ON CONSTRAINT QUALIFICATIONS IN DIRECTIONALLY DIFFERENTIABLE MULTIOBJECTIVE OPTIMIZATION PROBLEMS

GIORGIO GIORGI\textsuperscript{1}, BIENVENIDO JIMÉNEZ\textsuperscript{2} AND VINCENTE NOVO\textsuperscript{3}

Communicated by Jean Abadie

Abstract. We consider a multiobjective optimization problem with a feasible set defined by inequality and equality constraints such that all functions are, at least, Dini differentiable (in some cases, Hadamard differentiable and sometimes, quasiconvex). Several constraint qualifications are given in such a way that generalize both the qualifications introduced by Maeda and the classical ones, when the functions are differentiable. The relationships between them are analyzed. Finally, we give several Kuhn-Tucker type necessary conditions for a point to be Pareto minimum under the weaker constraint qualifications here proposed.

Keywords. Multiobjective optimization problems, constraint qualification, necessary conditions for Pareto minimum, Lagrange multipliers, tangent cone, Dini differentiable functions, Hadamard differentiable functions, quasiconvex functions.

Mathematics Subject Classification. 90C29, 90C46.
1. Introduction

Constraint qualifications have a significant role in optimization problems, since they allow us to guarantee the effective intervention of the objective function in the Fritz John type necessary conditions for a point to be an optimum. Since the first decade of the 50’s, the study of these qualifications has been the aim of several researchers with different approaches, proposing various regularity conditions.

Maeda [10] studies multiobjective optimization problems with differentiable functions between finite-dimensional spaces and gives a Kuhn-Tucker type necessary condition for a Pareto optimum of a function over a feasible set defined by inequality constraints, assuring that the multipliers of the objective function are all positive under a regularity condition, called generalized Guignard constraint qualification. He also studies other qualifications, showing that this one is the weakest.

Preda and Chitescu [13] develop, at first, results similar to those obtained by Maeda, considering Dini-quasiconvex and directionally differentiable functions. But owing to the requirement on the objective functions to be Dini-quasiconvex and Dini-quasiconcave with convex and concave Dini derivatives, their necessary optimality conditions (Ths. 3.1 and 3.2) are very restrictive. On the other hand, the necessary condition expressed in Theorem 3.2, assuring the existence of positive multipliers for the objective functions, has a mistake that will be corrected in this paper.

Jiménez and Novo [7] extend the results obtained by Maeda for differentiable functions, by considering equality constraints, not considered by Maeda nor by Preda and Chitescu. They also introduced new qualifications that are sufficient conditions for what the afore mentioned papers called generalized Guignard constraint qualification.

In the present paper, the results obtained by Maeda, Preda and Chitescu and Jiménez and Novo are extended, by considering Dini or Hadamard differentiable functions and equality constraints. Furthermore, new qualifications are also introduced and the relationships between them are studied, thus obtaining a scheme which generalizes the ones of Bazaraa and Shetty [2], Figure 6.4, Maeda [10], Figure 1, Preda and Chitescu [13], Figure 1, and Jiménez and Novo [7], Figure 1.

This paper is structured as follows: Section 2 contains the definitions and notations we use and some previous results. In Section 3 several constraint qualifications are proposed and the relationships between them are studied. Finally, in Section 4, several necessary optimality conditions of the Kuhn-Tucker type are obtained, i.e. such that they assure the positivity of the multipliers under the weaker qualifications proposed.
2. Notations and preliminaries

Let $x$ and $y$ be two points of $\mathbb{R}^n$. Throughout this paper, we use the following notations.

$$x \leq y \text{ if } x_i \leq y_i, \ i = 1, \ldots, n.$$  
$$x < y \text{ if } x_i < y_i, \ i = 1, \ldots, n.$$  

Let $S$ be a subset of $\mathbb{R}^n$. As usual, $\text{cl} \ S$, $\text{co} \ S$, cone $S$ and $\text{lin} \ S$ will denote the closure, convex hull, generated cone and generated subspace by $S$, respectively. $B(x_0, \delta)$ is the open ball of center $x_0$ and radius $\delta > 0$.

Given a function $f : \mathbb{R}^n \to \mathbb{R}^p$, the following multiobjective optimization problem is considered

$$(\text{MOP}) \ \text{Min} \{f(x) : x \in S\}.$$  

It is said that the point $x_0 \in S$ is a local Pareto minimum, denoted $x_0 \in \text{LMin}(f, S)$, if there exists a neighborhood of $x_0$, $B(x_0, \delta)$, such that

$$S_f \cap S \cap B(x_0, \delta) = \emptyset,$$  

where $S_f = \{x \in \mathbb{R}^n : f(x) \leq f(x_0), \ f(x) \neq f(x_0)\}$.

The usual concepts of Pareto minimum, weak Pareto minimum and local weak Pareto minimum are also used. They will be denoted by $\text{Min}(f, S)$, $\text{WMin}(f, S)$ and $\text{LWMin}(f, S)$, respectively.

Because of the difficulties in verifying condition (1), different approximations at $x_0$ of the sets $S$ and $S_f$ are normally used, which have a simpler structure and are easier to obtain. The tangent cones are the approximations more usually used.

**Definition 2.1.** Let $S \subset \mathbb{R}^n$, $x_0 \in \text{cl} S$.

(a) The tangent cone to $S$ at the point $x_0$ is

$$T(S, x_0) = \{v \in \mathbb{R}^n : \exists k \to 0^+, \ \exists x_k \in S \text{ such that } (x_k - x_0)/t_k \to v\}.$$  

(b) The cone of attainable directions is

$$A(S, x_0) = \{v \in \mathbb{R}^n : \forall t_k \to 0^+, \ \exists x_k \in S \text{ such that } (x_k - x_0)/t_k \to v\}.$$  

(c) The cone of linear directions is

$$Z(S, x_0) = \{v \in \mathbb{R}^n : \exists \delta > 0 \text{ such that } x_0 + tv \in S \ \forall t \in (0, \delta)\}.$$  

For these cones, we have the following inclusions

$$Z(S, x_0) \subset A(S, x_0) \subset T(S, x_0).$$  

A complete and rigorous analysis of these cones in a greater detail can be found in Bazaraa and Shetty [2] and in Aubin and Frankowska [1].

Let $D \subset \mathbb{R}^n$. Then the polar cone to $D$ is $D^* = \{\xi \in \mathbb{R}^n : \langle \xi, d \rangle \leq 0 \ \forall d \in D\}$.

The normal cone to $S$ at $x_0$ is the polar to the tangent cone, i.e., $N(S, x_0) = T(S, x_0)^*$.  

Note that if the sets are defined through function constraints, their approximation is realized through the cones defined by the directional derivatives of the functions.
Definition 2.2. Let $f : \mathbb{R}^n \to \mathbb{R}^p$, $x_0, v \in \mathbb{R}^n$.

(a) The Dini derivative (or directional derivative) of $f$ at $x_0$ in the direction $v$ is

$$Df(x_0, v) = \lim_{t \to 0^+} \frac{f(x_0 + tv) - f(x_0)}{t}.$$  

(b) The Hadamard derivative of $f$ at $x_0$ in the direction $v$ is

$$df(x_0, v) = \lim_{(t, u) \to (0^+, v)} \frac{f(x_0 + tu) - f(x_0)}{t}.$$  

(c) $f$ is Dini differentiable (respectively Hadamard differentiable) at $x_0$ if its Dini derivative (resp. Hadamard derivative) exists for all the directions.

The following properties are well-known:
- if $f$ is Fréchet differentiable at $x_0$ with Fréchet differential $\nabla f(x_0)$, then $df(x_0, v) = \nabla f(x_0)v$;
- if $df(x_0, v)$ exists, then also $Df(x_0, v)$ exists and they are equal.

Definition 2.3. The Dini subdifferential of a Dini differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ at $x_0$ is

$$\partial_D f(x_0) = \{ \xi \in \mathbb{R}^n : \langle \xi, v \rangle \leq Df(x_0, v) \forall v \in \mathbb{R}^n \}.$$  

If $Df(x_0, v)$ is a convex function in $v$, then there exists the subdifferential (in the Convex Analysis sense) of this function at $v = 0$: $\partial Df(x_0, \cdot)(0)$. This set is nonempty, compact and convex and $\partial_D f(x_0) = \partial Df(x_0, \cdot)(0)$.

In this paper, the following generalized convexity notions will be used.

Definition 2.4. Let $\Gamma \subset \mathbb{R}^n$ be a convex set, $f : \Gamma \to \mathbb{R}$, and $x_0 \in \Gamma$.

(a) $f$ is quasiconvex at $x_0$ if $\forall x \in \Gamma$, $f(x) \leq f(x_0) \Rightarrow f(\lambda x + (1 - \lambda)x_0) \leq f(x_0)$ $\forall \lambda \in (0, 1)$;
(b) $f$ is quasiconcave at $x_0$ if $-f$ is quasiconvex at $x_0$;
(c) $f$ is quasilinear at $x_0$ if $f$ is quasiconvex and quasiconcave at $x_0$;
(d) $f$ is pseudoconvex at $x_0$ if $\forall x \in \Gamma$, $f(x) < f(x_0) \Rightarrow Df(x_0, x - x_0) < 0$;
(e) $f$ is pseudoconcave at $x_0$ if $-f$ is pseudoconvex at $x_0$. $f$ is pseudolinear at $x_0$ if $f$ is pseudoconvex and pseudoconcave at $x_0$;
(f) $f$ is linearlike at $x_0$ if $f(x) = f(x_0) + Df(x_0, x - x_0) \forall x \in \Gamma$;
(g) $f$ is Dini-quasiconvex at $x_0$ if $\forall x \in \Gamma$, $f(x) \leq f(x_0) \Rightarrow Df(x_0, x - x_0) \leq 0$;
(h) $f$ is Dini-quasiconcave at $x_0$ if $f$ and $-f$ are Dini-quasiconvex at $x_0$;
(i) $f$ is quasiconvex on $\Gamma$ if $f$ is quasiconvex at each point of $\Gamma$. The other concepts here introduced can be defined on a set in a similar way.

In the next proposition we summarize some properties of the generalized convex functions previously introduced.

Proposition 2.1. Let $\Gamma \subset \mathbb{R}^n$ be a convex set, $f : \Gamma \to \mathbb{R}$, and $x_0 \in \Gamma$.

(a) [3] Th. 3.5.2) $f$ is quasiconvex on $\Gamma$ if and only if the level sets $\Gamma_\alpha = \{ x \in \Gamma : f(x) \leq \alpha \}$ are convex for all $\alpha \in \mathbb{R}$.
(b) Let $f$ be Dini differentiable at $x_0$. If $f$ is quasiconvex at $x_0$, then $f$ is Dini-quasiconvex at $x_0$.

(c) ([5] Th. 3.5) if $f$ is pseudoconvex at $x_0$ and continuous on $\Gamma$, then $f$ is quasi-convex at $x_0$.

(d) ([5] Th. 3.2) if $f$ is continuous and Dini-quasiconvex on $\Gamma$, then $f$ is quasi-convex on $\Gamma$.

**Remark 2.1.** The following implications can easily be proved for a linear type function:

(i) if $f$ is linearlike at $x_0$, then $f$ is pseudolinear at $x_0$ and quasilinear at $x_0$;

(ii) if $f$ is quasilinear at $x_0$ and Dini differentiable at $x_0$, then $f$ is Dini-quasilinear at $x_0$.

The second implication follows from Proposition 2.1(b). The converse of (i) does not hold. It can be proved, for instance, with the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x| + x^2$ and the point $x_0 = 0$.

Moreover, the reverse of Proposition 2.1(b) does not hold (i.e., if $f$ is Dini-quasiconvex at $x_0$, then $f$ is quasiconvex at $x_0$ over any neighborhood of $x_0$ (it is sufficient to consider the points $x_n = (\delta_n, \delta_n^2)$ with $\delta_n \to 0^+$), then $f(x_n) = f(x_0)$ and $Df(x_0, x_n - x_0) > 0$). Hence, thanks to Proposition 2.1(b), $f$ is not quasiconvex at $x_0$. Furthermore, this function is pseudolinear at $x_0$ and, consequently, it is not true that a pseudolinear function at $x_0$ is quasilinear at $x_0$ (or Dini-quasilinear at $x_0$).

The function $f(x) = x^3$ is quasiconvex at $x_0 = 0$ but it is not pseudoconvex.

Finally, none of the linear types guarantees by itself the continuity of the derivative. Example 3.1 in [4] Chapter 1 shows this fact.

We say that the convex sets $B_j, j \in J = \{1, \ldots, m\}$, of $\mathbb{R}^n$ are positively linearly independent (p.l.i.) if

$$0 \in \sum_{j \in J} \lambda_j B_j, \quad \lambda \geq 0 \quad \Rightarrow \quad \lambda = 0,$$

i.e., if $0 \notin \text{co}(\cup_{j \in J} B_j)$. 

has no solution

We have

By means of this notation, as the cones

Proposition 2.2 (generalized Tucker alternative theorem). Let \( f_1, \ldots, f_p, g_1, \ldots, g_m \) be sublinear functions from \( \mathbb{R}^n \) to \( \mathbb{R} \) and \( h_1, \ldots, h_r \) linear functions from \( \mathbb{R}^n \) to \( \mathbb{R} \) given by \( h_k(v) = \langle c_k, v \rangle \), \( k \in K = \{1, \ldots, r\} \). Suppose that for each \( i \in \{1, \ldots, p\} \) the cone

is closed. Then, the following statements are equivalent:

(a) The system

\[
  f(v) \leq 0, \ f(v) \neq 0, \ g(v) \leq 0, \ h(v) = 0
\]

has no solution \( v \in \mathbb{R}^n \).

(b) There exist \( \lambda, \mu, \nu \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r \) such that \( \lambda > 0 \), \( \mu \geq 0 \) and

\[
  0 \in \sum_{i=1}^{p} \lambda_i \partial f_i(0) + \sum_{j=1}^{m} \mu_j \partial g_j(0) + \sum_{k=1}^{r} \nu_k c_k.
\]

Proof. To transform this proposition into the one of Ishizuka, it is enough to consider

\[ A_i = \partial f_i(0), \ i = 1, \ldots, p, \ B_j = \partial g_j(0), \ j = 1, \ldots, m, \ B_{m+k} = \text{co}\{ -c_k, c_k \}, \ k = 1, \ldots, r, \]

which implies

\[
  f_i(v) = \text{Max}_{a \in A_i}(a, v), \ i = 1, \ldots, p, \ g_j(v) = \text{Max}_{b \in B_j}(b, v), \ j = 1, \ldots, m.
\]

Let

\[
  g_{m+k}(v) = \text{Max}_{c \in \text{co}\{ -c_k, c_k \}}(c, v) = \text{Max}\{ -\langle c_k, v \rangle, \langle c_k, v \rangle \} = |h_k(v)|, \ k = 1, \ldots, r.
\]

We have \( \partial g_{m+k}(0) = B_{m+k} \) and the equation \( h_k(v) = 0 \) is equivalent to \( g_{m+k}(v) \leq 0 \). By means of this notation, as the cones \( D_i \) are closed, according to Ishizuka’s Proposition 2.2 [6], (a) is equivalent to

(c) There exist \( \lambda, \mu, \alpha \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r \) such that \( \lambda > 0, \ (\mu, \alpha) \geq 0 \) and

\[
  0 \in \sum_{i=1}^{p} \lambda_i \partial f_i(0) + \sum_{j=1}^{m} \mu_j \partial g_j(0) + \sum_{k=1}^{r} \alpha_k [-c_k, c_k].
\]

Taking into account that \( c \in \alpha_k [-c_k, c_k] \) for some \( \alpha_k \geq 0 \) if and only if there exists \( \nu_k \in \mathbb{R} \) such that \( c = \nu_k c_k \), Proposition (c) is equivalent to (b).

In order to decide if the cones \( D_i \) are closed, we have the following criterion.

Remark 2.2. Note that if \( 0 \notin \text{co}(\cup_{j \neq i} A_j \cup \cup_{j=1}^{m} B_j) + \text{lin}\{c_k : k \in K\} \), then \( D_i \) is closed. This follows from Proposition 3.6 in [8]. Note that if \( 0 \notin C = \text{co}(\cup_{i=1}^{p} A_i \cup \cup_{j=1}^{m} B_j) + \text{lin}\{c_k : k \in K\} \), then the \( p \) cones \( D_i \) are closed. But this condition is incompatible with Proposition 2.2(b) and, consequently, with Proposition (a). As a matter of fact, if \( 0 \notin C \) and \( u = \text{Proy}_C(0) \), then the vector \( v = -u \) is a solution of the system in (a).
Now we consider a set $S$ defined by equality and inequality constraints and a point of $S$ at which we need to obtain the tangent cone. This is done in Proposition 2.6.

From now on, we shall assume that the feasible set of problem (MOP) is defined by

$$S = \{ x \in \mathbb{R}^n : g(x) \leq 0, \ h(x) = 0 \}, \quad (3)$$

where $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^r$, whose component functions are, respectively, $g_j$, $j \in J = \{1, \ldots, m\}$, $h_k$, $k \in K = \{1, \ldots, r\}$. We shall adopt the following notation. Given $x_0 \in S$, the active index set at $x_0$ is $J_0 = \{ j \in J : g_j(x_0) = 0 \}$. The sets defined by the constraints $g$ and $h$ are denoted, respectively, by $G = \{ x \in \mathbb{R}^n : g(x) \leq 0 \}$, $H = \{ x \in \mathbb{R}^n : h(x) = 0 \}$, so $S = G \cap H$.

We suppose that all functions considered are continuous at $x_0$ and that the active constraints are Dini differentiable at $x_0$. The cones that we shall use in order to approximate $S$ at $x_0$ are (linearized cones):

$$C_0(S) = \{ v \in \mathbb{R}^n : Dg_j(x_0, v) < 0 \ \forall j \in J_0, \ Dh_k(x_0, v) = 0 \ \forall k \in K \},$$

$$C(S) = \{ v \in \mathbb{R}^n : Dg_j(x_0, v) \leq 0 \ \forall j \in J_0, \ Dh_k(x_0, v) = 0 \ \forall k \in K \}.$$

$C_0(G)$ and $C(G)$ are defined in an analogous way and we denote $K(H) = \text{Ker} Dh(x_0, \cdot)$. Consequently, $C_0(S) = C_0(G) \cap K(H)$ and $C(S) = C(G) \cap K(H)$.

Our aim is to obtain the inclusions

$$C_0(S) \subset T(S, x_0) \subset C(S). \quad (4)$$

This is done in the following propositions.

**Proposition 2.3** ([12], (Prop. 3.1)). Let $Dg_j(x_0, \cdot)$, $j \in J_0$ be convex, $Dh(x_0, \cdot)$ linear and $C_0(S) \neq \emptyset$. Then $\text{cl} C_0(S) = C(S)$.

**Proposition 2.4.** Suppose that for each $j \in J_0$, either $g_j$ is Hadamard differentiable at $x_0$ or $g_j$ is Dini-quasiconvex at $x_0$ and $Dg_j(x_0, \cdot)$ is continuous on $\mathbb{R}^n$, and for each $k \in K$, either $h_k$ is Hadamard differentiable at $x_0$ or Dini-quasilinear at $x_0$ with $Dh_k(x_0, \cdot)$ continuous. Then

$$T(S, x_0) \subset C(S).$$

The proof of the previous proposition is similar to that of Lemma 3.2 in [9].

**Proposition 2.5.** If there is no equality constraints, $S = G$, and the functions $g_j$, $j \in J_0$, are Dini differentiable at $x_0$, then

$$C_0(G) \subset Z(G, x_0) \subset \begin{cases} A(G, x_0) \subset T(G, x_0) \\ C(G) \end{cases}.$$
Proposition 2.6 [9] (Cor. 3.5). Let us suppose the following:
(a) \( h \) is continuous on a neighborhood of \( x_0 \), Fréchet differentiable at \( x_0 \) and \( \{\nabla h_k(x_0) : k \in K\} \) is linearly independent;
(b) for each \( j \in J_0 \), \( g_j \) is either Dini-quasiconvex and continuous on a neighborhood of \( x_0 \) or Hadamard differentiable at \( x_0 \), in both cases with convex derivative at \( x_0 \);
(c) \( C_0(S) \neq \emptyset \).
Then
\[
\text{cl} \ C_0(S) = A(S, x_0) = T(S, x_0) = C(S).
\]

Note that, by [8], Theorem 3.9, we have that \( \{\nabla h_k(x_0) : k \in K\} \) is linearly independent and \( C_0(S) \neq \emptyset \) if and only if the following implication is true:
\[
0 \in \sum_{j \in J_0} \mu_j \partial D g_j(x_0) + \sum_{k=1}^r \nu_k \nabla h_k(x_0), \quad \mu \geq 0 \Rightarrow \mu = 0, \nu = 0,
\]
which is constraint qualification (CQ2) in [9]. By Proposition 2.1(d), if \( g_j \) is Dini-quasiconvex and continuous on a neighborhood of \( x_0 \), then \( g_j \) is quasiconvex on such a neighborhood.

3. Constraint qualifications in multiobjective optimization

Let us consider the multiobjective optimization problem
\[
\text{(MOP)} \quad \text{Min}\{f(x) : x \in S\},
\]
where the feasible set \( S \) is given by (3) and \( f : \mathbb{R}^n \to \mathbb{R}^p \) has component functions \( f_i, i \in I = \{1, \ldots, p\} \).

By keeping the notation of Section 2, given \( x_0 \in S \), the following sets are considered:
\( F = \{x : f(x) \leq f(x_0)\} \), \( S^0 = F \cap S \) and for each \( i \in I \), \( F^i = \{x : f_i(x) \leq f_i(x_0) \forall j \in I \setminus \{i\}\} \) and \( S^i = F^i \cap S \). Obviously \( F = \cap_{i=1}^p F^i \) and \( S^0 = \cap_{i=1}^p S^i \). Since the sets given above are defined by function constraints, the corresponding linearized cones can be defined. Let us remark that for the set \( F \) all functions \( f_i, i \in I \), are active at \( x_0 \) and for the set \( F^i \) the same is true for the functions \( f_j, j \in I \setminus \{i\} \). We have \( C_0(S^0) = C_0(F^i) \cap C_0(G) \cap K(H), \)
\[ C(S^0) = C(F^i) \cap C(G) \cap K(H) \] and similar expressions for \( C_0(S^0) \) and \( C(S^0) \).

It is a known result that \( x_0 \) is a local Pareto minimum to problem (MOP) if and only if for each \( i = 1, \ldots, p \), \( x_0 \) is a local minimum of the scalar problem
\[
\text{(P_i)} \quad \text{Min}\{f_i(x) : x \in S^i\}.
\]

We consider now different qualifications for problem (MOP) in the approaches of Maeda [10], Freda and Chiteşcu [13] and Jiménez and Novo [7]. The implications between the various qualifications are also analyzed.
We suppose that all functions are Dini differentiable at \( x_0 \), unless we specify another thing.

Let us consider the following hypotheses:

(H0) \( h \) is continuous on a neighborhood of \( x_0 \), Fréchet differentiable at \( x_0 \) and \( \{\nabla h_k(x_0) : k \in K\} \) is linearly independent.

(H1) Each function of the set \( \{f_i, g_j : i \in I, j \in J_0\} \) is either Dini-quasiconvex at \( x_0 \) with continuous derivative on \( \mathbb{R}^n \) or Hadamard differentiable at \( x_0 \).

(H2) Each function of the set \( \{f_i, g_j : i \in I, j \in J_0\} \) is either Dini-quasiconvex and continuous on a neighborhood of \( x_0 \) or Hadamard differentiable at \( x_0 \).

(H3) For each \( i = 1, \ldots, p, T(S^i, x_0) \subset C(S^i) \) holds true.

(H4) Each function in the set \( \{Df_i(x_0, \cdot), Dg_j(x_0, \cdot) : i \in I, j \in J_0\} \) is convex.

**Definition 3.1.** The next constraint qualifications are considered:

1. **Generalized Guignard (GGCQ):** \( C(S^0) = \cap_{i=1}^p \text{clco} T(S^i, x_0) \).
2. **Abadie (ACQ):** \( C(S^0) = T(S^0, x_0) \) and (H3).
3. **Generalized Abadie (GACQ):** \( C(S^0) = \bigcap_{i=1}^p T(S^i, x_0) \) and (H3).
4. **Global Cottle (GCCQ):** \( C_0(F) \cap C_0(S) \neq \emptyset \), (H0) and (H4).
5. **Cottle (CCQ):** for each \( i = 1 \ldots p, C_0(S^i) \neq \emptyset \) (H0) and (H4).
6. **Slater type.**
   a) **Slater (SCQ):** \( f_i, i \in I, g_j, j \in J_0 \), are pseudoconvex at \( x_0 \); \( h_k, k \in K \), are Dini-quasilinear at \( x_0 \), (H0), (H4) and for each \( i = 1, \ldots, p \) there exists \( x_i \in \mathbb{R}^n \) such that
   \[
   f_j(x_i) < f_j(x_0) \forall j \neq i, \ g_j(x_i) < 0 \forall j \in J_0, \text{ and } h_k(x_i) = 0 \forall k \in K.
   \]  
   (5)
   b) **Differentiable Slater (DSCQ):** \( f_i, i \in I, g_j, j \in J_0 \), are pseudoconvex at \( x_0 \), (H0), (H4) and for each \( i = 1, \ldots, p \) there exists \( x_i \in \mathbb{R}^n \) such that
   \[
   f_j(x_i) < f_j(x_0) \forall j \neq i, \ g_j(x_i) < 0 \forall j \in J_0, \text{ and } x_i - x_0 \in K(H).
   \]  
   (6)
7. **Linearlike (LLCQ):** \( f_i, g_j, h_k, i \in I, j \in J_0, k \in K \), are all linearlike at \( x_0 \) with continuous derivative.
8. **Linearlike objectives (LLO):** \( f_i, i \in I \), are linearlike at \( x_0 \) with convex derivative, each \( g_j, j \in J_0 \), has convex derivative and is either Hadamard differentiable or Dini-quasiconvex at \( x_0 \); \( h_k, k \in K \), are affine and \( C(F) \cap C_0(G) \cap K(H) \neq \emptyset \).
9. **Mangasarian-Fromovitz.** Each qualification in this group must verify (H0) and (H4):
   a) With positively linearly independent objectives (PIOMF): \( C(F) \cap C_0(S) \neq \emptyset \) and \( C_0(F) \cap K(H) \neq \emptyset \).
   b) With quasi-independent objectives (QIOMF): \( C(F) \cap C_0(S) \neq \emptyset \) and for each \( i = 1, \ldots, p \) we have that \( C_0(F^i) \cap K(H) \neq \emptyset \).
   c) With positively linearly independent constraints (PICMF): \( C_0(F) \cap C(S) \neq \emptyset \) and \( C_0(G) \cap K(H) \neq \emptyset \).
   d) With \( C(F) \cap C_0(G) \cap K(H) \neq \emptyset \) and for each \( i = 1, \ldots, p, C(F) \cap C_0(G) \cap K(H) \notin \text{Ker} \ Df_i(x_0, \cdot) \). We speak in this case of **Preda-Chitescu Mangasarian-Fromovitz qualification (PCMF)**.
10. Zangwill (ZCQ): \( \text{cl } Z(S^0, x_0) = C(S^0) \) and (H3).
11. Kuhn-Tucker (KTCQ): \( A(S^0, x_0) = C(S^0) \) and (H3).
12. Reverse (RCQ): \( f_i, g_j, i \in I, j \in J_0 \), are pseudoconcave at \( x_0 \), (H1) and \( h_k, k \in K \), are linearlike at \( x_0 \) with continuous derivative.

Lemma 3.1.

(i) If \( h \) is linearlike at \( x_0 \) with continuous Dini derivative, then \( h \) is Hadamard differentiable at \( x_0 \).

(ii) If \( h \) is pseudolinear and Dini-quasilinear at \( x_0 \), then

\[
Z(H, x_0) = T(H, x_0) = K(H);
\]

\( H = x_0 + K(H) \).

(iii) If \( g_j, j \in J_0 \), are pseudoconcave at \( x_0 \), then \( Z(G, x_0) = C(G) \).

Proof.

(i) It is an elementary exercise.

(ii) (a) \( Z(H, x_0) \subset T(H, x_0) \) is true for all sets \( H \) and \( T(H, x_0) \subset K(H) \) by Proposition 2.4. We now prove that \( K(H) \subset Z(H, x_0) \). Let \( v \in K(H) \), then \( Dh(x_0, v) = 0 \) and therefore \( Dh(x_0, x_0 + tv) - x_0 \geq 0 \forall t > 0 \). Since \( h \) is pseudoconvex, \( h(x_0 + tv) \geq h(x_0) = 0 \forall t > 0 \), and analogously, due to the pseudoconcavity, \( h(x_0 + tv) \leq h(x_0) = 0 \). Consequently \( h(x_0 + tv) = 0 \), i.e., \( x_0 + tv \in H \forall t > 0 \), which implies \( v \in Z(H, x_0) \).

(b) We have just proved that \( x_0 + tv \in H \forall t > 0 \). Taking \( t = 1 \), we have \( v \in H - x_0 \) and thus \( K(H) \subset H - x_0 \). Now we prove the reverse inclusion. Let \( x \in H \), hence \( h(x) - h(x_0) \leq 0 \). Since \( h \) is Dini-quasiconvex at \( x_0 \), it follows \( Dh(x_0, x - x_0) \leq 0 \). Likewise with \( -h \), we get \( -Dh(x_0, x - x_0) \leq 0 \). Consequently, \( Dh(x_0, x - x_0) = 0 \), which means, \( x - x_0 \in K(H) \).

(iii) From Proposition 2.5, we only have to prove that \( C(G) \subset Z(G, x_0) \). Let \( v \in C(G) \), then \( Dg_j(x_0, v) \leq 0 \forall j \in J_0 \). By pseudoconcavity, \( g_j(x_0 + tv) \leq g_j(x_0) = 0 \forall t > 0 \). If \( j \in J \setminus J_0 \), by the continuity of \( g_j \) we have \( g_j(x_0 + tv) < 0 \) for all \( t \) small enough. Therefore, \( x_0 + tv \in G \), and consequently \( v \in Z(G, x_0) \), thus completing the proof. 

We remark that just by using the definition, we get that if \( h \) is linearlike at \( x_0 \) with linear Dini derivative, then \( h \) is affine. If \( h \) is linearlike at \( x_0 \) with continuous Dini derivative, then part (ii) of Lemma 3.1 holds true (according to Rem. 2.1, \( h \) is pseudolinear and Dini-quasilinear at \( x_0 \)).

In Theorem 3.1 below, the relationship between the different constraint qualifications are established. In order to prove the theorem we need a previous lemma. The inclusion relationships in the lemma are obvious and the proof of the second part is similar to that of Proposition 2.3.

Lemma 3.2. If \( Dh(x_0, \cdot) \) is linear and \( Df_i(x_0, \cdot), \ Dg_j(x_0, \cdot), \ i \in I, j \in J_0 \) are convex, then

\[
C_0(S^0) = C_0(F) \cap C_0(G) \cap K(H) \subset \left\{ \begin{array}{l}
C(F) \cap C_0(G) \cap K(H) \\
C_0(F) \cap C(G) \cap K(H)
\end{array} \right\}
\]

\[
\subset C(F) \cap C(G) \cap K(H) = C(S^0),
\]
and if some of the sets $C_0(S^0)$, $C(F) \cap C_0(G) \cap K(H)$ and $C_0(F) \cap C(G) \cap K(H)$ is nonempty then its closure is $C(S^0)$.

**Theorem 3.1.** The following implications are verified:

1. Linearlike $\Rightarrow$ Reverse $\Rightarrow$ Zangwill.
2. Linearlike objectives $\Rightarrow$ Zangwill.
3. Slater $\Rightarrow$ Differentiable Slater $\Rightarrow$ Cottle.
4. PICMF $\Leftrightarrow$ Global Cottle $\Leftrightarrow$ PIOMF $\Leftrightarrow$ PCMF.
5. Global Cottle $\Rightarrow$ QIOMF $\Rightarrow$ Cottle.
6. Cottle and (H2) $\Rightarrow$ Generalized Abadie.
7. Global Cottle and (H2) $\Rightarrow$ Kuhn-Tucker.
8. a) Zangwill $\Rightarrow$ Kuhn-Tucker $\Rightarrow$ Abadie $\Rightarrow$ Generalized Abadie.
   b) Generalized Abadie and (H4) $\Rightarrow$ Generalized Guignard.

The results above are summarized in Figure 1, which generalizes the similar figures in [2], Figure 6.4, [10], Figure 1, [13], Figure 1 and [7], Figure 1.

**Proof.**

1. a) Linearlike $\Rightarrow$ Reverse. It suffices to observe that if a function is linearlike, then it is pseudoconcave and, since it has continuous derivative, by Lemma 3.1(i), it follows that it is Hadamard differentiable, which implies (H1).

   b) Reverse $\Rightarrow$ Zangwill. Lemma 3.1(iii) shows that $Z(F \cap G, x_0) = C(F \cap G)$, and by Lemma 3.1(ii), $Z(H, x_0) = K(H)$. Thus

   $$Z(S^0) = Z(F \cap G \cap H, x_0) = Z(F \cap G, x_0) \cap Z(H, x_0) = C(F \cap G) \cap K(H) = C(S^0).$$

   Since (H1) is true, (H3) follows from Proposition 2.4.

2. Linearlike objectives $\Rightarrow$ Zangwill. From Proposition 2.5 it follows $C_0(G) \subset Z(G, x_0)$. Since $f$ is linearlike at $x_0$, by means of Lemma 3.1(iii), we get $Z(F, x_0) = \ldots$
$C(F)$. As $h$ is affine, $Z(H,x_0) = K(H)$. Hence we have

$$C(F) \cap C_0(G) \cap K(H) \subset Z(F,x_0) \cap Z(G,x_0) \cap Z(H,x_0) = Z(S^0,x_0) \subset T(S^0,x_0).$$

(7)

As $f$ is linearlike with continuous derivative, from Lemma 3.1(i), $f$ is Hadamard differentiable. Since $g_j$, $j \in J_0$, is Hadamard differentiable or Dini-quasiconvex, by Proposition 2.4, $T(S^T,x_0) \subset C(S^i)$ for all $i$ and $T(S^0,x_0) \subset C(S^0)$. Taking this last inclusion and (7) into account and using Lemma 3.2, we can conclude that $\text{cl} Z(S^0,x_0) = C(S^0)$.

3. a) Slater $\Rightarrow$ Differentiable Slater. For each problem (P)$\ast$ there exists $x_i$, verifying (5). In particular, $h_k(x_i) = h_k(x_0)$, and by the Dini-quasiconvexity of $h_k$ and $-h_k$, $\nabla h_k(x_0)(x_i - x_0) = 0$, $\forall k \in K$, which means that $x_i - x_0 \in K(H)$. 

b) Differentiable Slater $\Rightarrow$ Cottle. For each $i = 1, \ldots, p$, there exists $x_i$, verifying (6). Because of the pseudoconvexity of $f_j$ and $g_j$, we have

$$Df_j(x_0, x_i - x_0) < 0 \forall j \neq i, \; Dg_j(x_0, x_i - x_0) < 0 \forall j \in J_0.$$

By hypothesis, $x_i - x_0 \in K(H)$, and consequently $x_i - x_0 \in C_0(S^i)$.

4. a) PICMF $\Rightarrow$ Global Cottle. By assumption we have $C_0(G) \cap K(H) \neq \emptyset$, and hence from Proposition 2.3

$$\text{cl}[C_0(G) \cap K(H)] = C(G) \cap K(H).$$

Let $v \in C_0(F) \cap [C(G) \cap K(H)]$ (this set is nonempty by assumption). Since $C_0(F)$ is open, there exists a neighborhood $B(v)$ of $v$, such that $B(v) \subset C_0(F)$, and from (8), $B(v) \cap [C_0(G) \cap K(H)] \neq \emptyset$. So, we can state that $C_0(F) \cap \bar{C_0(G)} \cap K(H) \neq \emptyset$ and hence we have the global Cottle qualification.

b) Global Cottle $\Rightarrow$ PICMF. By hypothesis, $C_0(F) \cap C_0(G) \cap K(H) \neq \emptyset$, and therefore $C_0(F) \cap C(S) \neq \emptyset$ and $C_0(G) \cap K(H) \neq \emptyset$, that is, we have PICMF.

c) Global Cottle $\Leftrightarrow$ PIOMF. It is enough to note that $f$ and $g$ have a symmetric role in PIOMF and PICMF. So, if PICMF $\Leftrightarrow$ Global Cottle, then also PIOMF $\Leftrightarrow$ Global Cottle.

d) Global Cottle $\Leftrightarrow$ PCMF.

$\Rightarrow$ It is immediate, because there exists $v \in C_0(F) \cap C_0(G) \cap K(H)$. Then, $Df_i(x_0, v) < 0 \forall i \in I$.

$\Leftarrow$ By hypothesis, $\forall i \in I \exists v_i \in C(F) \cap C_0(G) \cap K(H)$ such that $Df_i(x_0, v_i) < 0$. Let be $v = \sum_{i=1}^p \lambda_i v_i$ with $\lambda_i = 1/p$. We shall see that $v \in C_0(F) \cap C_0(G) \cap K(H)$, which implies global Cottle.

By the convexity of $Df_j(x_0, \cdot)$ we have $Df_j(x_0, v) \leq \sum_{i=1}^p \lambda_i Df_j(x_0, v_i) < 0$, since $Df_j(x_0, v_i) \leq 0 \forall j \neq i$ and $Df_j(x_0, v_i) < 0$. Analogously, $Dg_j(x_0, v) < 0 \forall j \in J_0$ and by the linearity of $\nabla h_k(x_0)$, $\nabla h_k(x_0)v = 0$.

5. a) Global Cottle $\Rightarrow$ QIOMF. As global Cottle is equivalent to PIOMF, it is enough to get that PIOMF implies QIOMF. But this is obvious, because if $C_0(F) \cap K(H) \neq \emptyset$, then for each $i = 1, \ldots, p$ $C_0(F^i) \cap K(H) \neq \emptyset$. 
b) QIOMF ⇒ Cottle. Assume that there exists $i \in I$ such that $C_0(S^i) = \emptyset$. This means that there is no solution $v \in \mathbb{R}^n$ of the system

\[
\begin{align*}
Df_j(x_0, v) &< 0 \quad \forall j \neq i \\
Dg_j(x_0, v) &< 0 \quad \forall j \in J_0 \\
\nabla h_k(x_0)v & = 0 \quad \forall k \in K.
\end{align*}
\]

Using Theorem 3.5 in [8] we obtain that there exists $(\lambda, \mu, \nu) \in \mathbb{R}^{p-1} \times \mathbb{R}^{J_0} \times \mathbb{R}^r$ such that $(\lambda, \mu) \geq 0$, $(\lambda, \mu) \neq 0$ and

\[
\sum_{j \neq i} \lambda_j Df_j(x_0, v) + \sum_{j \in J_0} \mu_j Dg_j(x_0, v) + \sum_{k=1}^r \nu_k \nabla h_k(x_0)v \geq 0 \quad \forall v \in \mathbb{R}^n.
\]

(9)

By hypothesis, there exists $u \in C(F) \cap C_0(G) \cap K(H)$. If for some $j \in J_0$, $\mu_j > 0$, then $\sum_{j \neq i} \lambda_j Df_j(x_0, u) + \sum_{j \in J_0} \mu_j Dg_j(x_0, u) < 0$, in contrast with the result obtained in (9) with $v = u$. Thus $\mu = 0$ and in (9) we have therefore that $\sum_{j \neq i} \lambda_j Df_j(x_0, v) + \sum_{k=1}^r \nu_k \nabla h_k(x_0)v \geq 0 \quad \forall v \in \mathbb{R}^n$. By hypothesis, there exists $w \in C_0(F^i) \cap K(H)$ and an analogous argument shows that $\lambda = 0$, which is a contradiction.

6. Cottle and (H2) ⇒ Generalized Abadie. It is enough to apply Proposition 2.6 to each set $S^i$.

7. Global Cottle and (H2) ⇒ Kuhn-Tucker. To prove this result, it is sufficient to take the implication Global Cottle ⇒ Cottle into account and to apply Proposition 2.6 to each $S^i$ and to $S^0$.

8. a) Zangwill ⇒ Kuhn-Tucker ⇒ Abadie ⇒ Generalized Abadie. Since $S^0 \subset S^i \forall i \in I$, from the isotonicity of the tangent cone and from (H3) it follows that

\[
T(S^i, x_0) \subset \cap_{i=1}^p T(S^i, x_0) \subset \cap_{i=1}^p C(S^i) = C(S^0).
\]

Now, from equation (2) applied to $S^0$, the three implications follow.

b) Generalized Abadie and (H4) ⇒ Generalized Guignard. Obviously, $T(S^i, x_0) \subset \text{cl co} T(S^i, x_0) \subset C(S^i)$ (the last inclusion is due to the convexity of the derivatives and to (H3)). Therefore we get

\[
\cap_{i=1}^p T(S^i, x_0) \subset \cap_{i=1}^p \text{cl co} T(S^i, x_0) \subset \cap_{i=1}^p C(S^i) = C(S^0)
\]

and the implication is then evident. 

\[\Box\]

**Remark 3.1.**

(1) It is known that if $f$ is Hadamard differentiable at $x_0$ and $x_0$ is a local Pareto minimum of $f$ over $S$, then

\[
C_0(F) \cap T(S, x_0) = \emptyset.
\]

(10)
(2) If for some \( i \in I \), \( f_i \) is Hadamard differentiable at \( x_0 \), (H0), (H2) and (H4) hold, and \( x_0 \in \text{LMin}(f,S) \), then \( C_0(S^0) = \emptyset \) (and consequently, Global Cottle qualification is not satisfied at \( x_0 \)). Indeed one has that \( x_0 \) is a local solution to problem (P_i), i.e., \( x_0 \in \text{LMin}(f_i,S_i) \), and by the previous remark,
\[
C_0(f_i) \cap T(S^i,x_0) = \emptyset, \tag{11}
\]
where \( C_0(f_i) = \{ v \in \mathbb{R}^n : df_i(x_0,v) < 0 \} \).
Assume that \( C_0(S^0) \neq \emptyset \), then \( C_0(S^i) \neq \emptyset \). By Proposition 2.6, \( T(S^i,x_0) = C(S^i) \). From (11), it follows that \( C_0(f_i) \cap C_0(S^i) \subset C_0(f_i) \cap C(S^i) = \emptyset \). But \( C_0(S^0) = C_0(f_i) \cap C_0(S^i) = \emptyset \) and we have a contradiction.

So, Global Cottle is not a true constraint qualification when some \( f_i \) is Hadamard differentiable and (H2) holds.

(3) If there is no equality constraints, then global Cottle is not verified at a local Pareto minimum (from Prop. 2.5 and Lem. 4.1 in the next section) and, consequently, neither is (AMFCQ) in [13].

4. Optimality conditions under generalized qualifications

In this section Kuhn-Tucker type necessary optimality conditions are given for a point to be local Pareto minimum. These conditions are obtained both in primal form and in dual form, with a feasible set defined by inequality and equality constraints, the objective functions and the constraints being, at least, Dini differentiable. In order to obtain the positivity of the multipliers associated with the vector-valued objective function, a generalized constraint qualification will be assumed. In this way we generalize Maeda’s results [10], which are valid for differentiable functions and without equality constraints, and Preda and Chitescu’s [13] who consider a problem with Dini differentiable functions and without equality constraints. We generalize also the results of Jiménez and Novo [7], valid for differentiable problems with equality constraints.

**Theorem 4.1.** Let \( f \) be Hadamard differentiable at \( x_0 \), \( g_j \), \( j \in J_0 \), Dini differentiable at \( x_0 \) and \( h \) Fréchet differentiable at \( x_0 \) and suppose that the generalized Abadie qualification is verified. If \( x_0 \in \text{LMin}(f,S) \), then there exists no solution \( v \in \mathbb{R}^n \) of the system
\[
\begin{align*}
Df(x_0,v) &\leq 0, \quad Df(x_0,v) \neq 0 \\
Dg_j(x_0,v) &\leq 0 \quad \forall j \in J_0 \\
\nabla h(x_0)v & = 0,
\end{align*}
\tag{12}
\]
i.e., \( x_0 \) is a proper local solution to problem (MOP) in the sense of Kuhn-Tucker.
Proof. Assume that the conclusion is not true. Then there exist \( v \in \mathbb{R}^n \) and \( i \in \{1, \ldots, p\} \) such that

\[
\begin{cases}
Df_i(x_0, v) < 0 \\
Df_j(x_0, v) \leq 0 \ \forall j \neq i \\
v \in C(S).
\end{cases}
\] (13)

Thus \( v \in C(S^0) \) and, by the generalized Abadie qualification, \( v \in T(S', x_0) \). Since \( x_0 \) is a local Pareto minimum, it also is a local minimum of each scalar problem (P\(_j\)), in particular, \( x_0 \in \text{LMin}(f_i, S') \). As \( f_i \) is Hadamard differentiable, we have \( Df_i(x_0, u) \geq 0 \ \forall u \in T(S', x_0) \). Taking \( u = v \), then \( Df_i(x_0, v) \geq 0 \), in contradiction to (13).

**Theorem 4.2.** Let \( f \) and \( h \) be Fréchet differentiable at \( x_0 \) and \( g_j, \ j \in J_0 \), Dini differentiable at \( x_0 \) and suppose that the generalized Guignard qualification is verified. If \( x_0 \in \text{LMin}(f, S) \), then there is no solution \( v \in \mathbb{R}^n \) of the system (12).

Proof. Assume that (13) is true for some \( i \in \{1, \ldots, p\} \) and \( v \in \mathbb{R}^n \). Thanks to the generalized Guignard qualification, \( v \in \text{clco} \ T(S', x_0) \). Since \( x_0 \in \text{LMin}(f_i, S') \) and \( f_i \) is Fréchet differentiable, we obtain \( \nabla f_i(x_0)u \geq 0 \ \forall u \in T(S', x_0) \) (\( Df_i(x_0, u) = \nabla f_i(x_0)u \)). By the linearity and the continuity of \( \nabla f_i(x_0)(\cdot) \), it follows that \( \nabla f_i(x_0)u \geq 0 \ \forall u \in \text{clco} \ T(S', x_0) \). Taking \( u = v \) we have a contradiction to (13).

It is possible to obtain the dual form of these two last theorems by applying the generalized Tucker alternative theorem (Prop. 2.2).

**Theorem 4.3.** Assume the hypothesis of Theorems 4.1 or 4.2 and let the derivatives \( Df(x_0, \cdot) \) and \( Dg_j(x_0, \cdot) \), \( j \in J_0 \), be convex. If the cones

\[
D_i = \text{cone co}(\cup_{j \neq i} \partial_D f_j(x_0)) + \text{cone co}(\cup_{j \in J_0} \partial_D g_j(x_0)) + \text{lin}\{\nabla h_k(x_0) : k \in K\}
\] (14)

\( i = 1, \ldots, p \), are closed, then there exists \((\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r \) such that

\[
\begin{align*}
(a) & \ \lambda > 0, \ \mu \geq 0, \ \mu_j g_j(x_0) = 0, \ j = 1, \ldots, m; \\
(b) & \ \sum_{i=1}^{p} \lambda_i Df_i(x_0, v) + \sum_{j=1}^{m} \mu_j Dg_j(x_0, v) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0)v \geq 0 \ \forall v \in \mathbb{R}^n.
\end{align*}
\] (15)

As usual, we take \( \mu_j = 0 \) if \( g_j(x_0) < 0 \). Note that condition (b) is equivalent to

\[
0 \leq \sum_{i=1}^{p} \lambda_i \partial_D f_i(x_0) + \sum_{j=1}^{m} \mu_j \partial_D g_j(x_0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0).
\] (16)
If we denote by $L$ the Lagrangian function: 

$$L = \sum_{i=1}^{p} \lambda_i f_i + \sum_{j=1}^{m} \mu_j g_j + \sum_{k=1}^{r} \nu_k h_k,$$

then (16) is equivalent to

$$0 \in \partial_D L(x_0).$$

Theorem 4.3 generalizes Corollary 8 in [7] by Jiménez and Novo.

Now we investigate about the conditions on the functions of the problem, which assume that the cones (14) are closed. One of these criteria is given below.

**Proposition 4.1.** If for each $i = 1, \ldots, p$, $C_0(S^i) \neq \emptyset$, then the cones $D_i$, $i = 1, \ldots, p$, given by (14), are closed.

This follows from Proposition 3.6 in [8].

We remark that if the Cottle qualification holds, then it is unnecessary to use the generalized Tucker alternative theorem to obtain positive multipliers, since this result can directly be obtained.

**Proposition 4.2.** Let $f$ be Hadamard differentiable at $x_0$ with convex derivative and suppose that the Cottle qualification and (H2) are satisfied. If $x_0 \in \text{LMin}(f,S)$, then conditions (15) hold.

**Proof.** From Theorem 3.1, part 6, the generalized Abadie qualification is verified and, by Theorem 4.1, the system (12) does not admit solution.

For each $i = 1, \ldots, p$ we have $x_0 \in \text{LMin}(f_i,S^i)$, hence $Df_i(x_0, v) \geq 0 \forall v \in T(S^i, x_0)$. As it was seen in the proof of Theorem 3.1, part 6, for each $i = 1, \ldots, p$, $T(S^i, x_0) = C(S^i)$. Therefore, none of the $p$ systems ($i = 1, \ldots, p$):

$$\begin{cases} 
Df_i(x_0, v) < 0 \\
Df_j(x_0, v) \leq 0 \forall j \neq i \\
Dg_j(x_0, v) \leq 0 \forall j \in J_0 \\
\nabla h_k(x_0)v = 0 \forall k \in K
\end{cases}$$

has a solution $v \in \mathbb{R}^n$. Let us consider the convex problem

$$(\text{CP}_i) \quad \alpha_i = \min \{Df_i(x_0, v) : Df_i(x_0, v) \leq 0 \forall j \neq i, \quad Dg_j(x_0, v) \leq 0 \forall j \in J_0, \quad \nabla h_k(x_0)v = 0 \forall k \in K\}.$$

Because of the incompatibility of the system (18) above, we have $\alpha_i \geq 0$. Since $v = 0$ is a feasible solution and $Df_i(x_0, 0) = 0$, it is $\alpha_i = 0$. From Theorem 28.2 in [14] (we can use it because $C_0(S^i) \neq \emptyset$), it follows that there exist $\lambda_{ij} \geq 0$, $j \neq i$; $\mu_{ij} \geq 0$, $j \in J_0$; $\nu_{ik} \in \mathbb{R}$, $k \in K$ such that

$$Df_i(x_0, v) + \sum_{j=1, j \neq i}^{p} \lambda_{ij} Df_j(x_0, v) + \sum_{j \in J_0} \mu_{ij} Dg_j(x_0, v) + \sum_{k=1}^{r} \nu_{ik} \nabla h_k(x_0)v \geq 0$$
for all \( v \in \mathbb{R}^n \), and for \( i = 1, \ldots, p \). Adding over \( i = 1, \ldots, p \), we have

\[
\sum_{i=1}^{p} \lambda_i Df_i(x_0, v) + \sum_{j \in J_0} \mu_j Dg_j(x_0, v) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0)v \geq 0 \text{ } \forall v \in \mathbb{R}^n,
\]

where, in order to simplify, we have denoted \( \lambda_i = 1 + \sum_{j=1,j \neq i}^{p} \lambda_{ij}, i = 1, \ldots, p \);
\( \mu_j = \sum_{i=1}^{p} \mu_{ij}, j \in J_0 \); \( \nu_k = \sum_{i=1}^{p} \nu_{ik}, k = 1, \ldots, r \), and obviously we have \( \lambda > 0 \), and \( \mu \geq 0 \).

As a consequence of Theorem 4.3 we obtain the following corollary, which extends Maeda’s Theorem 3.2 [10] for a problem with differentiable functions and also equality constraints.

**Corollary 4.1** ([7], (Cor. 8)). Let \( f, g \) and \( h \) be Fréchet differentiable at \( x_0 \) and suppose that the generalized Guignard qualification is verified. If \( x_0 \in \text{LMin}(f, S) \), then there exist \( (\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r \) such that

(a) \( \lambda > 0, \mu \geq 0, \mu_j g(x_0) = 0, j = 1, \ldots, m \);
(b) \( \sum_{i=1}^{p} \lambda_i \nabla f_i(x_0) + \sum_{j=1}^{m} \mu_j \nabla g_j(x_0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) = 0 \).

**Proof.** Under these assumptions, the condition “for each \( i = 1, \ldots, p \), \( D_i \) is closed” is verified, since

\[
D_i = \text{cone} \{ \nabla f_j(x_0) : j \neq i \} \cup \{ \nabla g_j(x_0) : j \in J_0 \} + \text{lin} \{ \nabla h_k(x_0) : k \in K \}
\]

is a polyhedral convex cone and, therefore, it is closed. \( \square \)

The following example shows that Cottle qualification may not be verified; however, we can apply Theorem 4.3.

**Example 4.1.** In \( \mathbb{R}^3 \), let \( x_0 = (0, 0, 0) \), \( f_1 = 2x - 2z \), \( f_2 = -2y \) and let \( g \) be the support function of the set \( B = \{(x, y, z) : x^2 + (y-2)^2 + z^2 \leq 2, z \geq 0 \} \). We obtain the following expression of \( g \):

\[
g(x, y, z) = \begin{cases} 
2y + \sqrt{2x^2 + 2y^2 + 2z^2} & \text{if } z \geq 0 \\
2y + \sqrt{2x^2 + 2y^2} & \text{if } z < 0.
\end{cases}
\]

Obviously \( Dg(x_0, v) = g(v) \) and \( \partial Dg(x_0) = B \). The feasible set is \( G = \{(x, y, z) : g(x, y, z) \leq 0 \} \) and the point \( x_0 \) is a Pareto minimum of \( f = (f_1, f_2) \) over \( G \). Cottle qualification is not verified because \( C_0(S^2) = \{ v : \nabla f_2(x_0) v < 0, Dg(x_0, v) < 0 \} = \emptyset \), but we can apply Theorem 4.3, since the cones \( D_1 = \text{cone} \{ \{(0, -2, 0) \} \cup B \} = \{(x, y, z) : z \geq 0 \} \) and \( D_2 = \text{cone} \{ (2, 0, -2) \} \cup B \) are closed \( (D_2 \) is closed because \( C_0(S^2) \neq \emptyset \), since \( (-1, -2, 0) \in C_0(S^2) \)). So (16) holds, with \( (\lambda_1, \lambda_2, \mu) = (1, 2, 2) \) and \( b = (-1, 2, 1) \in \partial Dg(x_0) \).

Finally, we establish necessary optimality conditions without equality constraints; moreover, we do not require the objective functions to be Hadamard differentiable as in Theorems 4.1 and 4.2.
In Lemma 4.1 and Theorem 4.4 the following definition is used: \( \varphi : \mathbb{R}^n \to \mathbb{R} \) is quasiconvex at \( x_0 \) on a neighborhood of \( x_0 \) if there exists \( \delta > 0 \) such that
\[ x \in B(x_0, \delta), \varphi(x) \leq \varphi(x_0) \Rightarrow \varphi(\lambda x_0 + (1 - \lambda)x) \leq \varphi(x_0) \quad \forall \lambda \in (0, 1). \]

**Lemma 4.1.** If \( f \) has continuous Dini derivative at \( x_0, g_j, j \in J_0, \) are quasiconvex at \( x_0 \) on a neighborhood of \( x_0 \) and \( x_0 \in \text{LMin}(f, G), \) then
\[ T(G, x_0) \cap C_0(F) = \emptyset. \]

**Proof.** Assume that \( T(G, x_0) \cap C_0(F) \neq \emptyset \) and choose \( v \in T(G, x_0) \cap C_0(F). \) Thus, \( Df(x_0, v) \neq 0. \) Using Lemma 4.10 in [8], \( \text{cl} \ Z(G, x_0) = T(G, x_0). \) Hence there exist \( v_n \in Z(G, x_0) \) such that \( v_n \to v. \) By the continuity of \( Df(x_0, \cdot), \) we can suppose that \( Df(x_0, v_n) < 0 \forall n \in \mathbb{N}. \) By the definition of Dini derivative, \( Df(x_0, v_n) = \lim_{\epsilon \to 0^+} (f(x_0 + t v_n) - f(x_0))/t < 0. \) Therefore, there exists \( \delta_n > 0 \) such that \( \forall t \in (0, \delta_n), f(x_0 + tv_n) - f(x_0) < 0. \) As \( v_n \in Z(G, x_0), \) there exists \( \eta_n > 0 \) such that \( \forall t \in (0, \eta_n), x_0 + tv_n \in G. \) Let us choose \( \varepsilon_n \) such that \( 0 < \varepsilon_n \leq \text{Min}\{\delta_n, \eta_n\} \) and \( \varepsilon_n \to 0^+. \) The sequence \( x_n = x_0 + \varepsilon_n v_n \to x_0, x_n \in G \) and \( f(x_n) < f(x_0), \) in contradiction with the minimality of \( x_0. \)

**Theorem 4.4.** Let \( S = G \) and assume that:
(a) \( f_i, i \in I, g_j, j \in J_0, \) are quasiconvex at \( x_0 \) on a neighborhood of \( x_0. \)
(b) \( f_i, i \in I, g_j, j \in J_0, \) are Dini differentiable at \( x_0, \) with \( Df_i(x_0, \cdot), i \in I, \) linear and \( Dg_j(x_0, \cdot), j \in J_0, \) convex.
(c) The generalized Guignard qualification holds.
If \( x_0 \in \text{LMin}(f, G), \) then the system
\[
\begin{align*}
Df_i(x_0, v) &\leq 0, \quad Df_i(x_0, v) \neq 0 \\
Dg_j(x_0, v) &\leq 0 \quad \forall j \in J_0
\end{align*}
\]
has no solution \( v \in \mathbb{R}^n. \)

**Proof.** Assume the thesis does not hold. Then there exist \( v \in \mathbb{R}^n \) and \( i \in \{1, \ldots, p\} \) such that
\[
\begin{align*}
Df_i(x_0, v) &< 0 \\
Df_j(x_0, v) &\leq 0 \quad \forall j \neq i \\
Dg_j(x_0, v) &\leq 0 \quad \forall j \in J_0.
\end{align*}
\]
(19)
Therefore \( v \in C(S^i) \) and, using condition (c), \( v \in \cap_{j=1}^p \text{cl co} T(S^j, x_0). \) Consequently, \( v \in \text{cl co} T(S^i, x_0). \) Since \( x_0 \in \text{LMin}(f, S), \) it follows that \( x_0 \in \text{LMin}(f_i, S^i). \) Since \( Df_i(x_0, \cdot) \) is continuous (it is linear) and as hypothesis (a) holds, we can apply Lemma 4.1, obtaining \( Df_i(x_0, u) \geq 0 \forall u \in T(S^i, x_0). \) Moreover, by the linearity, we deduce that \( Df_j(x_0, u) \geq 0 \forall u \in \text{cl co} T(S^j, x_0). \) In particular, taking \( u = v, \) we obtain \( Df_i(x_0, v) \geq 0, \) contradicting (19).

This theorem improves Theorem 3.1 in Preda and Chitescu [13] because quasiconcavity of \( f_i \) is not required.
The following theorem can be proved in a similar. From this result, we obtain subsequently (Th. 4.6) necessary optimality conditions in dual form.

**Theorem 4.5.** Assume the hypotheses of Theorem 4.4, with \( Df_i(x_0, \cdot) \), \( i \in I \), convex (instead of linear) and

(\text{c'}) The generalized Abadie qualification holds, (instead of (c)). Then the same conclusion of Theorem 4.4 holds.

**Theorem 4.6.** Suppose that the hypotheses of Theorems 4.4 or 4.5 hold true. If for each \( i = 1, \ldots, p \) the cone

\[
D_i = \text{cone} \left( \cup_{j \neq i} \partial D f_j(x_0) \right) + \text{cone} \left( \cup_{j \in I_0} \partial D g_j(x_0) \right)
\]

is closed, then there exists \((\lambda, \mu) \in \mathbb{R}^p \times \mathbb{R}^m\) such that

(a) \( \lambda > 0, \mu \geq 0, \mu_j g_j(x_0) = 0, \quad j = 1, \ldots, m; \)

(b) \( \sum_{i=1}^{p} \lambda_i Df_i(x_0, v) + \sum_{j=1}^{m} \mu_j Dg_j(x_0, v) \geq 0 \quad \forall v \in \mathbb{R}^n. \)

Expressions similar to equations (16) and (17) can also be obtained for (b).

This theorem corrects Theorem 3.2 in Preda and Chitescu [13], which is not true, as the following counterexample shows.

**Example 4.2.** Let us consider the problem

\[
\text{Min } f(x) \text{ subject to } g(x) \leq 0,
\]

where \( f : \mathbb{R}^3 \to \mathbb{R}^2 \) is given by \( f(x, y, z) = (x, -y) \) and \( g(x, y, z) = y + z + \sqrt{x^2 + y^2 + z^2} \). We are going to study the minimality conditions at the point \( x_0 = (0, 0, 0) \).

The feasible set \( S = \{(x, y, z) : g \leq 0\} \) is the regular cone (in the sense of elementary geometry) with axis the halfline \((x, y, z) = \lambda(0, -1, -1), \lambda \geq 0\) and whose generatrices form an angle of \( \pi/4 \) with the axis of the cone. We have

\[
S^1 = \{(x, y, z) : f_2 \leq 0, \quad g \leq 0\} = \{(0, 0, z) : z \leq 0\}, \\
S^2 = \{(x, y, z) : f_1 \leq 0, \quad g \leq 0\} = S \cap \{x \leq 0\}, \\
S^0 = \{(x, y, z) : f_1 \leq 0, \quad f_2 \leq 0, \quad g \leq 0\} = S^1, \\
\nabla f_1(x_0) = (1, 0, 0), \nabla f_2(x_0) = (0, -1, 0), \quad Dg(x_0, v) = g(v).
\]

We are under the hypotheses of Theorem 3.2 in Preda and Chitescu [13]:

1. The point \( x_0 \) is a Pareto minimum, since \( f(x_0) = (0, 0) \) and if \((x, y, z) \in S \), then \( f(x, y, z) - f(x_0) \in -\mathbb{R}_+^2 \setminus \{0\} \) is not true (note that \( S \) is inside the dihedral \( \{(x, y, z) : y \leq 0, \quad z \leq 0\} \) and it cuts the plane \( y = 0 \) only in the halfline \( OZ^- \)).
2. The generalized Guignard qualification is true, because \( \text{cl} \text{co} T(S^1, x_0) \cap \text{cl} \text{co} T(S^2, x_0) = S^1 \) and \( C(S^0) = S^0 = S^1 \).
3. \( f_1, f_2 \) are linear, and consequently they are quasiconvex and quasiconcave; \( g \) is convex, and therefore, it also is quasiconvex.
4. \( f_1, f_2 \) are Fréchet differentiable, thus their Dini derivatives are linear, and hence, concave and convex. \( g \) is Hadamard differentiable and its derivative at \( x_0 \) is \( g \) itself, which is a convex function.
However, there exist no $\lambda_1 > 0$, $\lambda_2 > 0$, $\mu \geq 0$ such that
\[
\lambda_1 \nabla f_1(x_0)v + \lambda_2 \nabla f_2(x_0)v + \mu Dg(x_0,v) \geq 0 \quad \forall v \in \mathbb{R}^3.
\]  
(20)

In fact, the unique solution with $\mu = 0$ is $\lambda_1 = \lambda_2 = 0$. Then, let $\mu > 0$; we can assume $\mu = 1$ and put $v = (x, y, z)$. Then (20) is equivalent to
\[
\langle (-\lambda_1, \lambda_2 - 1, -1), (x, y, z) \rangle \leq \| (x, y, z) \| \quad \forall (x, y, z) \in \mathbb{R}^3.
\]
This expression means that $(-\lambda_1, \lambda_2 - 1, -1)$ is a subgradient at 0 of the convex function $x \mapsto \| x \|$. But $\partial \| (0) = \mathop{cl} B(0, 1)$ and therefore $\lambda_1^2 + (\lambda_2 - 1)^2 + 1 \leq 1$, and the only solution is $(\lambda_1, \lambda_2) = (0, 1)$.

Note that Theorem 4.6 cannot be applied, because the cone
\[
D_1 = \text{cone} \{ \nabla f_2(x_0) \} + \text{cone} \partial Dg(x_0) = \{ (x, y, z) : z > 0 \text{ or } (z = 0, x = 0) \}
\]
is not closed, being $\partial Dg(x_0) = \mathop{cl} B(b_0, 1)$ with $b_0 = (0, 1, 1)$.

REFERENCES