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NOTE ON THE CIRCULAR CUBIC AND BI-CIRCULAR QUARTICS WITH FOUR ASSIGNED CYCLIC POINTS

Nota II (*) *di* HARIDAS BAGCHI *e di* BISWARUP MUKHERJI

(*a Calcutta*).

Introduction. - This short paper consists of two sections, dealing respectively with:

(a) the (bicursal) circular cubic, uniquely defined by four assigned «cognate» (1) cyclic points,

and (b) the family of ∞^1 (bicursal) bicircular quartics through four assigned «cognate» (1) cyclic points.

Sec I discusses, among other things, a geometrical construction for a (bicursal) circular cubic,

(i) when four «cognate» cyclic points are known;

and (ii) when a «circle of inversion» and its related «focal parabola» are known.

Then Sec II disposes of the aggregate of ∞^1 (bicursal) bicircular quartics through four assigned «cognate» cyclic points, with special reference to:

(i) the locus of the remaining twelve cyclic points,

(ii) the locus of the sixteen ordinary foci,

(iii) the locus of the two double foci,

and (iv) the envelope of the eight bitangents.

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(1) A (bicursal) circular cubic or a (bicursal) bicircular quartic is known to have sixteen foci and sixteen cyclic points, situated, four by four, on the four circles of inversion. In the present context two or more cyclic points (or foci), *situated on the same circle of inversion*, have been spoken ef as mutually «cognate».

Finally there is an addendum, bearing on the *infinitude* of circular cubics through three given «cognate» cyclic points.

The paper is believed to embody some amount of original matter.

SECTION I.

(i) Circular cubic, defined by four «cognate» cyclic points.

(ii) Circular cubic, defined by a circle of inversion and the associated focal parabola.

1. - When nine points in a plane are assigned beforehand, a plane cubic Γ is, in general, uniquely defined and it admits of (Euclidean) construction *by points* after the manner of CHASLES (²), CLEBSCH (³), GRASSMANN (⁴) and others. Supposing two of the given points to move off to infinity (as a special case) we can readily deduce the geometrical construction of a circular cubic, when it is required to pass through *seven* assigned points.

A circular cubic or a bicircular quartic has 16 foci and 16 cyclic points, lying, 4 by 4, on the four circles of inversion. We shall speak of two or more cyclic points or foci as *cognate* when they lie on the same circle of inversion.

2. - Let it be required to determine a circular cubic Γ, having four *assigned* concyclic points for a tetrad of cognate cyclic points. Plainly, to be given a cyclic point on an arbitrary curve (which may or may not be a circular cubic) amounts to *two* conditions and so to be given four cognate cyclic points (situated on a circle) is tantamount to 7 $(= 4 \times 2 - 1)$ independent conditions. Hence four assigned cognate cyclic points $(\alpha, \beta, \gamma, \delta)$ will, in general, determine a circular cubic Γ and that *uniquely*.

(²) *Comptes Rendus*, Vol. 36 (1853).
(³) Math Ann. Bd. 5 (1872), P 425.
(⁴) Crelle Bd 31 (1846), P 111 and Bd 36 (1848). P 117.

Before we discuss the actual construction of this cubic Γ, it is worth while to mention the following lemmas (the truth of which is well-known):

(i) that the centre O of the circle Π, which contains the four cyclic points $(\alpha, \beta, \gamma, \delta)$ is one of the four centres of inversion of Γ and the three other centres of inversion are the three diagonal points A, B, C of the complete quadrangle $\alpha\beta\gamma\delta$; and (ii) that the four circles of inversion of Γ consist of Π and the three polar circles (Π_1, Π_2, Π_3) of the three triangles OBC, OCA, OAB.

When the four cyclic points $\alpha, \beta, \gamma, \delta$ are *pre-assigned,* four other points (on Γ), *viz.,* O, A, B, C are automatically defined, so that *any seven* of these $8\,(= 4 + 4)$ points will suffice to construct Γ *by points.*

It may also be mentioned that, when four cognate cyclic points $\alpha, \beta, \gamma, \delta$, — lying on the circle (of inversion) Π, — are given, the associated focal parabola Σ also lends itself to (geometrical) construction by points. For the tangents p, q, r, s, drawn to Π at $\alpha, \beta, \gamma, \delta$ being tangents to Σ (5) also, this latter conic can be easily constructed by points by an application of BRIANCHON' s Theorem.

3. Next let us proceed to construct a circular cubic Γ when one of its four circles of inversion (say, Π) and the associated focal parabola (say, Σ) are given beforehand. If $\alpha, \beta, \gamma, \delta$ denote the points of contact of Π with the four tangents which it has in common with Σ, we can readily infer that $\alpha, \beta, \gamma, \delta$ are the four cognate cyclic points of Γ that lie on Π. The four concyclic cyclic points $(\alpha, \beta, \gamma, \delta)$ being thus determined, the final geometrical construction can be framed on the lines, suggested in the previous article. It should, however, be carefully noted that, for the geometrical construction to be actually

(5) Refer to (i) H. HILTON: *Plane Algebraic Curves* (1920) Ex 15, P 309 and Ex 6, P 223.

(ii) H. BAGCHI: *Circles of double contact of a Bicircular Quartic, Bull. Cal. Math. Soc.* Vol XL, P 216.

feasible, it is imperative that all the four cyclic points on Π are *real*, a condition, which requires that Π and Σ should be *non intersecting* and that Π should lie on the *convex* side of Σ.

Section II.

(One-parameter family of bi-circular quartics, passing through four assigned cyclic points).

4. - As remarked in Art 2, to be given four cognate cyclic points on a circular cubic or on a bicircular quartic amounts to *seven* independent conditions. Now inasmuch as a bicircular quartic is determined by *eight* independent conditions, it follows that altogether a set of ∞^1 bicircular quartics can be described so as to have four assigned points α, β, γ, δ (lying on a circle Π) for a set of cognate cyclic points. In order to study the characteristic properties of this infinitude of bicircular quartics, we notice that these have the *same* four circles of inversion, which are none other than the four circles Π, Π_1, Π_2, Π_3, mentioned in § 2 and which depend solely upon the configuration of the four assigned cyclic points α, β, γ, δ.

Hence attending to the familiar lemmas:

(a that the four «circles of inversion» of a bicircular quartic (Ω) contain, four by four,

(i) all the sixteen cyclic points, and (ii) all the sixteen foci, and (b) that the eight bitangents of Ω consist of four pairs of lines, having the four «centres of inversion» for their respective points of intersection,

we deduce immediately that, for the ∞^1 bicircular quartics, having four given «cognate» cyclic points (α, β, γ, δ) in common,

(1) *the complete locus of the remaining twelve cyclic points consists of three (fixed) circles* Π_1, Π_2, Π_3,

(2) *the complete locus of the sixteen foci consists of four (fixed) circles, viz.* Π, Π_1, Π_2, Π_3,

and (3) *the envelope of the eight bitangents is a degenerate curve of class four, consisting merely of the four points* O, A, B, C.

The locus of the *double foci* of this system of bicircular quartics will be scrutinised in the succeeding article.

5. - We now propose to investigate the locus of the double foci of this infinitude of bicircular quartics. To that end we observe in the first place that if p, q, r, s be the tangents drawn to Π at α, β, γ, δ, the system of focal conics of the ∞^1 bicircular quartics can be characterised as a pencil of line-conics, having p, q, r, s for common tangents. Consequently the locus of the double foci of a set of ∞^1 bicircular quartics, having four cognate cyclic points in common, is precisely the same as that of the foci of a *specialised* variety of line-conics (*i.e.*, conics touching four fixed lines), the speciality consisting in that the pencil includes a circle (say, Π) as a member.

If now the Cartesian equation of this circle Π be represented in the form:

$$x^2 + y^2 = c^2 ,$$

the equations to p, q, r, s may be written as:

$$(1) \qquad \alpha_r \equiv x \cos \varepsilon_r + y \sin \varepsilon_r - c = 0 , \qquad (r = 1, 2, 3, 4).$$

For obvious reasons the four lines must conform to an *identical* relation, *viz*.

$$\sum_{r=1}^{4} A_r \, \alpha_r = 0.$$

In view of (1), this relation gives rise to:

$$(2), (3), (4) \quad \sum_{r=1}^{4} A_r = 0, \sum_{r=1}^{4} A_r \cos \varepsilon_r = 0 \quad \text{and} \quad \sum_{r=1}^{4} A_r \sin \varepsilon_r = 0 .$$

We may now rely upon SALMON [*Treatise on Conic Sections* (1879), P 275, Ex 15] to conclude that the locus (Ξ) of the foci of the pencil of line–conics, having α_1, α_2, α_3, α_4 for *common* lines, is given by:

$$(5) \qquad \frac{A_1}{\alpha_1} + \frac{A_2}{\alpha_2} + \frac{A_3}{\alpha_3} + \frac{A_4}{\alpha_4} = 0.$$

The very form of (5) goes to shew that Ξ is a cubic, passing through all the six vertices of the complete quadrilateral formed by α_1, α_2, α_3, α_4, the opposite vertices being *co-tangential*. To obtain other particulars about Ξ, we may multiply (5) by $\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4$, so as to clear it of fractions, and then the set of homogeneous terms of the third degree is easily seen to be:

$$= \sum A_1 (x \cos \varepsilon_2 + y \sin \varepsilon_2)\ (x \cos \varepsilon_3 + y \sin \varepsilon_3)\ (x \sin \varepsilon_4 + + y \sin \varepsilon_4).$$

As soon as we set $y = \pm i\, x$, this expression reduces to:

$$x^3\, e^{\pm i(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)}\ (A_1\, e^{+i\varepsilon_1} + A_2\, e^{+i\varepsilon_2} + A_3\, e^{\mp i\varepsilon_3} + + A_4\, e^{\mp i\varepsilon_4}),$$

which *is* $= 0$, *independently of* x, by (3) and (4).

This fact marks out Ξ as a *circular* cubic. Further remarking that, because of (2), the equation (5) is satisfied by $x = y = 0$, we infer that the circular cubic Ξ goes through the centre of Π.

That is to say, the locus of the foci of the pencil of conics, touching four given lines, circumscribed about a circle Π, is a circular cubic Ξ, which passes through all the six vertices of the complete quadrilateral formed by the four lines and goes through the centre of Π. Returning then to our former topic, we infer immediately that *the locus of the double foci of the one-parameter family of bicircular quartics, defined by* *four assigned cyclic points* α, β, γ, δ, *which lie on a circle* Π *with* 0 *as centre, is a circular cubic, passing through* 0 *and also through the six vertices of the complete quadrilateral, formed by the tangents to* Π *at* α, β, γ, δ.

6. - Recapitulating some of the results of the previous articles and retaining the notations and conventions adopted in Arts 2–5, we can now summarise our final conclusions in the form of a substantive proposition as follows:

For the system of ∞^1 *bi-circular quartics, defined by four assigned «cognate» cyclic points, seated on a circle* Π,

(i) *the four circles of inversion* Π, Π_1, Π_2, Π_3, *are the same for all*;

(ii) Π_1, Π_2, Π_3, *taken together, represent the complete locus of the remaining twelve cyclic points*;

(iii) *all the four circles of inversion, taken together, represent the complete locus of the ordinary foci (of the quartics)*;

(iv) *the locus of the double foci (of the quartics) is a* [illegible] *circular cubic* Ξ, *which passes through* 0 *and also through the six vertices of the complete quadrilateral, formed by the tangents to* Π *at* α, β, γ, δ;

(v) *the opposite vertices of the complete quadrilateral of* (iv) *are colangential points on* Ξ;

and (vi) *the eight bitangents of any of the quartics pass, two by two, through four fixed points, viz. the four centres of inversion* O, A, B, C.

Addendum

(System of ∞^1 circular cubics through three given "cognate" cyclic points)

Inasmuch as the pre-assignment of the positions of three «cognate» cyclic points (say, P, Q, R) amounts to 3×2 or 6 conditions, the totality of possible circular cubics is ∞^1. If Π

denotes the associated circle of inversion, *viz.* PQR, the tangents drawn to it at P, Q, R must all touch the focal parabola of *any one* of the infinitude of circular cubics. Hence remembering that the circum circle of a triangle, circumscribed about a parabola, goes through its focus, we arrive at the following proposition:

For the one-paramater family of circular cubics with three assigned «cognate» cyclic points, the locus of the double foci is a certain circle.