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A sample problem


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ON THE SOLUTION OF A CLASS OF LOCATION PROBLEMS.
A SAMPLE PROBLEM (*)

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Abstract. — In Section 2 of this paper the optimality conditions for the EMFL (Euclidean Multi-Facility Location) minisum problem are shown to be a useful tool for finding the analytical solution of many simple problems. In Section 3 the problem of connecting by means of two new facilities the vertices of an isocèle triangle is completely solved. © Elsevier, Paris

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1. INTRODUCTION

After the optimality conditions for the general EMFL minisum problem have been stated (see [1], [2], [3]), the analytical solution of many simple problems has become possible. The form of the optimality conditions presented in [2] gives an explicit expression of the subdifferential of the objective function and represents a useful tool for this purpose, suggesting the method proposed in Section 2. The problem solved in Section 3, as well as the case considered in [4], show that the analytical solution of some symmetrical problems may present simple and aesthetic features.

2. THE OPTIMALITY CONDITIONS AND THE ANALYTICAL SOLUTION OF THE EMFL MINISUM PROBLEM

The optimality conditions in the form presented in [2] can be summarized as follows.

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Let $X_1, \ldots, X_m$ ($X_j = (x_j, y_j)$) be the New Facilities, $A_1, \ldots, A_n$ the Existing Facilities, and $H$ the graph having $X_j, A_r$ as nodes, and the links interconnecting them as edges. The edges are assumed to be ordered, for example in lexicographic order $X_1X_2, X_1X_3 \ldots X_1X_m, X_2X_3, \ldots, X_{m-1}X_m, X_1A_1, \ldots, X_mA_n$ (of course, only the edges present in $H$ appear in the sequence).

The EMFL minisum problem can be stated as

$$F(X) = F(X_1, \ldots, X_m) = \sum_{(j,r) \in \Omega} w_{jr}\|X_j - X_r\|_2$$

$$+ \sum_{(j,r) \in \overline{\Omega}} \overline{w}_{jr}\|X_j - A_r\|_2 = \min$$

where $\Omega, \overline{\Omega}$ are respectively the sets of the pairs $(j, r)$ such that the edge $X_jX_r$ (or $X_jA_r$) exists in $H$, and $w_{jr}$ (or $\overline{w}_{jr}$) are the corresponding positive weights. Let $\bar{X} = [\bar{X}_1, \ldots, \bar{X}_m]$ be a given point of $R^{2m}$. Two facilities $\bar{X}_j, \bar{X}_r$ (or $\bar{X}_j, A_d$) adjacent in $H$ are “Interacting” if they overlap, i.e. $\bar{X}_j = \bar{X}_r$ (or $\bar{X}_j = A_d$). The corresponding zero-length edges $\bar{X}_j\bar{X}_r$ (or $\bar{X}_jA_d$) are “active edges”. As has been proved in [2], the components of the subdifferential set $\partial F(\bar{X}) = [x_1^*, y_1^*, \ldots, x_m^*, y_m^*]$ can be characterized as follows:

$$\begin{bmatrix} x_j^* \\ y_j^* \end{bmatrix} = \begin{bmatrix} g_{jx} \\ g_{jy} \end{bmatrix} - \sum_{r \in \Sigma_j^-} \begin{bmatrix} u_{rj} \\ v_{rj} \end{bmatrix} + \sum_{r \in \Sigma_j^+} \begin{bmatrix} u_{jr} \\ v_{jr} \end{bmatrix} + \begin{bmatrix} \bar{u}_{jd} \\ \bar{v}_{jd} \end{bmatrix},$$

$$(j = 1, \ldots, m)$$

$$u_{jr}^2 + v_{jr}^2 \leq w_{jr}^2, \quad \forall r \in \Sigma_j^+, \quad \bar{u}_{jd}^2 + \bar{v}_{jd}^2 \leq \bar{w}_{jd}^2$$

where:

- $g_{jx} = \frac{\partial F}{\partial x_j} \big|_{\bar{X}}, g_{jy} = \frac{\partial F}{\partial y_j} \big|_{\bar{X}}$ are the partial derivatives with respect to $x_j, y_j$ of the differentiable part of $F(X)$, i.e. the sums with respect to $r$ of the derivatives of $w_{jr}\|X_j - X_r\|_2$ and $\overline{w}_{jr}\|X_j - A_r\|_2$, restricted to the non-interacting pairs $X_jX_r$ and $X_jA_r$,

- $\Sigma_j^-, \Sigma_j^+$ are the sets of the indices $r$ (respectively $r < j$ for $\Sigma_j^-$ and $r > j$ for $\Sigma_j^+$) of the facilities $X_r$ interacting with $X_j$. Hence, each pair of variables $(u_{rj}, v_{rj})$ [or $(u_{jr}, v_{jr})]$ is associated with an active edge $X_rX_j$ [or $X_jX_r$],

- the pair $(\bar{u}_{jd}, \bar{v}_{jd})$ is associated with the edge $X_jA_d$ (if a facility $A_d$ interacting with $X_j$ exists).
A $R^{2m}$ point $\tilde{X}$ is a minimizer for (1) iff $0 \in \partial F(\tilde{X})$, i.e. for any active edge a pair $(u_{jr}, v_{jr})$ or $(\bar{u}_{jd}, \bar{v}_{jd})$ exists which satisfies the inequalities (3) and the linear system of $2m$ equations obtained by setting (2) to zero. These conditions reduce to $\text{Grad}[F(X)] = 0$ if no pair of interacting facilities exists. Let $z$ be the number of active edges in $H(\tilde{X})$, and $(j_1, r_1), (j_2, r_2), \ldots, (j_z, r_z)$ be the ordered list of the pairs of indices of the corresponding interacting facilities. Since for any $j$ the first components in (2) contains only $g_{jx}, u_{jr}, u_{jz}, \bar{u}_{jd}$ and the second component only $g_{jy}, v_{jr}, \bar{v}_{jd}$, then (2) can be split into two parts $G_x + AU, G_y + AV$, and the related linear system $\partial F(\tilde{X}) = 0$ into two independent subsystems

$$G_x + AU = 0, \quad G_y + AV = 0 \quad (4)$$

where:

- $U(z) = [u_{j_1, r_1}, u_{j_2, r_2}, \ldots, u_{j_z, r_z}]^T; V(z) = [v_{j_1, r_1}, v_{j_2, r_2}, \ldots, v_{j_z, r_z}]^T$
- $G_x(m) = [g_{1x}, g_{2x}, \ldots, g_{mx}]^T, G_y(m) = [g_{1y}, g_{2y}, \ldots, g_{my}]^T$
- $A(mxz)$ is the coefficient matrix, whose entries are 1, -1, 0. In order to determine them, consider for $k = 1, 2, \ldots, z$ the pairs $u_{j_k, r_k}, v_{j_k, r_k}$. If $X_{j_k}, X_{r_k}$ are new facilities (with $j_k < r_k$ since they are ordered), construct the $k$th column of $A$ by setting 1 in row $j_k$, -1 in row $r_k$, and 0 elsewhere. If the $k$th pair is of the type $\bar{u}_{j_k, d_k}, \bar{v}_{j_k, d_k}$, then the $k$th column of $A$ must have 1 in row $j_k$ and 0 elsewhere.

In the current literature, the new facilities $\tilde{X}_j$ of a given $R^{2m}$ point $\tilde{X}$ are classified as belonging to one of the following categories:

(i) Isolated points. $\tilde{X}_j$ is an isolated point if it does not interact with other facilities.

(ii) Coinciding points. $\tilde{X}_j$ is a coinciding point if it interacts with a facility $A_d$ only.

(iii) Isolated cluster. A group of interacting new facilities is an isolated cluster if their common location is distinct from all other facility locations.

(iv) Coinciding cluster. A group of interacting new facilities is a coinciding cluster if their common location coincides with a facility $A_d$ (interacting at least with a facility $X_j$ of the cluster).

As has been proved in [2], the systems (4) can be split into as many independent subsystems as there are isolated points, coinciding points, and clusters in the point $\tilde{X}$ to be tested for optimality. Let $K$ be a cluster, and let us suppose that $K$ is formed by all the new facilities of $H$ (if not so, a pair of subsystems must be considered instead of the systems (4)). If $K$ is
a tree, then the systems have a unique solution which can be computed by means of a recursive algorithm [2]. If \( K \) contains cycles, the systems have in general more unknowns than equations, so that the solutions depend on \( p = z - r \) (with \( r \) rank of \( A \)) arbitrary parameters. As will be shown in the problem in Section 3 (region \( R_1 \)), in the case of \( p = 1 \) the solutions of the two systems can be expressed by means of two parameters \( \alpha, \beta \), and the corresponding inequalities (3) represent circles of \( (\alpha, \beta) \) plane. Therefore, (3) are satisfied if these circles have a common intersection.

If the solution of (1) is considered as function of the weights, it can be completely described by determining the sets of weights corresponding to any possible type of solution (i.e. any combination of isolated points, coinciding points, and clusters), and then partitioning the space of the weights into regions corresponding to these sets. The weights can be normalized so that one of them is 1.

3. THE CONSIDERED PROBLEM AND ITS SOLUTION

The symmetric EMFL problem shown in Figure 1 is considered. The Existing Facilities are \( A_0 = (0, 0), A_1 = (a, h), A_2 = (-a, h) \), the New Facilities \( X_1 = (x_1, y_1), X_2 = (x_2, y_2) \), and the weights are \( \mu \) for the edges \( A_0X_1, A_0X_2 \), 1 for \( X_1A_1, X_2A_2 \), and \( \nu \) for \( X_1X_2 \). \( L = \sqrt{a^2 + h^2} \) is the side of the triangle, and \( \vartheta = 2 \arctg (a/h) \) is the angle \( A_1A_0A_2 \) \( [\vartheta \in (0, 2\pi)] \).

![Figure 1.](image)

It is required to locate \( X_1 \) and \( X_2 \) so as

\[
F(X) = F(X_1, X_2) = \mu[f_1(X) + f_2(X)] + \nu f_3(X) + f_4(X) + f_5(X) = \min
\]

(5)
with

\[ f_1(X) = \sqrt{x_1^2 + y_1^2}, \quad f_2(X) = \sqrt{x_2^2 + y_2^2}, \]
\[ f_3(X) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}, \]
\[ f_4(X) = \sqrt{(a - x_1)^2 + (h - y_1)^2}, \]
\[ f_5(X) = \sqrt{(-a - x_2)^2 + (h - y_2)^2}. \]

The solution of (5) is symmetric with respect the y-axis. A proof is given by the following

**Theorem 1:** Let \( X_1, X_2 \) be a solution of (5) with \( X_1 = (x, y) \), and let \( \mu, \nu > 0 \). Then \( X_2 = (-x, y) \).

**Proof:** Let us first assume that \( X_1, X_2 \) are isolated points, and consider the two Weber problems having respectively \( X_1, X_2 \) as new facilities, and \( (A_0, A_1, X_2), (A_0, X_1, A_2) \) as existing facilities (see Fig. 1). Such problems must be also solved when \( X \) is optimal: in fact, if not so \( X_1 \) (or \( X_2 \)) could be moved from its position, with a decrease of \( F(X) \). For a property of the 3-point Weber problem, the angles

\[ \alpha_1 = A_0 X_1 A_1, \quad \alpha_2 = A_1 X_1 X_2, \quad \alpha_3 = X_2 X_1 A_0, \]
\[ \beta_1 = A_2 X_2 A_0, \quad \beta_2 = A_2 X_2 X_1, \quad \beta_3 = X_1 X_2 A_0 \]

depend at optimality on the weights only, and their cosines (see [5]) are

\[
\begin{align*}
\cos \alpha_1 &= \frac{\nu^2 - 1 - \mu^2}{2\mu}, & \cos \alpha_2 &= \frac{(\mu^2 - 1 - \nu^2)}{2\nu}, \\
\cos \alpha_3 &= \frac{(1 - \mu^2 - \nu^2)}{2\mu\nu}, & \cos \beta_j &= \cos \alpha_j (j = 1, 2, 3).
\end{align*}
\]

Therefore \( \alpha_j = \beta_j \ (j = 1, 2, 3) \). But these equalities can hold only if the solution is symmetric, i.e. if \( X_1 = (x, y), X_2 = (-x, y) \).

If \( X_1, X_2 \) are non isolated, two clearly symmetric cases are possible:

(i) \( X_1 = X_2 \) belonging to the y-axis;

(ii) \( X_1 = A_1, X_2 = A_2 \). Both can be thought as a limit of a case of isolated points.

The problem (5) allows of four different types of solution, corresponding to the regions \( R_1, R_2, R_3, R_4 \) of the space of the weights as shown in Figure 2 for the special case of \( \theta = \pi/4 \). The coordinates of \( P, S, Q \) are

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Figure 2.

$P(1, 0), S(0, 1)$ and $Q(s, c)$, with $s = \sin(\vartheta/2), c = \cos(\vartheta/2)$. The shape of the regions depends on $\vartheta$, but it is independent of $h$. It can be observed that $Q$ moves along the arc of circumference $\mu = \cos(\vartheta/2), \nu = \sin(\vartheta/2)$ as $\vartheta$ varies in $(0, 2\pi)$.

The following statements define the regions and the corresponding type of solution.

1. $R_1$ is delimited by the ray $(S, +\infty)$ of the $\mu$-axis, by the ray $(Q, +\infty)$ parallel to the $\nu$-axis, and by the arc $SQ$ of the hyperbole

\[
\mu^2 - \nu^2 + 2\nu s = 1
\]

If $(\nu, \mu) \in R_1$, the coinciding cluster $X_1 = X_2 = A_0$ solves the problem (5).

2. $R_2$ is bounded by the segments $OP, OS$ and by the arc $PS$ of the ellipse

\[
\mu^2 + \nu^2 + 2\mu \nu s = 1
\]

If $(\nu, \mu) \in R_2$ the solution is in the coinciding points $X_1 = A_1, X_2 = A_2$.

3. The region $R_3$ is bounded by the ray $(Q, +\infty)$ parallel to the $\nu$ axis and by the arc $PQ$ of the curve

\[
\frac{s^2}{s^2 + (c - y^*)^2} + \left[ \frac{y^* - c}{\sqrt{s^2 + (c - y^*)^2}} + \mu \right]^2 = \nu^2
\]

with

\[
y^* = c + s \frac{\mu}{\mu^2 - 1} \sqrt{1 - \mu^2}, \quad (0 \leq \mu \leq c)
\]
If \((\nu, \mu) \in R_3\) the solution is an "Isolated Cluster" \(X_1 = X_2 = (0, y)\) located on the \(y\)-axis. The ordinate is \(y = Ly^*\), with \(L\) side of the triangle.

4. The region \(R_4\) is bounded by the arcs \(SQ, PS, PQ\) of (7), (8) and (9). If \((\nu, \mu) \in R_4\) the problem is solved by the isolated points \(X_1(x, y), X_2(-x, y)\), with

\[
x = \frac{m_2 a - h}{m_2 - m_1}, \quad y = m_1 x
\]

\[
m_1 = \frac{\sqrt{4\nu^2\mu^2 - (1 - \mu^2 - \nu^2)^2}}{1 - \mu^2 - \nu^2}, \quad m_2 = -\frac{\sqrt{\mu^2 - (1 - \nu^2)^2}}{\mu^2 - 1 - \nu^2}
\]

These statements will now be proved.

1. As concerns the cluster \(X_1 = X_2 = A_0\), tacking into account that

\[
g_{1x} = \partial f_4/\partial x_1|_{A_0} = -s, \quad g_{1y} = \partial f_4/\partial y_1|_{A_0} = -c
\]

\[
g_{2x} = \partial f_5/\partial x_2|_{A_0} = s, \quad g_{2y} = \partial f_5/\partial y_2|_{A_0} = -c
\]

the optimality conditions (4), (3) can be written as

\[
\begin{aligned}
&u_{12} + \bar{u}_{10} = s, \\
&-u_{12} + \bar{u}_{20} = -s,
\end{aligned} \quad \begin{aligned}
&v_{12} + \bar{v}_{10} = c, \\
&-v_{12} + \bar{v}_{20} = c
\end{aligned}
\]

\[
\begin{aligned}
u_{12}^2 + v_{12}^2 \leq \nu^2, \quad \bar{u}_{10}^2 + \bar{v}_{10}^2 \leq \mu^2, \quad \bar{u}_{20}^2 + \bar{v}_{20}^2 \leq \mu^2
\end{aligned}
\]

(13')

Let \(\alpha, \beta\) be two real parameters. Then the general solutions of (13) are

\[
U = \begin{bmatrix} u_{12} \\ \bar{u}_{10} \\ \bar{u}_{20} \end{bmatrix} = U_1 + U_2 = \begin{bmatrix} \alpha + s \\ -\alpha \\ \alpha \end{bmatrix},
\]

\[
V = \begin{bmatrix} v_{12} \\ \bar{v}_{10} \\ \bar{v}_{20} \end{bmatrix} = V_1 + V_2 = \begin{bmatrix} \beta \\ -\beta + c \\ \beta + c \end{bmatrix}
\]
where \( U_1 = [\alpha, -\alpha, \alpha], V_1 = [\beta, -\beta, \beta] \) are the general solutions of the homogeneous parts, and \( U_2 = [s, 0, 0], V_2 = [0, c, c] \) are particular solutions of the complete systems. Hence the inequalities (13') can be written as

\[
\begin{align*}
(\alpha + s)^2 + \beta^2 & \leq \nu^2 \\
\alpha^2 + (-\beta + c)^2 & \leq \mu^2 \\
\alpha^2 + (\beta + c)^2 & \leq \mu^2
\end{align*}
\]

which in an auxiliary \((\alpha, \beta)\) plane define three circles \( C_1, C_2, C_3 \) of centres \( C_1(-s, 0), C_2(0, c), C_3(0, -c) \), and radii \( \nu, \mu, \mu \) (see Fig. 3).

![Figure 3.](image-url)

It follows by simple inspection of the isosceles triangle \( C_1C_2C_3 \), that \( \mu \geq c \) is a necessary condition for \( C_1, C_2, C_3 \) to have a common intersection. Let us assume that \( \mu \geq c \) is satisfied, and \( \nu = C_1P \). Then the circles intersect only if \( \mu^2 \geq OC_3^2 + OP^2 \), i.e. \( \mu^2 \geq c^2 + (s - \nu)^2 \) or \( \mu^2 \geq 1 + \nu^2 - 2\nu s \), which is the hyperbole (7). If both these conditions are satisfied then \((\mu, \nu) \in R_1\) and (13), (13') are also satisfied, so that the cluster \( X_1 = X_2 = A_0 \) solves the problem.

2. The conditions (4), (3) of optimality for \( X_1 = A_1, X_2 = A_2 \) can be written as follows:

\[
\begin{align*}
\bar{u} & = -g_{1x}, & \bar{v} & = -g_{1y} \Rightarrow g_{1x}^2 + g_{1y}^2 \leq 1 & \text{(14)} \\
\bar{u} & = -g_{2x}, & \bar{v} & = -g_{2y} \Rightarrow g_{2x}^2 + g_{2y}^2 \leq 1 & \text{(15)}
\end{align*}
\]

Let us first consider (14) (condition for \( X_1 = A_1 \)). After computing the derivatives

\[
g_{1x} = \mu \frac{\partial f_1}{\partial x_1} \bigg|_{A_1} + \nu \frac{\partial f_3}{\partial x_1} \bigg|_{A_1, A_2}, \quad g_{1y} = \mu \frac{\partial f_1}{\partial y_1} \bigg|_{A_1} + \nu \frac{\partial f_3}{\partial y_1} \bigg|_{A_1, A_2}
\]
in $A_1 = (a, h)$ and $A_2 = (-a, h)$ and by using some well known trigonometric identities, the inequality in (14) becomes $\mu^2 + \nu^2 + 2\mu\nu s \leq 1$, which is the ellipsis delimiting the region $R_2$. The same result can be obtained by considering (15) instead of (14).

3. If the solution is not in the vertices, two cases can occur: an isolated cluster located on the $y$-axis or two isolated points $X_1 \neq X_2$. Let us consider the first case. The optimality conditions (4), (3) are:

$$\begin{align*}
\left\{ \begin{array}{l}
u_{12} = -g_{1x} \\
u_{12} = -g_{2x}
\end{array} \right., \quad \left\{ \begin{array}{l}
u_{12} = -g_{1y} \\
u_{12} = -g_{2y}
\end{array} \right.
\end{align*}$$

(16)

$$u_{12}^2 + v_{12}^2 \leq \nu^2 \quad \Rightarrow \quad g_{1x}^2 + g_{1y}^2 \leq \nu^2$$

and the consistency of the systems requires

$$g_{1x} + g_{2x} = 0, \quad g_{1y} + g_{2y} = 0 \quad (17)$$

Since a solution on the $y$-axis is required, we can set $X_1 = (0, y)$, $X_2 = (0, y)$ in the derivatives, thus obtaining (after introducing the notation $X^* = (X_1, X_2)$):

$$g_{1x} = \frac{\partial f_4}{\partial x_1} \bigg|_{X^*} + \mu \frac{\partial f_1}{\partial x_1} \bigg|_{X^*} = \frac{-a}{\sqrt{a^2 + (h-y)^2}},$$

$$g_{1y} = \frac{\partial f_4}{\partial y_1} \bigg|_{X^*} + \mu \frac{\partial f_1}{\partial y_1} \bigg|_{X^*} = \frac{y-h}{\sqrt{a^2 + (h-y)^2}} + \mu,$$

$$g_{2x} = \frac{\partial f_5}{\partial x_2} \bigg|_{X^*} + \mu \frac{\partial f_2}{\partial x_2} \bigg|_{X^*} = \frac{a}{\sqrt{a^2 + (h-y)^2}},$$

$$g_{2y} = \frac{\partial f_5}{\partial y_2} \bigg|_{X^*} + \mu \frac{\partial f_2}{\partial y_2} \bigg|_{X^*} = \frac{y-h}{\sqrt{a^2 + (h-y)^2}} + \mu.$$

Therefore, the first of (17) is always satisfied, and the second reduces to $g_{1y} = 0$ (or to $g_{2y} = 0$). Moreover the inequality in (16) becomes

$$\frac{a^2}{a^2 + (h-y)^2} + \left[ \frac{y-h}{\sqrt{a^2 + (h-y)^2}} + \mu \right]^2 \leq \nu^2 \quad (18)$$

Since $h = Lc$ and $a = Ls$, the solution $y \in (0, h)$ of the equation $g_{1y} = 0$ can be written as

$$y = L \left( c + s \frac{\mu}{\mu^2 - 1} \sqrt{1 - \mu^2} \right) = Ly^* \quad (19)$$

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By setting $y = Ly^*$ and taking the equality sign (18) can be rewritten in the form (9), (10) which is the left bound of the region $R_3$. The upper and lower bounds are given by the inequalities $0 \leq \mu \leq c$, which follow from (19) since $\lim_{\mu \to c} y = 0$, $\lim_{\mu \to 0} y = Lc = h$. Hence the isolated cluster $X_1 = X_2$ is optimal if $(\mu, \nu) \in R_3$.

4. The remaining case of isolated points $X_1 \neq X_2$ corresponds to the remaining region $R_4$. The systems (4) reduce to the gradient system $G_x = 0$, $G_y = 0$, which can be numerically solved. However a more simple way to obtain the solution is to intersect the two straight lines passing through $A_0 X_1$, and $A_1 X_1$:

$$y = m_1 x, \quad y - h = m_2 (x - a)$$

whose coefficients $m_1$ and $m_2$ can be easily computed, since the three angles in $X_1$ are known at optimality (their cosines are given by (6)). The solution is expressed by (11), (12).

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