EVRIPIDIS BAMPIS

The complexity of short schedules for uet bipartite graphs

RAIRO. Recherche opérationnelle, tome 33, n°3 (1999), p. 367-370

<http://www.numdam.org/item?id=RO_1999__33_3_367_0>
THE COMPLEXITY OF SHORT SCHEDULES
FOR UET BIPARTITE GRAPHS (*)

by Evripidis BAMPIS (1)

Communicated by Philippe CHRÉTIENNE

Abstract. – We show that the problem of deciding if there is a schedule of length three for the
multiprocessor scheduling problem on identical machines and unit execution time tasks in NP-
complete even for bipartite graphs, i.e. for precedence graphs of depth one. This complexity result
extends a classical result of Lenstra and Rinnoy Kan [5].

Keywords: Schedule, multiprocessor, bipartite graph, complexity.

1. INTRODUCTION

This note deals with the difficulty of obtaining near-optimal solutions
for the multiprocessor scheduling problem subject to precedence delays:
Given $n$ unit execution time tasks whose precedence constrains form a
directed acyclic graph and $m$ identical machines the objective is to find a
schedule minimizing the makespan, i.e. the time at which the last task of
the precedence graph completes its execution. Using the three field notation
[3] this problem can be denoted as:

$$P|\text{prec, } p_j = 1|C_{\text{max}}.$$
This problem has been extensively studied in the literature. Graham [1] proved that any list scheduling strategy is a 2-approximation algorithm. Ullman [8] proved that this problem is NP-hard. Furthermore, Lenstra and Rinnoy Kan [5] showed that the problem of deciding if there is a feasible schedule of length three is NP-complete, and thus there is no hope of finding an approximation algorithm with performance guarantee less than $\frac{4}{3}$, unless $P = NP$. The proof is based on a reduction of the CLIQUE problem and was presented in some of the most classical books of Computer Science (see for instance [6]). We extend this classical result by showing that it is true even if the precedence graph is a bipartite graph, i.e. a precedence graph with depth one.

2. THE NP-COMPLETENESS RESULT

In this section we prove the following theorem:

**Theorem:** The problem $P|prec = \text{bipartite}, \ p_j = 1|C_{\text{max}} \leq 3$ is NP-complete.

**Proof:** The proof is based on the balanced bipartite independent set problem (BBIS) whose NP-completeness is clearly implied by the well known NP-complete problem balanced bipartite complete graph (BBCG) [2].

**Instance of BBIS:** An undirected balanced bipartite graph $B = (X \cup Y, E)$, with $|X| = |Y| = n$, and an integer $k$.

**Question:** Is there in $B$, an independent set with $k$ vertices in $X$ and $k$ vertices in $Y$?

If such an independent set exists, we call it balanced independent set of order $k$.

The problem remains NP-complete even if $k = \frac{n}{2}$ [9]. In the following we consider BBIS as the source problem.

Given any instance of BBIS, i.e. a balanced bipartite graph $B = (X \cup Y, E)$, with $|X| = |Y| = n$, we construct an instance of our scheduling problem in the following way:

There are $m = n$ identical processors to execute $3n$ unit execution time tasks. The precedence graph $G$ that we consider contains:

1. $B = (X \cup Y, E)$, with $|X| = |Y| = n$, and where the edges between $X$ and $Y$ are replaced by the corresponding arcs directed from the vertices of $X$ towards the vertices of $Y$.

2. A complete bipartite graph $A = (W \cup Z, E')$, with $|W| = |Z| = \frac{n}{2}$ and where all the vertices of $W$ precede all the vertices of $Z$.
3. All the vertices of $W$ precede all the vertices of $Y$ and all the vertices of $X$ precede all the vertices of $Z$.

Notice that $G$ is bipartite, and that the corresponding instance of $P|\text{prec} = \text{bipartite}, p_j = 1|C_{\text{max}} \leq 3$ may be computed in polynomial time.

We have now to show that $B$ contains a balanced independent set with $\frac{n}{2}$ vertices in $X$ and $\frac{n}{2}$ vertices in $Y$ if and only if there is a feasible schedule of $G$ with length three.

Let us first consider that $B$ contains such a balanced independent set, call it $S = X_1 \cup Y_1$ with $X_1 \subset X$, $Y_1 \subset Y$, and $|X_1| = |Y_1| = \frac{n}{2}$. Then there is a schedule of $G$ with length three:

$t = 1$: execute all the tasks of $X - X_1$ and all the tasks of $W$.

$t = 2$: execute all the tasks of $X_1$ and all the tasks of $Y_1$.

$t = 3$: execute all the tasks of $Y - Y_1$ and all the tasks of $Z$.

It is easy to verify that this schedule is feasible since all the precedence constraints are satisfied. The feasibility of the schedule is guaranteed from the existence of the a balanced independent set $(X_1, Y_1)$.

Conversely, let us now assume that there is a feasible schedule of $G$ of length three. We have to show that any schedule of this length necessitates the existence of a balanced independent set $X'_1 \cup Y'_1$ with $X'_1 \subset X$, $Y'_1 \subset Y$, and $|X'_1| = |Y'_1| = \frac{n}{2}$ in $B$.

The key point of the proof is that all the tasks of $W$ (resp. $Z$) have to be executed during the first (resp. third) time unit. This is clear since every task of $W$ (resp. $Z$) precedes (resp. is preceded by) exactly $\frac{3n}{2}$ tasks and there is only $n$ machines. Thus, only $\frac{n}{2}$ tasks of $X$ can be executed at the first time unit. Notice that no task of $Y$ can be executed in the first time unit because of the constraints coming from $W$. Similarly, all the task of $X$ have to finish no later than the second time unit since they precede all the tasks of $Z$. Given that during the third time unit there are exactly $\frac{n}{2}$ free machines, we get that there are exactly $\frac{n}{2}$ tasks of $Y$ and $\frac{n}{2}$ tasks of $X$ executed in the second time unit. From the feasibility of the schedule we can conclude that these tasks form a balanced independent set $X'_1 \cup Y'_1$ with $X'_1 \subset X$, $Y'_1 \subset Y$, and $|X'_1| = |Y'_1| = \frac{n}{2}$ in $B$, as required.

3. CONCLUDING REMARKS

We proved that the problem $P|\text{prec} = \text{bipartite}, p_j = 1|C_{\text{max}} \leq 3$ is $NP$-complete. This problem is interesting to compare to $P|\text{prec}, p_j = 1, c_{ij} = 1|C_{\text{max}} \leq 3$, which is polynomial [7], and to $P|\text{prec} = \text{bipartite}, p_j = 1, C_{ij} = 1|C_{\text{max}} \leq 4$ which is $NP$-complete [4].

vol. 33, n° 3, 1999
REFERENCES