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On higher differences. Nota I

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ON HIGHER DIFFERENCES

Nota I (*)

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I. Definitions and Direct Theorems.

1. Notations.

(a) $x^{(n)} = x(x-1)(x-2)\dots(x-n+1)$;

$$x^{(-n)} = \frac{1}{(x+n-1)(x+n-2)\dots(x+1)x}\,;\; x^{(o)} = 1\,.$$

(b) $(ca^m)_n = (ca^m-1)(ca^{m-1}-1)\dots(ca^{m-n+1}-1)$,

where only the index of a, called the base, changes.

Evidently when $m=0$

$$(c)_n = (c-1)\left(\frac{c}{a}-1\right)\dots\left(\frac{c}{a^{n-1}}-1\right)$$

and when $c=1$

$$(a^m)_n = (a^m-1)(a^{m-1}-1)\dots(a^{m-n+1}-1)$$

so that

$$(a^m)_n = 0, \text{ if } m \text{ is a positive integer } \geq 0 \text{ and } < n.$$

We take $(ca^m)_0 = 1$.

(c) ${}^nS_p$ = sum of the products of n numbers $1, a, a^2, \dots, a^{n-1}$ taken p at a time. ${}^nS_0 = 1$, ${}^nS_p = 0$ if p is < 0 or $> n$.

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We write for brevity,

$$ {}^{n}O_{p}^{-} = {}^{n}S_{p}/{}^{p}S_{p}\,. $$

where evidently

$$ {}^{p}S_{p} = a^{\frac{1}{2}p(p-1)}. $$

(d) ${}^{n}R_{p}$ = sum of the products of n arbitrary numbers, $\omega_0, \omega_1, \omega_2, \ldots, \omega_{n-1}$ taken p at a time.

We take ${}^{n}R_0 = 1$, ${}^{n}R_p = 0$ if p is < 0 or $> n$.

2. Basic principles.

The basic principles of the subject of Higher Differences may be simply expressed by some theorems as given below:

(i). Theorem.

$$ {}^{n}S_{p} = \frac{(a^{n})_{n}}{(a^{p})_{p}(a^{n-p})_{n-p}}\,{}^{p}S_{p} \tag{1} $$

This was established in an earlier paper[1]).
[In particular, for $a \to 1$, we have ${}^{n}S_{p} = {}^{n}C_{p}$].

(ii). Theorem. If $w_0, w_1, w_2, \ldots, w_{n-1}$ be used as the successive multipliers, then the first element of the n^{th} order of differences from the series $u_0, u_1, u_2, \ldots,$ is

$$ \sum_{p=0}^{n} (-)^{p}\, u_{p}\, {}^{n}R_{p} \tag{2} $$

[If we take the series

$$ u_0\,, \quad u_1\,, \quad u_2\,, \ldots \tag{3} $$

and multiply each of its elements by w_0 and then substract the product from the immediately preceding element, then

1) Chakrabarti, S. C., *On the Evaluation of Some Factorable Continuants.* « Bul. Cal. Math. Soc. », XIV, (1923-24), 91-106.

we have another series, viz,

$$u_0 - {}^1R_1u_1 , \quad u_1 - {}^1R_1u_2 , \quad \ldots \tag{4}$$

which is called the first order of differences from the series (3). Similarly, by using w_1 as the multiplier, from the series (4), we have the series

$$u_0 - {}^2R_1u_1 + {}^2R_2u_2 , \quad u_1 - {}^2R_1u_2 + {}^2R_2u_3 , \ldots$$

which is called the second order of differences of the series (3). In like way, by using w_2, w_3 etc. as successive multipliers the successive orders of differences may be obtained from (3). Then it can be shewn by induction that the expression (2) is the first element of the n^{th} order of differences from the series (3).

(iiii). As a particular case of (2), we have

THEOREM. When 1, a, a^2, ..., a^{n-1} are the successive multipliers, the first element of the n^{th} order of differences from the series u_0, u_1, u_2, ..., is

$$\sum_{p=0}^{n} (-)^p u_p {}^nS_p \tag{5}$$

In the case of Finite Differences, $a = 1$ and (5) takes the well-known form

$$\sum_{p=0}^{n} (-)^p u_p {}^nC_p .$$

(iv). THEOREM

$$\sum_{p=0}^{n} (-)^p {}^nS_p = 0 . \tag{6}$$

This follows readily from (5), if the series 1, 1, 1, ..., be taken in the place of u_0, u_1, u_2, ...

3. Introduction.

In 1922-23 [1]), while working on the evaluation of some factorable continuants, I came accross the algebraic relationships given in § 2, which led me to develop the subject of Higher Differences. This subject is a generalization of that of Finite Differences. In Finite Differences[2]), we make use of the multipliers 1, 1, 1, ..., while in Higher Differences, as shewn in § 2, we employ 1, a, a^2, ... as multipliers or more generally still we use the multipliers, w_0, w_1, w_2, ... The object of this paper is to make a systematic study of the subject. I believe that the results obtained here are new.

4. The Operator A^r.

If u_x be a function of x, then we define the operator A^r by the relations:

$$A^r u_x = A^{r-1} u_{x+1} - a^{r-1} A^{r-1} u_x, \quad A^0 u_x = u_x .$$

In particular

$$A' u_x = u_{x+1} - u_x, \; A^2 u_x = A' u_{x+1} - a A' u_x = \sum_{p=0}^{2} (-)^p \, u_{x+2-p} \, {}^2S_p .$$

Thus in general, we have

THEOREM

$$A^n \; u_x = \sum_{p=0}^{n} (-)^p \, u_{x+n-p} \, {}^nS_p \tag{7}$$

This can be proved by induction.

We often write A for A'.

NOTE. (7) is a particular case of (5), u_{x+n}, u_{x+n-1} etc. being used in the place of u_0, u_1, etc.

COR.

$$\Delta^n \; u_x = \sum_{p=0}^{n} (-)^p \, u_{x+n-p} \, {}^nC_p, \quad \text{for } A^r = \Delta^r \text{ if } a = 1.$$

This is a well-known result of F. D.

5. The Operator A_r.

If u_x be a function of x, the operator A_r is defined by the relations:

$$A_r u_x = A_{r-1} u_{ax} - a^{r-1} A_{r-1} u_x, \quad A_0 u_x = u_x .$$

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2) BOOLE, G., *A treatise on the Calculus of Finite Differences*, 3rd Ed., 1931.

In particular

$$A_1 u_x = u_{ax} - u_x \quad , \quad A_2 u_x = \sum_{p=0}^{n} (-)^p \, u_{a^{2-p}x} \, {}^2S_p .$$

Thus generally, we have

THEOREM

$$A_n u_x = \sum_{p=0}^{n} (-)^p \, u_{a^{n-p}x} \, {}^nS_p \tag{8}$$

This, like (7), can also be proved by induction.

NOTE. (8) is a particular case of (5), $u_{a^n x}$, $u_{a^{n-1}x}$ etc. being used in the place of u_0, u_1, etc.

6. Operations with A^r.

(i). $A^r(u_x \pm v_x) = A^r u_x \pm A^r v_x .$ (9)

For

$$A^r(u_x \pm v_x) = \sum_{p=0}^{r} (-)^p (u_{x+r-p} \pm v_{x+r-p}) \, {}^rS_p , \qquad \text{by (7).}$$

(ii). $A^r c u_x = c A^r u_x$, c being a constant. (10)
This is easily proved.

(iii). $A^r a^{nx} = a^{\frac{1}{2} r(r-1)} (a^n)_r a^{nx}$ (11)

For

$$\begin{aligned} A^r a^{nx} &= A^{r-1} a^{n(x+1)} - a^{r-1} A^{r-1} a^{nx} \\ &= (a^n - a^{r-1}) A^{r-1} a^{nx} , \qquad \text{by (10)} \\ &= (a^n - a^{r-1})(a^n - a^{r-2}) \ldots (a^n - 1) a^{nx} \end{aligned}$$

In particular

$$A^n a^{nx} = a^{\frac{1}{2} n(n-1)} (a^n)_n a^{nx} \tag{12}$$

while

$$A^{n+k} a^{nx} = 0 \quad , \quad k = 1, 2, 3, \ldots . \tag{13}$$

COR. 1.

$$A^n (a^x + b)^n = a^{\frac{1}{2} n(n-1)} (a^n)_n a^{nx}$$

$$A^{n+k} (a^x + b)^n = 0 \quad , \quad k = 1, 2, 3, \ldots .$$

COR. 2.

$$A^n (a^x)_n = A^n a^{nx} / a^{\frac{1}{2} n(n-1)} .$$

COR. 3.

$$\Delta^n x^n = n!$$

For

$$\Delta^n x^n = \underset{a \to 1}{\mathrm{Lt}}\ A^n \left(\frac{a^x - 1}{a - 1}\right)^n = \underset{a \to 1}{\mathrm{Lt}} \frac{A^n a^{nx}}{(a-1)^n} = n!$$

(iv). $A^m (a^x)_n = (a^n)_m (a^x)_{n-m} a^{m(x-n+m)}$, $n > m$

7. Operations with A_r.

(i). In (i) and (ii), § 6, A^r can be replaced by A_r and the results hold good.

(ii). As an analogue of (iii), § 6, we have

$$A_r x^n = a^{\frac{1}{2} r(r-1)} (a^n)_r\, x^n . \tag{14}$$

For

$$\begin{aligned} A_r x^n &= A_{r-1} (ax)^n - a^{r-1} A_{r-1} x^n \\ &= (a^n - a^{r-1}) A_{r-1} x^n \end{aligned}$$

In particular,

$$A_n x^n = a^{\frac{1}{2} n(n-1)} (a^n)_n\, x^n \tag{15}$$

$$A_{n+k}\, x^n = 0 \quad , \quad k = 1, 2, 3, \dots . \tag{16}$$

COR. 1.

$$A_n x^{(n)} = A_n x^n .$$

COR. 2.

$$A_n (x)_n = (a^n)_n\, x^n .$$

(iii). $A_m (x)_n = \dfrac{(a^n)_m}{a^{m(n-m)}} (x)_{n-m}\, x^m$, $n \geq m$. (17)

8. Some more results.

(i). If $u_x = \dfrac{1}{(a^{x+m-1})_m}$

Then

$$A^1 u_x = \frac{1}{(a^{x+m})_m} - \frac{1}{(a^{x+m-1})_m} = \frac{(a^{-m}-1)a^{x+m}}{(a^{x+m})_{m+1}}$$

$$A^2 u_x = \frac{(a^{-m})_2\, a^{2(x+m+1)}}{(a^{x+m+1})_{m+2}}$$

and in general

$$A^n u_x = \frac{(a^{-m})_n\, a^{n(x+m+n-1)}}{(a^{x+m+n-1})_{m+n}} \tag{18}$$

COR.

$$\Delta^n x^{(-m)} = (-)^n (m+n-1)^{(n)} x^{(-m-n)} .$$

[To deduce this result of F. D. from (18), take $u_x = \dfrac{(a-1)^m}{(a^{x+m-1})_m}$ and find $\underset{a \to 1}{\mathrm{Lt}}\, A^n u_x$].

(ii). If $u_x = \sum_{p=0}^{\infty} (-)^p \dfrac{(a^{x+p-1})_p}{(a^p)_p}$ [3]

then

$$A^n u_x = (-)^n a^{n(x+n-1)} u_{x+n} \tag{19}$$

(iii). If $u_x = \dfrac{1}{(a^{m-1}x)_m}$

then

$$A_n u_x = (-)^n \frac{(a^{m+n-1})_n\, a^{\frac{1}{2}n(n-1)} x^n}{(a^{m+n-1}x)_{m+n}} \tag{20}$$

(iv). If $u_x = \sum_{p=0}^{\infty} (-)^p \dfrac{(a^{p-1}x)_p}{(a^p)_p}$

then

$$A_n u_x = (-)^n\; a^{n(n-1)} x^n\; u_{a^n x} \tag{21}$$

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3) The conditions for convergency of infinite series being easily obtainable, are not given in this paper.

9. An analogue of Maclaurin's Theorem.

THEOREM. (22)

If $\varphi(x)$ be any rational and integral function of x of degree n in x, then

$$\varphi(x) = \sum_{p=0}^{n} \frac{A_p\varphi(1)}{(a^p)_p}(x)_p, \qquad [A_0\varphi(1) = \varphi(1)]$$

where $A_p\,\varphi(1)$ stands for the value of $A_p\varphi(x)$ when $x = 1$.

PROOF. Assume

$$\varphi(x) = \lambda_0 + \lambda_1(x)_1 + \lambda_2(x)_2 + \dots + \lambda_n(x)_n. \tag{23}$$

Then on both sides of (23), operate with $A_1, A_2, \dots, A_n$ and then put $x = 1$ in (23) and each of the relations obtained by operating. Then we have

$$(a^p)_p\lambda_p = A_p\varphi(1), \; p = 0, 1, 2, \dots, n.$$

This proves (22).

If $\psi(x)$ be any rational and integral function of a^x of degree n in a^x, then similarly as (22) by operating with $A^1, A^2, \dots, A^n$, it can be shewn that

$$\psi(x) = \sum_{p=0}^{n} \frac{A^p\,\psi(0)}{(a^p)_p}(a^x)_p \tag{24}$$

where $A^p\;\psi(0)$ stands for the value of $A^p\;\psi(x)$ when $x = 0$.

NOTE. In fact, if $\psi(x)$ be the function obtained by putting a^x for x in a rational and integral function $\varphi(x)$ of degree n in x, then

$$A_p\varphi(1) = A^p\;\psi(0).$$

[The results (22) and (24) are similar to the well-known theorem of F. D., viz,

$$\varphi(x) = \sum_{p=0}^{n} \frac{\Delta^p\varphi(0)}{p\,!}\,x^{(p)}$$

where $\Delta^p\varphi(0)$ stands for the value of $\Delta^p\varphi(x)$ when $x = 0$].

Similarly as (22), we can also have

THEOREM

$$\varphi(x) = \sum_{p=0}^{n} \frac{A_p\varphi(h)}{(a^p)_p}\left(\frac{x}{h}\right)_p \tag{25}$$

where $A_p\varphi(h)$ stands for the value of $A_p\varphi(x)$ when $x = h$.

10. THEOREM.

$$u_{x+n} = (\sum_{p=0}^{n} {}^nO_p^- A^p)u_x \tag{26}$$

This can be proved by induction as follows:

$$u_{x+n+1} = A^{n+1}u_x + \sum_{p=0}^{n} (-)^p\, {}^{n+1}S_{p+1}u_{x+n-p}, \quad \text{by (7)}$$

$$= A^{n+1}u_x + \sum_{p=0}^{n} (-)^p\, {}^{n+1}S_{p+1}(\sum_{t=0}^{n-p} {}^{n-p}O_t^- A^t)u_x$$

$$= A^{n+1}u_x + \sum_{p=0}^{n} (-)^p(\sum_{t=0}^{n-p} {}^{n+1}S_{p+1}{}^{n-p}O_t^- A^t)u_x$$

$$= [A^{n+1} + \sum_{p=0}^{n} (-)^p(\sum_{t=0}^{n-p} {}^{n-t+1}S_{p+1}{}^{n+1}O_t^- A^t)]u_x, \text{ by (1).}$$

$$= [A^{n+1} + {}^{n+1}O_0^- \sum_{p=0}^{n} (-)^p\, {}^{n+1}S_{p+1}A^0 + {}^{n+1}O_1^- \sum_{p=0}^{n-1} (-)^p\, {}^nS_{p+1}A^1$$

$$+ {}^{n+1}O_2^- \sum_{p=0}^{n-2} (-)^p\, {}^{n-1}S_{p+1}A^2 + \dots + {}^{n+1}O_n^- \sum_{p=0}^{0} (-)^p\, {}^1S_{p+1}A^n]u_x$$

[Here the upper limits of the summations go on diminishing similarly as those of S, for ${}^nS_r = 0$ if $r > n$].

$$\therefore\ u_{x+n+1} = (\sum_{p=0}^{n+1} {}^{n+1}O_p^- A^p)u_x, \text{ for } \sum_{p=0}^{k-1} (-)^p\, {}^kS_{1+p} = 1, \text{ by (6)}$$

COR.

$$u_{x+n} = \{ \sum_{p=0}^{n} {}^nC_p\Delta^p \}\, u_x$$

This theorem of F. D., follows from (26) if $a \to 1$.

As an analogue of (26), we have

THEOREM. (27)

$$u_{a^n x} = (\sum_{p=0}^{n} {}^nO_p^- A_p) u_x$$

The proof is similar to that of (26).

11. In addition to the operators A^r and A_r, we now introduce two other operators F^r and F_r which stand respectively for $\sum_{p=0}^{r} {}^rO_p^- A^p$ and $\sum_{p=0}^{r} {}^rO_p^- A_p$, so that

$$u_{x+r} = F^r u_x \quad \text{and} \quad u_{a^r x} = F_r u_x \qquad [F^r C = C, \text{ a constant}].$$

(i). The F's obey the ordinary laws of indices and are analogues of E. Thus

$$F^2 \cdot F^3 u_x = F^5 u_x \quad \text{and} \quad F_2 \cdot F_3 u_x = F_5 u_x \quad \text{but} \quad A^2 \cdot A^3 \not\equiv A^5 u_x.$$

(ii). In terms of F

$$A^r u_x = (F' - 1)(F' - a)(F' - a^2) \ldots (F' - a^{r-1}) u_x$$

and

$$A_r u_x = (F_1 - 1)(F_1 - a)(F_1 - a^2) \ldots (F_1 - a^{r-1}) u_x.$$

(iii). Operations with F's are easy to perform. Thus

$$(F_1 - 1)^3 u_x = (F_3 - 3F_2 + 3F_1 - 1) u_x = u_{a^3 x} - 3u_{a^2 x} + 3u_{ax} - u_x$$

$$(F^1 + 1)^3 x = (F^3 + 3F^2 + 3F^1 + 1) x$$

$$= x + 3 + 3(x + 2) + 3(x + 1) + x = 8x + 12.$$

12. THEOREMS. According as n is odd or even

(i). $\sum_{p=0}^{n} (-)^p \, {}^nO_p^- = 0$ or $(-)^{\frac{n}{2}} (a^{n-1} - 1)(a^{n-3} - 1) \ldots (a - 1)$ (28)

and

(ii). $\sum_{p=0}^{n} (-)^p a^{n-p} \, {}^nO_p^- = (-)^k (a^n - 1)(a^{n-2} - 1) \ldots (a - 1)$ (29)

or $(-)^k (a^{n-1} - 1)(a^{n-3} - 1) \ldots (a - 1)$

where k is the integral part of $\frac{n}{2}$.

PROOF. Denote the left-sides of (i) and (ii) by T_n and Q_n respectively, then

$$T_{n+1} = T_n - Q_n, \quad \text{for} \quad {}^nO_r^- + {}^nO_{r-1}^- a^{n-r+1} = {}^{n+1}O_r^-$$

and

$$Q_{n+1} = a^{n+1} T_n - Q_n, \quad \text{for} \quad a^r \, {}^nO_r^- + {}^nO_{r-1}^- = {}^{n+1}O_r^-$$

These two relations also hold if the right-sides of (i) and (ii) be denoted by T_n and Q_n respectively.

Hence both (i) and (ii) are proved by induction.

13. The operations with A_1, A_2 etc. are somewhat analogous to differentiations.

(i). (a) If $u_x = x^n$, $A_m u_x = (a^n)_m {}^mS_m x^n$, $n > m$

$A_n u_x = (a^n)_n {}^nS_n x^n$; and $A_{n+k} u_x = 0$, $k = 1, 2, \ldots$

(b) If we define

$$e_x = 1 + \frac{x}{(a^1)_1} + \frac{x^2}{(a^2)_2} + \frac{x^3}{(a^3)_3} + \cdots$$

then

$$A_n e_x = {}^nS_n x^n \, e_x .$$

NOTE. It is easy to show by (28), that

$$e_x e_{-x} = 1 - \frac{x^2}{a^2-1} + \frac{x^4}{(a^4-1)(a^2-1)} - \frac{x^6}{(a^6-1)(a^4-1)(a^2-1)} + \cdots \tag{30}$$

and

$$e_{ix} e_{-ix} = 1 + \frac{x^2}{a^2-1} + \frac{x^4}{(a^4-1)(a^2-1)} + \frac{x^6}{(a^6-1)(a^4-1)(a^2-1)} + \cdots \tag{31}$$

(c) If we define

$$e_x^1 = 1 + \frac{{}^1S_1 x}{(a^1)_1} + \frac{{}^2S_2 x^2}{(a^2)_2} + \frac{{}^3S_3 x^3}{(a^3)_3} + \cdots$$

then

$$A_n e^1_x = a^{n(n-)} x^n e^1_{a^n x}$$

$$[e_x e^1_{-x} = e_{-x} e_x^1 = 1, \text{ by (6)}]$$

(ii). Let us define

$$Cs\, x = \frac{e_{ix} + e_{-ix}}{2} = 1 - \frac{x^2}{(a^2)_2} + \frac{x^4}{(a^4)_4} - \frac{x^6}{(a^6)_6} + \dots$$

$$\text{and } Sn\, x = \frac{e_{ix} - e_{-ix}}{2i} = \frac{x^1}{(a^1)_1} - \frac{x^3}{(a^3)_3} + \frac{x^5}{(a^5)_5} - \dots$$

then

(a) $A_n\, Cs\, x = (-)^k\, {}^nS_n\, x^n\, Sn\, x$ or $(-)^k\, {}^nS_n\, x^n\, Cs\, x$

according as n is odd or even, k being the integral part of $\frac{n+1}{2}$.

(b) $A_n\, Sn\, x = (-)^k\, {}^nS_n\, x^n\, Cs\, x$ or $(-)^k\, {}^nS_n\, x^n\, Sn\, x$

according as n is odd or even, k being the integral part of $\frac{n}{2}$.

(c) If $Sc\, x = \frac{1}{Cs\, x}$ then $A_1 Sc\, x = x\, Tn\, x \cdot Sc\,(ax)$

$$\text{where } Tn\, x = \frac{Sn\, x}{Cs\, x}.$$

(d) If $Co\text{-}Sc\, x = \frac{1}{Sn\, x}$,

$$\text{then } A_1\, Co\text{-}Sc\, x = -x\, Ct\, x\, Co\text{-}Sc\,(ax)$$

$$\text{where } Ct\, x = \frac{Cs\, x}{Sn\, x}.$$

(e) $A_1\, Tn\, x = x\, e_{ix}\, e_{-ix}\, Sc\,(ax) \cdot Sc\, x$

[For, the numerator of $\dfrac{Sn\,(ax)\, Cs\, x - Sn\, x\, Cs\,(ax)}{Cs\, x\, Cs\,(ax)}$

$$= x - x^3 \{ \sum_{p=0}^{3} (-)^p a^{3-p}\, {}^3O_p^- \}/(a^3)_3 + x^5 \{ \sum_{p=0}^{5} (-)^p a^{5-p}\, {}^5O_p^- \}/(a^5)_5 - \dots$$

$$= x\left\{1 + \frac{x^2}{a^2-1} + \frac{x^4}{(a^4-1)(a^2-1)} + \ldots\right\}, \text{ by (29)}$$

$= xe_{ix}e_{-ix}$, by (31)].

(f) $A_1 Ct\, x = -\, xe_{ix}e_{-ix}\, Co\text{-}Sc\,(ax)\, Co\text{-}Sc\, x\,.$

(g) If the multipliers be 1, a^2, a^4, a^6 etc., then

$$A_n e_x e_{-x} = (-)^n a^{n(n-1)} x^{2n}\, e_x\, e_{-x}$$

$$\text{and } A_n e_{ix} e_{-ix} = a^{n(n-1)} x^{2n} e_{ix}\, e_{-ix}$$

(h) If we define

$$\text{Lg }(x+1) = \frac{x}{a-1} - \frac{x^2}{a^2-1} + \frac{x^3}{a^3-1} - \ldots$$

then A_1 Lg $(x+1) = x - x^2 + x^3 - \ldots = \dfrac{x}{x+1}$,

and A_n Lg $(x + \text{I})$

$$= (-)^{n-1}(a^{n-1})_{n-1}{}^n S_n x^n / \{(a^{n-1}x+1)(a^{n-2}x+1) \ldots (x+1).$$

(k) Corresponding to the series

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \ldots$$

we may define

$$\lg x = \frac{x-1}{a-1} - \frac{(ax)_2}{a^2-1} + \frac{(a^2x)_3}{a^3-1} - \ldots$$

then

$$A_1 \lg x = x \sum_{p=0}^{\infty} (-)^p (a^p x)_p$$

and $A_n \lg x = (-)^{n-1} x^n a^{n(n-1)} \sum\limits_{p=0}^{\infty} (-)^p (a^{n-1+p})_{n-1} (a^{n-1+p}x)_p\,.$

NOTE. If we put a^x for x in each of the above results and operate with A^1, A^2, A^3 etc., we get an analogous result in each case.

14. A few results direct from the above theorems:

(i) $(1+A_1+A_2+\ldots)e_x=(1+{}^1S_1x+{}^2S_2x^2+\ldots)e_x$

(ii) $e_1e_A=e_F$ [For, $L\cdot S=1+\dfrac{1+A}{(a^1)_1}+\dfrac{1+{}^2O_1^-A+A^2}{(a^2)_2}+\ldots$]

(iii) $$u_0+\frac{x}{(a^1)_1}u_1+\frac{x^2}{(a^2)_2}u_2+\ldots=e_x\left\{u_0+\frac{x}{(a^1)_1}Au_0+\frac{x^2}{(a^2)_2}A^2u_0+\ldots\right\}.$$

[For,

$$e_x\left\{u_x+\frac{x}{(a^1)_1}Au_x+\frac{x^2}{(a^2)_2}A^2u_x+\ldots\right\}$$

$$=u_x+\frac{x}{(a^1)_1}(1+A)u_x+\frac{x^2}{(a^2)_2}(1+{}^2O_1^-A+A^2)u_x+\ldots$$

$$=u_x+\frac{x}{(a^1)_1}Fu_x+\frac{x^2}{(a^2)_2}F^2u_x+\ldots$$

$$=u_x+\frac{x}{(a^1)_1}u_{x+1}+\frac{x^2}{(a^2)_2}u_{x+2}+\ldots$$

Now take u_0 in the place of u_x].

(iv) $$e_t+\frac{e_{at}}{(a^1)_1}+\frac{e_{a^2t}}{(a^2)_2}+\ldots=e_1\left\{e_A\cdot 1+\frac{e_A\,a^{1\cdot 0}}{(a^1)_1}t+\frac{e_A\,a^{2\cdot 0}}{(a^2)_2}t^2+\ldots\right\}$$

where $e_1e_A\,a^{no}$ stands for the value of $e_1e_A\,a^{nx}$ when $x=0$.

[Here operate with e_1e_A (ie e_F) on $\{1+\dfrac{a^x}{(a^1)_1}t+\dfrac{a^{2x}}{(a^2)_2}t^2+\ldots$ and then put $x=0$ in the result $\}$.

(v) $$\sum_{p=0}^{n-1}(-)^p\frac{(a^p)_p}{(a^{r+p+1})_{p+2}a^{p(n-2)}}\,{}^{n-1}S_p=\frac{a^{n-1}}{(a^x-1)(a^{x+n}-1)}$$

[Let ${}^nS_p^1$ denote the sum of the products of n factors

$1, \frac{1}{a}, \frac{1}{a^2}, \dots \frac{1}{a^{n-1}}$ taken p at a time and replace ${}^{n-1}S_p/a^{p(n-2)}$ by its equivalent expression ${}^{n-1}S^1_p$. Then apply Th. 2, by using the multipliers $1, \frac{1}{a}, \frac{1}{a^2}, \dots$, etc. in the place of w_0, w_1, w_2, ..., etc. respectively].

(vi). If $A^n(a^0-1)^m$ stands for the value of $A^n(a^x-1)^m$ when $x=0$, then $A^n(a^0-1)^m$, $A^n(a^0-1)^{m+1}$, ..., form a recurring series whose scale of relation is

$$\sum_{p=0}^{n} (-)^p \lambda_{n-p} A^n(a^0-1)^{m+p} \equiv 0$$

where $\lambda_r =$ sum of the products of n factors $a-1$, a^2-1, a^3-1, ..., a^n-1 taken r at a time $[\lambda_0 = 1]$.