TWO NEW CLASSES OF TREES EMBEDDABLE INTO HYPERCUBES

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Abstract. The problem of embedding graphs into other graphs is much studied in the graph theory. In fact, much effort has been devoted to determining the conditions under which a graph $G$ is a subgraph of a graph $H$, having a particular structure. An important class to study is the set of graphs which are embeddable into a hypercube. This importance results from the remarkable properties of the hypercube and its use in several domains, such as: the coding theory, transfer of information, multicriteria rule, interconnection networks ... In this paper we are interested in defining two new classes of embedding trees into the hypercube for which the dimension is given.

1. INTRODUCTION

For a graph $G$, $V(G)$ and $E(G)$ denote the sets of vertices and edges of $G$ respectively. An embedding of the graph $G$ into the graph $H$ is an 1-1 mapping of $V(G)$ into $V(H)$ that preserves adjacency of vertices. Our particular interest concerns the case where $G$ is a tree and $H$ is a hypercube. A tree $T$ is a connected graph without circuits. A root of a tree is a particular vertex. A binary tree is a tree with a maximum degree less than or equal to 3. A tree $T$ is balanced if in the 2-coloring of $V(T)$, both color sets have the same cardinality. A hypercube

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of dimension $n$ (denoted by $Q_n$) is the graph whose vertices are all vectors of length $n$ consisting of 0’s and 1’s such that two vertices are adjacent if and only if they differ in exactly one coordinate. We note that

$$|V(Q_n)| = 2^n \quad \text{and} \quad |E(Q_n)| = n2^{n-1}.\]$$

The dimension of a tree $T$, denoted by $\dim T$, is the smallest $n$ such that $T$ is embeddable into $Q_n$. One of the results related to binary trees is given by Havel and Liebl [4]: for binary tree $T$ having $2^n$ vertices with $n \geq 3$, if $T$ is balanced and has $2$ vertices of degree $3$, then $T$ is embeddable into $Q_n$. Havel [3] and Nebesky [8] analyze the embedding of trees $D_n$, $\#D_n$, $\hat{D}_n$ and $\check{D}_n$ defined as follows:

If $n = 1$ then $D_1 = K_{1,2}$ (the complete bigraph), if $n > 1$ $D_n$ is the tree obtained from two disjoint copies $T$ and $T'$ of $D_{n-1}$ and a new vertex $v$ joined by one edge to the only vertex of degree $2$ of $T$ and by another edge to the analogous vertex of $T'$. Thus $D_n$ has $2^n$ endvertices, one vertex of degree $2$ called the root of $D_n$ and $2^n - 2$ vertices of degree $3$.

$\#D_n$ is obtained from two disjoint copies of $D_n$ such that their roots are joined by a new edge, this edge will be referred to as the axial edge of $\#D_n$. $\hat{D}_n$ has $2^{n+2} - 2$ vertices. $\hat{D}_n$ and $\check{D}_n$ are obtained from $\#D_n$ by inserting two new vertices of degree $2$ into the axial edge or into end-edge respectively. The trees $\#D_n$, $\hat{D}_n$ and $\check{D}_n$ are embeddable into $Q_{n+2}$ [8].

$B_n$ is obtained from $D_{n-1}$ by adding one edge to its root. $B_n$ has $2^{n-1} + 1$ endvertices and $2^n - 1$ vertices of degree $3$. In particular $B_1$ is the complete bigraph $K_{1,2}$. The cubical dimension of $B_n$ is $n + 1$ [6].

A caterpillar is a tree with at least $3$ vertices that becomes a path or single vertex when its endvertices are all removed. Havel and Liebl [4] have proved that any balanced caterpillar having $2^n$ vertices and maximum degree $3$ is embeddable into $Q_n$. In the same way, Harrary, Lewinter and Widulski [2] provided identical result for the caterpillar of maximum degree $4$.

A star is a tree with exactly one non endvertex $u$. A quasistar is a star such that its edges are subdivided. It is balanced if and only if it has just one odd ray. The degree of a quasistar is the number of edges incident to $u$. A quasistar of degree $k$ having $2^n$ vertices is called $k$-quasistar. A double star is obtained from two stars such that the junctions $u$ and $v$ are joined. A double quasistar is a subdivision of a double star in which the edge joined $u$ and $v$ is not subdivided. A double quasistar is balanced if the number of paths of odd order incident to $u$ is equal to the number of paths of odd order incident to $v$.

Let $uv$ be an edge. We design by $MD(a_1, ..., a_k)$ the graph formed by the edge $uv$ and $k \ (k \geq 1)$ distinct paths of orders respectively $a_1, ..., a_k$ ($a_i$ is a positive integer) where the extremities of each path are joined by an edge to $u$, the other by an edge to $v$ with $a_1 + ... + a_k = 2^n - 2$. $MD(a_1, ..., a_k)$ is balanced if $a_i$ is even for any $i$.

Recently Kobeissi [1] has shown that quasistar and double quasistar are embeddable into $Q_n$. The same Nebesky [9] has shown that any $MD$ $k$ balanced graph
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Figure 1. Complete binary tree $D_3$.

having $2^n$ vertices with $k \leq n - 1$ is embeddable into $Q_n$. As a result: any $k$ balanced quasistar (respectively double quasistar) with $2^n$ vertices such that $k \leq n$ (respectively $k \leq n - 1$) is embeddable into $Q_n$.

A tree $T$ is said to be $C_n$-valued, if the edges of $T$ are labelled by integers from \{1, ..., n\} in such a way that for any path $P$ of $T$ there is $k$ from \{1, ..., n\} such that an odd number of edges of $P$ are assigned $k$ [6].

Havel and Moravek [5] have used this notion in order to prove that a tree $T$ is embeddable into $Q_n$ if and only if there is a $C_n$-valuation of $T$.

2. EMBEDDING OF TWO NEW TREES

In this section, we introduce two new trees, denoted $D_n^{(k)}$ and $RD_n^{(2)}$ for which the cubical dimensions are given.

2.1. FIRST TYPE OF TREE

$D_n^{(k)}$ is obtained from the complete binary tree $D_n$ by the removal of $k$ endvertices where $k \leq 2^n$.

By removing only one vertex, the dimension of $D_n^{(1)}$ is equal to the dimension of $D_n$ for $n > 2$ as illustrated by the example given in Figure 1.

The tree $T_1$ obtained by removing vertex 15 is not embeddable into $Q_4$.

Proof. Let \{$x, N_0(x), N_1(x), N_2(x), ..., N_n(x)$\} the decomposition of hypercube $Q_n$ into levels from the vertex $x$ then we have $|N_i(x)| = \binom{n}{i}$. Hence for $Q_4$
we have:

\[
\begin{align*}
|N_0(x)| &= 1 \\
|N_1(x)| &= 4 \\
|N_2(x)| &= 6 \\
|N_3(x)| &= 4 \\
|N_4(x)| &= 1.
\end{align*}
\]

Following to this propriety and the definition of embedding, a tree T is embeddable into \(Q_4\) if for any \(x\) of T the decomposition of T into levels from \(x\) satisfied the conditions cited below:

\[
\begin{align*}
|N_0(x)| &\leq 1 \\
|N_1(x)| &\leq 4 \\
|N_2(x)| &\leq 6 \\
|N_3(x)| &\leq 4 \\
|N_4(x)| &\leq 1.
\end{align*}
\]

From these conditions, the tree \(T_1\) is not embeddable into \(Q_4\) (see Tab. 1 given in Appendix).

Note that \(\dim D_2^{(2)} = 3\) and \(\dim D_2^{(3)} = \dim D_2^{(4)} = 2\).

\[\square\]

**Theorem 1.** \(\dim D_n^{(k)} = n + 1\) for \(n > 1\), \(2 \leq k \leq 2^n\).

**Proof.** Firstly we prove that \(\dim D_2^{(2)} = n + 1\).

According to the proposition of HAVEL it is sufficient to determine a \(C_{n+1}\)-valuation of \(D_n^{(2)}\). The proof is by induction on \(n\). For \(n = 2\), a \(C_3\)-valuation is given in Figure 2. \[\square\]
As the complete binary trees are defined inductively, it is valuable for \( D_n^{(2)} \) and we can obtain \( D_n^{(2)} \) from \( D_{n-1} \) and \( D_{n-1}^{(2)} \) as shown in Figure 3.

If \( P \) belongs to \( D_{n-1} \), then \( \exists k \in \{1, ..., n+1\} \) such that an odd number of edges of \( P \) are assigned \( k \) because \( D_{n-1} \) is \( C_{n+1} \)-valued.

If \( P \) belongs to \( D_{n-1} \cup \text{ac} \), this tree is a \( B_n \) and then \( \exists k \in \{1, ..., n+1\} \) such that an odd number of edges of \( P \) are assigned \( k \).

If \( P = P_1 \cup \text{ab} \cup \text{ac} \cup \text{bd} \cup P_2 \) where \( P_1 \subseteq D_{n-2}^{(2)} \) and \( P_2 \subseteq D_{n-1} \) then we have the following (see Fig. 4):

- for the subpath \( P_1 \) : \( \exists k_1 \in \{1, ..., n-1\} \) such that an odd number of edges of \( P_1 \) are assigned \( k_1 \) ...(1);
- for the subpath \( P_2 \) : \( \exists k_2 \in \{1, ..., n+1\} \) such that an odd number of edges of \( P_2 \) are assigned \( k_2 \) ...(2);
- \( k_2 \neq n+1 \) otherwise the condition will not be satisfied for the tree \( B_n \) which is \( C_{n+1} \)-valued ...(3);
- hence from (1), (2) and (3) \( k \) may have the value \( n+1 \).

\[
\begin{align*}
\text{Figure 3. A } C_{n+1} \text{ valuation for } D_n^{(2)}. \\
\end{align*}
\]

\[
\begin{align*}
\text{Figure 4. A subgraph of } D_n^{(2)}.
\end{align*}
\]
– for the subpath $P_2$ : $\exists k_2 \in \{1, ..., n+1\}$ such that an odd number of edges of $P_2$ are assigned $k_2$...(2);
– $k_2 \neq n+1$ otherwise the condition will not be satisfied for the tree $B_n$ which is $C_{n+1}$ valued...(3);
– hence from (1), (2) and (3) $k$ may have the value 1.

![Figure 5. A subgraph of $D_n^{(2)}$.](image)

Also, there exists a $C_{n+1}$- valuation for $D_n^{(2)}$ and consequently this tree is embeddable into $Q_{n+1}$ for $n > 1$.

Secondly for $2 \leq k \leq 2^n$ we have
– $D_{n-1} = D_n^{(k)}$ for $k = 2^n$ so $D_{n-1} \subseteq D_n^{(k)}$ for $2 \leq k \leq 2^n$...(1);
– in other way $D_n^{(k)} \subseteq D_n^{(2)}$ for $2 \leq k \leq 2^n$...(2);
– from (1) and (2) we have $D_{n-1} \subseteq D_n^{(k)} \subseteq D_n^{(2)}$ then $D_{n-1} \leq \dim D_n^{(k)} \leq \dim D_n^{(2)}$;
– as $\dim D_n^{(2)} = \dim D_{n-1} = n+1$;
– so $\dim D_n^{(k)} = n+1$, $n > 2$, $k (2 \leq k \leq 2^n)$.

2.2. The second new type of tree

$RD_n^{(2)}$ is obtained from the complete binary tree $D_n$ by adding two edges to its root.

**Theorem 2.** $RD_n^{(2)}$ is embeddable into $Q_{n+2}$ and $\dim RD_n^{(2)} = n+2$, $n \geq 2$.

Note that $\dim RD_2^{(2)} = 4$.

**Proof.** By induction on $n$.

For $n = 2$.
A $C_4$-valuation is given in Figure 6. \hfill $\Box$

A $C_{n+2}$- valuation of $RD_n^{(2)}$ can be constructed as follows, see Figure 7.

**Remark 3.** We keep the same valuation of $RD_{n-1}^{(2)}$ for the subgraphs $D_{n-1}$ and show that for any path of $RD_n^{(2)}$, there exists an integer $k$ for which an odd number of edges are assigned $k$.

– If $P$ belongs to the subgraph in Figure 8:
  There exists an integer $k \in \{1, ..., n+2\}$ such that the propriety is verified because this graph is a $B_{n+1}$.
– If $P$ belongs to the subgraph of Figure 9:

So $P = P_1 \cup ac \cup ce \cup ef$ or $P_2 \cup bc \cup ce \cup ef$ with $P_1, P_2 \in D_{n-2}$. 
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Figure 6. A $C_4$-valuation of $RD_2^{(2)}$.

Figure 7. A $C_{n+2}$-valuation of $RD_n^{(2)}$.

Figure 8. A subgraph of $RD_n^{(2)}$. 
Figure 9. A subgraph of $RD_n^{(2)}$.

Figure 10. A $C_{n+1}$ valuation of $RD_{n-1}^{(2)}$.

Note that the $C_{n+1}$-valuation of $RD_{n-1}^{(2)}$ is defined as follows (see Fig. 10) then only one case is posed:

If for the path $P_1 \cup ac \cup cd$ or $P_2 \cup bc \cup cd$ $k = n - 1$ so for the path $P_1 \cup ac \cup ce \cup ef$ or $P_2 \cup bc \cup ce \cup ef$ the $k$ may have the value $n - 1$.

Then for the path $P$ of $RD_n^{(2)}$ there always exists an odd number of edges assigned $k$, so $\dim RD_n^{(2)} = n + 2n \geq 2$.

2.3. Conclusion

In this paper, we presented two types of tree that are embeddable in a hypercube. The special characteristics are given and analytical results provided.

The question we pose is how many edges can be added to the root of $D_n$ such that the graph obtained will be embeddable into $Q_{n+2}^2$. 
Table 1. The cardinality of levels of $T_1$.

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Appendix

$T_1$ is the tree obtained from $D_3$ by removing one endvertex. The cardinality of levels obtained from the decomposition of $T_1$ from the vertex $x$ such $x \in \{1, 2, 3, ..., 14\}$ is given in Table 1.

References