$ of trajectories, we suppose that W_p contains a congruence $\{V_q\}$, of arbitrary character, of Picard varieties V_q; and that, as before, the irreducible members of the congruence are non-singular and birationally equivalent. We suppose further that W_p contains a second congruence $\{V_{p-q}\}$ which is of Abelian type, and such that its irreducible members are non-singular and birationally equivalent [3]). As in § 3, we shall then have on W_p an involution I_d of sets $(V_q V_{p-q})$, where $d = [V_q V_{p-q}]$ is called the determinant of W_p.

We now assume that W_p may be mapped on the d-ple variety $W_p^* = \overline{V}_p^* \times \overline{V}_{p-q}^*$, where $\overline{V}_q^*$ and $\overline{V}_{p-q}$ are birationally equivalent to $\{V_{p-q}\}$ and $\{V_q\}$ respectively, in such a way that the branch locus is generated by varieties V_p^* and V_{p-q}^* corresponding respectively to members of $\{V_q\}$ and $\{V_{p-q}\}$ each counted a certain number of times; and we further assume that the correspondence between W_p and W_p^* possesses no exceptional features other than those which result from these hypotheses. A consequences of these assumptions is that both $\{V_q\}$ and $\{V_{p-q}\}$ will in general contain reducible va-

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2) In this connection we may note Andreotti's result (1) that a *compound* involution on a Picard variety, even if it has some plurigenus greater than zero, need not be generable by a finite group of automorphisms.

3) The variety V_{p-q} cannot be chosen arbitrarily, since it must contain the involution cut on it by $\{V_q\}$.

rieties $V_{q,s}$ and $V_{p-q,t}$, say, corresponding to generators of the branch locus, which are respectively $(s-1)$-ple and $(t-1)$-ple components of the coincidence locus of I_d.

We shall call the variety W_p so defined a *para-Abelian variety of type q*. It is clear that W_p does not admit the group of automorphisms which characterizes the pseudo-Abelian varieties; nevertheless it will appear that there are certain resemblances between the two classes of manifold, apart from their representations on multiple product varieties W_p^*.

To begin with, we observe that the *superficial irregularity* q_2 *of* W_p *satisfies the inequality* $q_2 \geq q'_2 + q_2''$, *where* q_2' and q_2'' denote the irregularities of the congruences on W^* which map $\{V_q\}$ and $\{V_{p-q}\}$ respectively; thus q_2' and q_2'' are the respective irregularities of $\{V_q\}$ and $\{V_{p-q}\}$. This result is stricly analogous to that of § 3.

Again, we remark that *every variety belonging to* $\{V_q\}$ *is para-Abelian (effective or virtual) of type q*. This follows at once from the definition.

Next, supposing (as usual), that W_q is free from exceptional varieties, we see that *the virtual canonical system* $|X_{p-1}|$ *of* W_p *belongs to* $\{V_q\}$*; also that* $|X_{p-1}|$ *contains as fixed* $(s-1)$*-ple component every hypersurface generated by varieties* $V_{q,s}$, *and passes* $(s-1)$*-ply through every manifold of lower dimension generated by those varieties.*

This proposition may be established by the method already adopted for the pseudo-Abelian varieties (**7**); it results from the fact that the variety V_q has a canonical hypersurface X_{q-1} of order zero.

It has been shown in (**7**) that canonical systems $\{X_k(W_p)\}$ $(k=0, 1, ..., p-1)$ of a pseudo-Abelian variety W_p all belong to the corresponding congruence $\{V_q\}$ or else have order zero. But the analogous property does not hold for para-Abelian varieties, as is already clear from an examination of the cases $p=2, 3$.

Another essential difference between the pseudo-Abelian and para-Abelian varieties is as follows. On the former variety the involution i_d cut by $\{V_q\}$ on the generic V_{p-1} is generable by a finite group $\mathcal{G}_d$ of automorphisms: on the latter this is not in general the case. We note, however, that there exists a subspecies of para-Abelian variety having the property that i_d is so generable and this type, from the point of view of the analytical representation, is the simplest to deal with; the discussion, in the cyclic case, of the analogous subspecies of parahyperelliptic threefold (**10**) may, with obvious modifications, be repeated here.

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