CLIQUE PARTITIONING OF INTERVAL GRAPHS
WITH SUBMODULAR COSTS ON THE CLIQUES

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Abstract. Given a graph $G = (V, E)$ and a “cost function” $f : 2^V \to \mathbb{R}$ (provided by an oracle), the problem [PCliqW] consists in finding a partition into cliques of $V(G)$ of minimum cost. Here, the cost of a partition is the sum of the costs of the cliques in the partition. We provide a polynomial time dynamic program for the case where $G$ is an interval graph and $f$ belongs to a subclass of submodular set functions, which we call “value-polymatroidal”. This provides a common solution for various generalizations of the coloring problem in co-interval graphs such as max-coloring, “Greene-Kleitman’s dual”, probabilist coloring and chromatic entropy. In the last two cases, this is the first polytime algorithm for co-interval graphs. In contrast, NP-hardness of related problems is discussed. We also describe an ILP formulation for [PCliqW] which gives a common polyhedral framework to express min-max relations such as $\chi = \alpha$ for perfect graphs and the polymatroid intersection theorem. This approach allows to provide a min-max formula for [PCliqW] if $G$ is the line-graph of a bipartite graph and $f$ is submodular. However, this approach fails to provide a min-max relation for [PCliqW] if $G$ is an interval graphs and $f$ is value-polymatroidal.

Keywords. Partition into cliques, Interval graphs, Circular arc graphs, Max-coloring, Probabilist coloring, Chromatic entropy, Partial $q$-coloring, Batch-scheduling, Submodular functions, Bipartite matchings, Split graphs.

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1. Introduction

Let $G = (V, E)$ be a simple graph. In the following, a clique of $G$ refers to a non-empty subset of vertices inducing a complete subgraph (not necessarily maximal with this property). Let $C(G)$ denote the set of cliques of $G$. A partition into cliques of $G$ is a partition $Q = (K_1, \ldots, K_k)$ of $V(G)$, where $K_1, \ldots, K_k \in C(G)$. In other words it is a coloring of $\overline{G}$, the complementary graph of $G$. Let $\mathcal{P}(G)$ denote the set of all partitions into cliques of $G$. A classical problem consists in determining $\chi(G)$, the minimum number of cliques necessary to partition $G$. In several applications however (see Sect. 3), there is a cost $f(C)$ associated to every clique $C \in C(G)$, and we are interested in partitioning $G$ into cliques, minimizing the sum of the costs of the cliques in the partition. Let $\chi(G, f)$ denote this minimum:

$$\chi(G, f) := \min_{Q \in \mathcal{P}(G)} \sum_{K \in Q} f(K).$$

In order to describe some properties of $f$, one may assume that $f$ is not only defined on cliques but is a set function on $V$, that is $f : 2^V \to \mathbb{R}$. This has no consequences for the definitions of $\chi(G, f)$ and [PCliqW] below. Notice that if $f(C) = 1$ for all cliques $C$, we get the classical problem of coloring $\overline{G}$ and we have $\chi(G, 1) = \chi(G)$. Determining $\chi(G, f)$ is therefore an NP-hard problem. Moreover, since $|C(G)|$ is usually exponential in $|V|$ (the complete graph $K_n$ on $n$ vertices has $|C(K_n)| = 2^n$), encoding $f$ itself raises complexity issues. In several applications however, both $G$ and $f$ have structural properties that allow to solve problem [PCliqW] in time polynomial in $|V|$.

[PCliqW] Partition into cliques with weights

**INPUT**: A graph $G = (V, E)$ and a value oracle, providing $f(K)$ in constant time for each $K \in C(G)$.

**OUTPUT**: A partition into cliques of cost $\chi(G, f)$.

[PCliqW] can also be described in terms of batch scheduling with compatibility graphs [12]. In this terminology (see [4] for batch scheduling problems not involving compatibility graphs and [16] for a classification of chromatic scheduling problems), each clique of a partition into cliques of $G$ is called a batch. The operating time of a batch $K$ is then $f(K)$ and our objective is to minimize the makespan $C_{\text{max}}$ (whence the batches are ordered arbitrarily on the batch machine). Talking about cliques and batches allows to distinguish easily between cliques of $G$ and cliques in a partition of $V(G)$. Two famous polytime cases of [PCliqW] are when

- $G$ is perfect and $f \equiv 1$ [17];
- $G$ is complete and $f$ is a submodular set function [17].

Our solution for [PCliqW] for interval graphs and value-polymatroidal functions can be seen as a compromise between these two classical cases. Moreover, [PCliqW] enjoys a simple min-max formula in both cases [17] ($\chi(G) = \alpha(G)$ in the first case and “Dilworth’s truncation” in the second). One could therefore expect a...
common generalized min-max formula to hold in other cases for which [P Cliq W] is polynomial. We deal with this issue in Section 7.

In Section 2, we define polymatroid rank functions and motivate the definition of value-polymatroidal set functions in the context of [P Cliq W]. In Section 3, we provide examples of value-polymatroidal set functions. In Section 4, we discuss value-polymatroidal functions whose values \( f(U) \) depend only on the size \(|U|\). In Section 5, we provide a dynamic program which solves [P Cliq W] for interval graphs in polytime if \( f \) is value-polymatroidal. The algorithm extends to the minimum cost partition problem for circular arc graphs, when we only consider cliques in which the arcs share a common point. As a counterpart, we mention NP-hardness of [P Cliq W] for interval graphs if \( f \) is only assumed to be polymatroidal \([2]\). In Section 6, we discuss NP-hardness of [P Cliq W] on split graphs for subclasses of value-polymatroidal set functions. In Section 7, we deal with some polyhedral issues and provide a min-max formula for [P Cliq W] in line-graphs of bipartite graphs.

2. Value-polymatroidal set functions

A set function \( f : P(V) \to \mathbb{R} \) is submodular if it satisfies one of the following equivalent properties \([17]\):

\[
\begin{align*}
    f(S \cup T) + f(S \cap T) &\leq f(S) + f(T) \quad \text{for all } S, T \subseteq V, \\
    f(S + u) + f(T) &\leq f(S + T) + f(T + u) \quad \text{for all } T \subseteq S \subseteq V \text{ and } u \in V \setminus S, \\
    f(S + u + v) + f(S) &\leq f(S + u) + f(S + v) \quad \text{for all } S \subseteq V \text{ and } u, v \in V \setminus S.
\end{align*}
\]

A set function \( f \) is non-negative if all its values are, non-decreasing if \( S \subseteq T \implies f(S) \leq f(T) \), subcardinal if \( f(U) \leq |U| \) for all \( U \subseteq V \). A polymatroid rank function is a submodular, non-negative, non-decreasing set function such that \( f(\emptyset) = 0 \). A matroid rank function is a subcardinal, integral polymatroid rank function.

In some graph classes, submodularity of \( f \) is enough to ensure polynomiality of [P Cliq W] (see Sect. 7 and \([16]\)). Although submodularity is not sufficient for interval graphs (see Th. 5.5), a stronger exchange property will do. We say that \( f \) is a value-polymatroidal set function if \( f(\emptyset) = 0 \), \( f \) is non-decreasing and for every \( S \) and \( T \) subsets of \( V \) such that \( f(S) \geq f(T) \) and every \( u \in V \setminus (T \cup S) \), we have

\[
f(S + u) + f(T) \leq f(S) + f(T + u).
\]

Proposition 2.1. Every value-polymatroidal set function is a polymatroid rank function.

Proof. Let \( f \) be value-polymatroidal. Since \( f \) is non-decreasing, we have \( f(S) \geq f(T) \) for every \( T \subseteq S \subseteq V \) and therefore \( f(S + u) + f(T) \leq f(S) + f(T + u) \) for every \( u \in V \setminus S \). \( \square \)
By a maximal clique, we mean a clique maximal for inclusion (not necessarily for cardinality). The main motivation behind the definition of value-polymatroidal set functions is given by the following proposition.

**Proposition 2.2.** For any graph $G$ and any value-polymatroidal set function $f$ on $V(G)$, there is a partition $Q$ of cost $\chi(G, f)$ in which one of the cliques in $Q$ is a maximal clique of $G$.

**Proof.** Let $Q$ be a minimum cost partition of $G$ and choose any clique $K \in Q$, such that $f(K) \geq f(T)$ for all $T \in Q$. If $K$ is not a maximal clique of $G$, there exists some $t \in V \setminus K$ such that $K + t$ is a clique in $G$. Now, $t$ belongs to some $T \in Q - K$. Since $f$ is non-decreasing, $f(K) \geq f(T) \geq f(T - t)$. Since $f$ is value-polymatroidal, $f(K + t) + f(T - t) \leq f(K) + f(T)$. Repeat the process until $K$ becomes a maximal clique of $G$. □

In general, rank functions of (poly)matroids are not value-polymatroidal, and the conclusion of Proposition 2.2 doesn’t hold as shown in Figure 1.

3. Examples of value-polymatroidal set functions

In this section we mention some (coloring) problems that have been studied in the literature, and that amount to solving $[\text{PCliqW}]$ for special subclasses of value-polymatroidal set functions. These problems are often formulated in terms of finding a minimum cost partition into stable sets, which is equivalent to $[\text{PCliqW}]$ by taking the complementary graph.
Maximum. Let $p : V \to \mathbb{R}_+$ and define
\[
f(U) := \max_{u \in U} p(u)
\]
for any $U \subseteq V$. Then $f$ is value-pomatroidal. Indeed, let $S, T \subseteq V$ with $f(S) \geq f(T)$, and let $u \in V \setminus (S \cup T)$. Then, since $p(s) = f(S) \geq f(T) = p(t)$ for some $s \in S$ and $t \in T$, we have
\[
f(S + u) + f(T) = \max\{p(s), p(u)\} + p(t) \leq p(s) + \max\{p(t), p(u)\} = f(S) + f(T + u).
\]
A set function arising as in (6) is called a max-batch cost function. When restricted to max-batch cost functions, the corresponding problem of finding a minimum cost partition into stable sets is called [max-coloring] and is strongly NP-hard for split graphs [3,8], for bipartite graphs [8] and for interval graphs [11]. However, [max-coloring] is polynomial for $P_4$-free graphs [8] as well as for co-interval graphs [2,9,12].

Independent probabilities. Let $q : V \to [0, 1]$ and for $U \subseteq V$, let
\[
f(U) := 1 - \prod_{u \in U} q(u).
\]
Let $S, T \subseteq V$ with $f(S) \geq f(T)$, and $u \in V \setminus (S \cup T)$. Write $f(S) = 1 - \sigma$ and $f(T) = 1 - \tau$ (so $\sigma \leq \tau$). Then
\[
f(S) + f(T + u) = (1 - \sigma) + (1 - q(u)\tau) \\
\geq (1 - q(u)\sigma) + (1 - \tau) = f(S + u) + f(T).
\]
Hence $f$ is value-pomatroidal. A set function arising as in (7) is a probabilistic cost function. Transitive references for applications of probabilist optimization can be found in [7].

When restricted to probabilistic cost functions, [PCliqW] is strongly NP-hard in split graphs [7]. The corresponding problem of partitioning into stable sets is called [probabilist coloring].

Chromatic Entropy. Let $p : V \to [0, 1]$ and for $U \subseteq V$, let
\[
c_U := \sum_{u \in U} p(u)
\]
\[
f'(U) := -c_U \log(c_U).
\]
If $c_V = 1$, $f'$ is a chromatic entropy cost function. Although $f'$ is not value-pomatroidal (it is not non-decreasing), the function $f := f' + c$ is value-pomatroidal as can be derived from the concavity of the function $x \mapsto x - x \log(x)$ [1]. Since for any partition $V = K_1 \cup \cdots \cup K_k$ of $V$ into cliques, we have $\sum_i f(K_i) = c(V) + \sum_i f'(K_i)$, the two functions $f'$ and $f$ yield the same optimal partitions.

The corresponding problem of partitioning into stable sets is called [chromatic entropy] [1,6] and is strongly NP-hard for interval graphs [6].
Uniform matroid and Partial $q$-coloring. Let $q \in \mathbb{N}$ and let
\[ f(U) := \min\{q, |U|\}. \]  

Then $f$ is value-polymatroidal, and the proof is left as an exercise since a more general statement is given with the next example. Functions arising this way are exactly the rank functions of uniform matroids. [PCliqW] with such a cost function arises in Greene-Kleitman's min-max relations stating that for any (co)-comparability graph $G$ and any integer $q$, the maximum cardinality $\alpha_q(G)$ of the union of $q$ stable sets of $G$ satisfies $\alpha_q(G) = \Upsilon(G, f)$ (see [5] and [17], Sects. 14.6 and 14.7 on unions of chains and antichains in posets and Sect. 66.5e on “$k$-perfect” graphs for more details and references).

Size-defined concave. Assume that $f(\emptyset) = 0$ and that
\[ f(U) := \psi(|U|) \]  

for some $\psi : \mathbb{N} \to \mathbb{R}_+$. Then $f$ is value-polymatroidal if and only if $f$ is the rank of a polymatroid and also if and only if $\psi$ has a non-decreasing concave extension on the real segment $[0, |V|]$ (see Sect. 4). The rank function of a uniform matroid is a special case.

4. Size-defined submodular set functions

In this section, we notice that if $f(U)$ only depends on $|U|$, then polymatroid ranks coincide with value-polymatroidal functions. Let $[a, b]$ denote the set of integers in the interval $[a, b]$. A set function $f$ on $V$ is size-defined if there exists a function $\psi : [0, |V|] \to \mathbb{R}$ such that $f(U) = \psi(|U|)$. The function $\psi$ is then the compact representation of $f$. Recall that a function $f : [a, b] \to \mathbb{R}$ is concave if for all $c, d \in [a, b]$ we have $f(c) + f(d) \leq 2f((c + d)/2)$

**Theorem 4.1.** Let $f$ be a size-defined, non-decreasing set function such that $f(\emptyset) = 0$ and $\psi$ be the compact representation of $f$. The following are equivalent:

(i) $f$ is value-polymatroidal
(ii) $f$ is a polymatroid rank function
(iii) $2\psi(i) \geq \psi(i - 1) + \psi(i + 1)$ for all $i \in [1, |V| - 1]$
(iv) $\psi(i + 1) - \psi(i) \geq \psi(j + 1) - \psi(j)$ for all $i, j \in [0, |V| - 1]$, with $i < j$
(v) $\exists \hat{\psi} : [0, |V|] \to \mathbb{R}$ concave such that $\psi(i) = \hat{\psi}(i)$ for $i \in [0, |V|]$

**Proof.** (i) $\implies$ (ii): Proposition 2.1
(ii) $\implies$ (iii): Use definition (4) of polymatroids with $|S| = i - 1$.
(iii) $\implies$ (iv): By induction on $j - i$. The case $j - i = 1$ being exactly iii).
Adding $\psi(i + 1) - \psi(i) \geq \psi(j + 1) - \psi(j)$ and $2\psi(j + 1) \geq \psi(j) + \psi(j + 2)$ gives $\psi(i + 1) - \psi(i) \geq \psi(j + 2) - \psi(j + 1)$.
(iv) $\implies$ (i): For $S, T \subseteq V$, since $f$ is size-defined and non-decreasing,
\[ f(S) \geq f(T) \iff \psi(|S|) \geq \psi(|T|) \iff |S| \geq |T| \]
Figure 2. Let $f$ be the probabilist cost defined by $p$. Vertex $d$ has maximum cost $f(\{d\}) = 1 - q(d) = 5/8$. However, in an optimal partition, vertex $d$ cannot be placed in a maximal clique since 
\[ 25/16 = f(\{a, b\}) + f(\{c, d\}) > \chi(G, f) = f(\{a, b, c\}) + f(\{d\}) \]
= 12/8.

Applying (iv) to $j = |S|$ and $i = |T|$ gives (i).

(v) $\implies$ (iii): Apply the concavity condition to $c = i - 1$ and $d = i + 1$.

(iii) $\implies$ (v): Take $\hat{\psi}$ as the piecewise linear interpolation of $f$ (for any $x \in [0..|V|]$, $\hat{\psi}(x) := \lambda f(\lfloor x \rfloor) + (1 - \lambda)f(\lceil x \rceil)$ for $\lambda := x - \lfloor x \rfloor$). One can check that the subgradient of $-\hat{\psi}$ is nondecreasing. 

\[ \square \]

5. Partition into cliques in interval and circular arc graphs

A graph $G = (V, E)$ is an interval graph [13,17] if there exists a set $\{\phi(v) \mid v \in V\}$ of closed intervals on the real line, such that two vertices $u$ and $v$ are adjacent in $G$ if and only if the two corresponding intervals $\phi(u)$ and $\phi(v)$ have nonempty intersection. Observe that any maximal clique $K$ in $G$ is of the form $\{v \in V \mid t \in \phi(v)\}$ for some endpoint $t$ of one of the intervals.

In [2,9,12], [PCliqW] is solved in polytime for interval graphs and max-batch cost functions. These algorithms use the fact that there exists an optimal solution in which a vertex of maximum cost is contained in a batch inducing a maximal clique. Based on this fact, a dynamic program is proposed. This fact is no longer true for value-polymatroidal costs as shown by the example in Figure 2. Nonetheless, based on Lemma 5.2, we describe a generalization of the algorithm proposed in [12], which provides an optimal solution for any value-polymatroidal cost function.
Theorem 5.1. For any interval graph $G = (V,E)$ and any value-polymatroidal set function $f$ on $V$ given by a value oracle, we can compute a partition into cliques of $G$ of cost $\overline{\chi}(G,f)$ in time $O(n^3)$.

Proof. Let $\{I_i = [a_i, b_i]\}_{i=1,\ldots,n}$ be a set of intervals on the real line representing graph $G$. We consider the set $X$ of endpoints of the intervals:

$$X = \{a_i\}_{i=1,\ldots,n} \cup \{b_i\}_{i=1,\ldots,n} = \{1, \ldots, q\}.$$ 

Let the subproblem $I(i,j)$ denote the set of all intervals completely contained in the closed interval $[i,j]$. For every pair of values $i \leq j \in X$, let $F(i,j) := \overline{\chi}(G[I(i,j)], f)$, be the optimum cost of a partition of the subgraph induced by $I(i,j)$ (by definition of $\overline{\chi}(G,f)$). $F(i,j) = 0$ if $I(i,j) = \emptyset$. Our Dynamic Programming approach is based on Lemma 5.2 below, which implies that we can separate the problem restricted to $I(i,j)$ into two subproblems.

Lemma 5.2. For every $i, j \in X$ there is an optimal partition into cliques of $G[I(i,j)]$ in which at least one batch induces a maximal clique of $G[I(i,j)]$.

Proof. Directly from Proposition 2.2

Given $i < z < j \in X$, let $K\hat{z}_{i,j}$ be the set of intervals of $I[i,j]$ containing point $z$. Notice that $K\hat{z}_{i,j}$ is a clique for all $i \leq z \leq j \in X$.

Lemma 5.3. For arbitrary fixed $i < j$ in $X$, the following recursion holds:

$$F(i,j) = \min_{z \in \mathbb{Z}[i,j]} \{f(K\hat{z}_{i,j}) + (F(i,z-1) + F(z+1,j))\}. \quad (12)$$

Proof. By Lemma 5.2, there is an optimal partition of $G[I(i,j)]$ in which a batch is a maximal clique $B^\ast$. All maximal cliques of $G[I(i,j)]$ are browsed while considering the minimum in (12). Hence $B^\ast = K\hat{z}_{i,j}$ for some $z^\ast$. Given such point $z^\ast$, every interval in $I[i,z^\ast - 1]$ has its terminal endpoint before the initial endpoint of every interval in $I[z^\ast + 1,j]$. Hence, the graph $G[I[i,j]\setminus B^\ast]$ decomposes into two disconnected subgraphs: $G[I[i,z^\ast - 1]]$ and $G[I[z^\ast + 1,j]]$. One can therefore solve the problems on these two subgraphs independently.

The Dynamic Programming algorithm starts from the initial conditions

$$F(i,i) = f(I[i,i]) \quad \text{for all } i = 1,\ldots,q.$$ 

Applying the recursion (12) with increasing subproblem width $x_j - x_i$, it computes an optimal schedule

$$S(x_i, x_j) = \begin{cases} \emptyset & \text{if } I[i,j] = \emptyset; \\ S(i, z^\ast - 1) \cup B^\ast \cup S(z^\ast + 1,j) & \text{otherwise}. \end{cases}$$

The optimum value is $\overline{\chi}(G,f) = F(1,q)$, and $S(1,q)$ is an optimal solution. Since there are $O(q^2) = O(n^2)$ subproblems and $O(q) = O(n)$ candidate values for $z$ in each subproblem, the resulting Dynamic Programming algorithm solves the problem in $O(n^3)$ time. This completes the proof of Theorem 5.1.
Theorem 5.1 and the associated algorithm can be extended in the following way. A graph $G = (V, E)$ is a circular arc graph [13] if there exists a set $\{\phi(v) \mid v \in V\}$ of closed arcs of the unit circle, such that two vertices $u$ and $v$ are adjacent in $G$ if and only if the two corresponding arcs $\phi(u)$ and $\phi(v)$ have nonempty intersection. Call a clique $K$ of $G$ a Helly clique if $\bigcap_{v \in K} \phi(v)$ is nonempty.

**Corollary 5.4.** For any circular arc graph $G$, and any value-polymatroidal function $f$ on $V(G)$ given by a value oracle, we can compute an optimum partition into Helly cliques in time $O(n^3)$.

**Proof.** Let $X$ be the set of endpoints of the arcs $\phi(v)$, (as in Theorem 5.1). For $i, j \in X$, let $I[i, j]$ be the set of arcs contained in the portion of the circle in clockwise order between $i$ and $j$. Note that after removing any maximal Helly clique, the remaining arcs are contained in some set $I[i, j]$. Compute all $O(n^2)$ values as in Theorem 5.1. Compute the best maximal Helly clique afterwards. $\square$

On the other hand, we have the following negative result:

**Theorem 5.5.** [2] $[PCliqW]$ is NP-hard even if $G$ is an interval graphs and $f$ is a polymatroid cost (even if $f$ is given by a rooted-TSP on a tree).

**Rooted-TSP on trees.** Let $T = (W, A)$ be a tree, $l : A \rightarrow \mathbb{N}$ and $r \in W$ be the root of $T$. For $U \subseteq W$, let $A(U)$ be the set of arcs spanning $U + r$ and $f(U) := 2\sum_{a \in A(U)} l(a)$. The function $f$ is called a rooted-TSP cost since it is the cost of visiting all nodes in $U \subseteq V$, moving along edges of $A$, starting and finishing the tour from node $r$ (see Fig. 3). Such a cost function can easily be shown to be polymatroidal\(^1\). Complementing Theorem 5.5, [2] gave a 2-approximation for $[PCliqW]$ when $G$ is an interval graphs and $f$ is rooted-TSP on a tree. This has applications in vehicle routing problems with time windows (where the length $l(a)$ represents a travel cost and we assume that the traveling times are negligible compared to the size of the time windows [9]).

### 6. Partition into cliques in split graphs

One may wonder if Proposition 2.2 could be applied in more general graphs than interval graphs. A property of interval graphs which is used to prove polynomiality in Theorem 5.1 is that they have a polynomial number of maximal cliques. In this section, we illustrate that this property is not sufficient to ensure polynmytime solvability of $[PCliqW]$ restricted to value-polymatroidal costs.

A graph $G = (V, E)$ is a split graph if $V$ can be partitioned into two sets $S$ and $K$ such that $S$ is a stable set and $K$ is a clique. Notice that split graphs have a polynomial number of maximal cliques (at most $|S| + 1$). However, [max-coloring] and [probabilist coloring] are (strongly) NP-hard in split graphs [3,8] and [7] respectively. Since the class of split graphs is self-complementary, $[PCliqW]$ is

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\(^1\)In fact, several characterizations of the graphs for which rooted TSP costs are polymatroidal for all edge length can be found in [15]. Based on [15], Jost [16] characterized these graphs as the graphs without $K_{2,3}$ minors.
Figure 3. A rooted tree with a length function \( l : A \to \mathbb{R} \). The cost associated with a subset \( U \subseteq V \) is twice the length of the arcs spanning \( U + r \). For example, \( f(\{a\}) = 4 \), \( f(\{a, b, f\}) = 12 \) and \( f(\{c, d, e, f\}) = 16 \).

also NP-hard if we restrict to maximum or probabilist cost functions. Moreover, Yannakakis and Gavril [18] proved that the maximum \( q \)-chromatic subgraph problem is NP-hard for split graph. Unsurprisingly then, Greene-Kleitman’s relation doesn’t hold for split graphs [5]. However, the “dual problem”, that is \([\text{PCliqW}]\) with \( f(U) := \min\{q, |U|\} \) is trivial. If \( q = 1 \) this is equivalent to find a partition of \( G \) into a minimum number of cliques. If \( q \geq 2 \), we may assume \( \omega(G) = |K| \) (in general, the bipartition \((S, K)\) of a split graph is not unique). Then the partition consisting of all elements of \( S \) alone and all vertices of \( K \) together in a unique class is optimal. This fact however, does not extend to size-defined cost functions.

**Theorem 6.1.** \([\text{PCliqW}]\) is strongly NP-hard even if we restrict \( G \) to be a split graph and \( f \) to be size-defined and value-polymatroidal.

*Proof.* We reduce the NP-complete problem [X3C] to \([\text{PCliqW}]\).

[X3C] **Exact three-set cover**

**INPUT :** A finite set \( X \) of size \( 3m \) and a set \( T \) of triples of \( X \).

**OUTPUT :** Does there exists a partition of \( X \) into \( m \) elements of \( T \)?

Given an instance of [X3C], build the split graph \( G = ((T, X), E) \) where \( G[T] \) is a stable set and \( G[X] \) a clique and \((t, x) \in E \) iff \( x \in t \). Let \( \psi(0) := 0 \), \( \psi(1) := \alpha = m + 1 \) and \( \psi(i) := \beta = m + 2 \) for all \( i \geq 2 \). We claim that there is a partition of cost not exceeding \( m\beta + (|T| - m)\alpha \) if and only if \( X \) has a partition into triples of \( T \). A partition into triples yields such a cost. Now, assume that \( X \) has no partition into
triples. Since $T$ induces a stable set, any partition of $V(G)$ into cliques contains at least $|T|$ classes. Those partitions which consist in exactly $|T|$ cliques, are of cost at least $(m+1)\beta + (|T| - (m+1))\alpha > m\beta + (|T| - m)\alpha$. Those consisting in at least $|T| + 1$ cliques are of cost at least $(|T| + 1)\alpha > m\beta + (|T| - m)\alpha$. □

7. ILP formulation and min-max formula for $\text{[PCliqW]}$

Seen as a partition problem, $\text{[PCliqW]}$ can be formulated as an integer linear program, with variables $y_C$ in $\mathbb{R}^{C(G)}$ (where $C(G)$ is the set of cliques of $G$):

(i) $\min f^T y$;  
(ii) $\sum_{C \ni v} y_C = 1$ for all $v \in V$;  
(iii) $y_C \in \{0, 1\}$ for all $C \in C(G)$.  

Clearly, if $f$ is non-negative, there is no advantage in taking $y_C > 1$. Therefore, $y_C \in \{0, 1\}$ can be replaced by $y_C \geq 0$ and $y_C \in \mathbb{Z}$. Also, if $f$ is non-decreasing, (13) (ii) can be replaced by $\sum_{C \ni v} y_C \geq 1$ (if $y_A = y_B = 1$, $A, B \in C(G)$ and $A \cap B \neq \emptyset$ then $B \setminus A$ is still a clique of $G$ and we can reset $y_B := 0$ and $y_B \setminus A := 1$).

If $f$ is non-negative and non-decreasing, the dual of the linear relaxation of (13) can therefore be written as maximizing $1^T x$ subject to:

(i) $\sum_{v \in C} x_v \leq f(C)$ for all $C \in C(G)$;  
(ii) $x_v \geq 0$ for all $v \in V(G)$.  

If $G$ is perfect and $f \equiv 1$, (14) is TDI. Also if $G$ is complete and $f$ is submodular, (14) is box-TDI. So in both cases, (14) yields a min-max formula for $\text{[PCliqW]}$. But there are other famous cases where (14) yields a min-max formula. Greene-Kleitman’s theorems can be restated in the following terms: if $G$ is a comparability graph or the complement of such a graph and if $f$ is the rank function of a uniform matroid, system (14) is TDI. Alternatively, Greene-Kleitman’s theorems can stated as the box-TDIness of (14) if $G$ is (co)-comparability and $f \equiv 1$ [5]. Note that cliques of the line-graph of a bipartite graph $G$ correspond to subsets of $\delta(v)$ (the set of edges incident with $v$), for some $v \in V(G)$. Now, a common generalization of the polymatroid intersection theorem, of Dilworth’s truncation and of min-max relations for bipartite $b$-matching can be stated as box-TDIness of (14) if $G$ is the line-graph of a bipartite multigraph and $f$ is submodular. More precisely we have (see Sect. 48.3 of [17] for an idea of the proof and Chapter 60 for extensions),

\footnote{An interpretation of system (14) within the framework of cooperative game theory with cooperation restricted to the cliques of a graph is described in [16].}
Let $f$ be the max-batch cost defined by $p$. An optimal solution to the linear relaxation of (13) is given by $y_C = 1/2$ if $C \in \{ \{v\}, \{b, v\}, \{a, b, c, a, d, e\}, \{c, d\}, \{e, w\}, \{w\}\}$ and $y_C = 0$ otherwise. The cost of this fractional partition is $13/2$. Optimality can be checked using an $x$ maximizing $1^T x$ subject to (14), for instance $x(a) := 3/2, x(e) = x(d) := 1/2$ and $x(b) = x(e) = x(v) = x(w) := 1$.

**Theorem 7.1** (submodular bipartite matchings polyhedron [16]). Let $G = ((A, B), E)$ be a bipartite multi-graph and for all $v \in A \cup B$ let $f_v$ be a submodular function on $\delta(v)$, then the following system is box-TDI

$$\sum_{e \in F} x_e \leq f_v(F) \text{ for all } v \in A \cup B \text{ and } \emptyset \neq F \subseteq \delta(v). \quad (15)$$

In view of these results, it seems reasonable to expect system (14) to provide other min-max relations for [PCliqW]. However, the linear relaxation of (13) does not always have an integral optimal solution, even if $G$ is an interval graph and $f$ is a value-polymatroidal set function as shown in Figure 4 (other examples for which $G$ is perfect, $f$ is a submodular but the linear relaxation of (13) has no integral optimal solution are provided in [16]).

8. CONCLUSION AND EXTENSION

Although we were able to compute an optimum solution for [PCliqW] when $G$ is an interval graph and $f$ is value-polymatroidal, we were unable to complement this result by a min-max formula. This issue could be linked with the following extension: consider the problem of multi-partition into cliques, that is, generalize the ILP (13) by replacing constraints (ii) by $\sum_{C \ni v} y_C = d_v$, where $d_v \in \mathbb{N}$ is the covering demand associated to vertex $v$. The complexity of this problem is left open and, to the best of our knowledge, is beyond the scope of our dynamic program. A polytime algorithm for this last problem might shed new light on the
structure of interval graphs and therefore be useful to solve various problems on interval graphs.

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