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Generalized optimal search paths for continuous univariate random variables


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GENERALIZED OPTIMAL SEARCH PATHS FOR CONTINUOUS
UNIVARIATE RANDOM VARIABLES (*)

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Abstract. — The purpose of this paper is to solve the Generalized Linear Search Problem for continuous random variables. This problem is concerned with finding a target located on a line. The position of the target is given by the value of a random variable which has a prior distribution. A searcher starts looking for the target from some point, moving along with an upper bound on his speed. The target being sought for might be in either direction from the starting point, so the searcher needs to change his direction many times before he attains his goal. With minimality of average time to target detection as the measure of optimality of search paths, we have obtained algorithms that find such paths for those targets which have absolutely continuous distributions. More detailed properties of optimal search paths are also studied. One of the main results is that these search paths are not minimal, in some cases, for some types of target distributions.

Keywords: Linear search; Optimization; Normal and bimodal normal distributions.

1. INTRODUCTION

The following problem has been considered in the literature. A target is assumed to be located on a line. Its position \( x \) is given by the value of a random variable \( X \), which has a known (or unknown) distribution \( F \).
A searcher starts looking for the target from some point $a_0$ on the line ($|a_0| < \infty$), moving along the line with an upper bound on his speed. The target being sought for might be in either direction from the starting point $a_0$, so the searcher would conduct his search in the following manner: Start at $a_0$ go to the left (right) as far as $a_1$. If the target is not found there, turn back and explore the right (left) part of $a_0$ as far as $a_2$. If the target is still not found, retrace the steps again to explore the left (right) part of $a_1$ as far as $a_3$, and so fourth until the target be detected. Let us define $c$ and $d$ as follows

$$c = \inf \{x: F(x) > 0\}, \quad d = \sup \{x: F(x) < 1\}.$$  

Then a search path may, in general, be represented by a sequence $A = \{a_i; i \geq 0\}$ with $a_{2i} \rightarrow c$ and $a_{2i-1} \rightarrow d$ as $i \rightarrow \infty$, or vice versa. Figure 1 gives an illustration of such search paths. Observe that the two search paths depicted in Figure 1 are duals and of sequential type. Moreover it is to be noted that these two search paths will give us several possible cases of search when we consider all relative positions, of the starting point $a_0$, to the origin (see [2]).

The problem is of interest because it may arise in many real world situations such as:

(i) Searching for lost persons or objects on roads (Beck [5], Beck and Newman [7], and Rousseeuw [13]).

(ii) Searching for a faulty unit in a large linear system such as electrical power lines, telephone lines, petrol or gas supply lines, and mining systems (Balkhi [2]).

(iii) Estimating a distribution parameter whose probability locations are given. The parameter, here, may be regarded as a target to be searched for.

In the above examples, and in many others of this type, (see [2]) the target distribution is given or to be estimated. It is possible, however, to study this problem as a game between the searcher and the target (see [7] and [11]).

Figure 1.
For any of such problems the path length of some search path \( A = \{ a_i ; i \geq 0 \} \), from the starting point \( a_0 \) until reaching the target \( x \), is considered as the cost of the search. By virtue of the randomness of the position of \( x \), it is clear that the cost of the search is, also, a random variable. The aim of the search is, then, to minimize its expected cost. Any search path that fulfills this aim is referred to as an optimal search path (O.S.P.). For all possible cases of search, the solution of this problem consists of two stages. The first is the establishment of the existence of (O.S.P.)'s. This stage has been, in fact, completed by many authors. A review of their results will be the subject of the next section. The second stage is the construction of (O.S.P.)'s. Concerning the case \( a_0 = 0 \) and the second stage Beck [6] and Franck [10] have indicated that a recursive formula for the entries \( a_i \)'s of a minimizing search path is available under proper differentiability conditions on the expected cost. But the solution there has not been given in a useful sense. Rousseeuw [13] has done some investigations about (O.S.P.)'s for the case \( a_0 = 0 \). But his work was concentrated on the Normal distribution and its analogous symmetric distributions only. Besides to the case \( a_0 = 0 \), there are, however, many other cases of possible search. Some of these cases have been previously considered in Balkhi [1]. Later Balkhi [2] has shown that there are only five cases of possible search one of which is the case \( a_0 = 0 \). The other four cover all possible cases of search for which \( a_0 \neq 0 \). The work of [2], in fact, has focussed on giving sufficient conditions under which there exists an (O.S.P.) for each possible case of search.

In this paper the construction of (O.S.P.)'s, for the only five possible cases of search considered in [2] and for regular target distributions (see definition 2.2), will be introduced in a unified way. The main properties of (O.S.P.)'s will be given some emphasis. An algorithm by which we can calculate (O.S.P.)'s together with an illustrative examples are also introduced. The numerical results of these examples will then show that some of the possible cases of search is better than some others in the sense that they give less expected cost. Justifying, thus, the generalization of this problem that have been previously considered by this author.

2. LITERATURE REVIEW

Authors in [5] to [8], [10], [11] and [13] have dealt with the case \( a_0 = 0 \) only. Under the name “The Generalized Linear Search Problem” (GLSP) Balkhi [2] has introduced this problem in more general approach by considering any starting point \( a_0 (|a_0| < \infty) \) other than the origin. The additional
assumption that the number of elements, of a search path \( A = \{a_i; i \geq 0\} \), between the origin and \( a_0 \), is finite, is also presumed in [2] (This assumption may be justified by [2] Lemma 3.8). It is shown, then, that we have only five possible cases of search, one of which is the case \( a_0 = 0 \). These cases are referred to as case \((k)\); \( k = 0, 1, 2, 3 \) and 4 [case \((0)\) is the case for which \( a_0 = 0 \)]. The class of search paths in case \((k)\) is denoted by \( Q_k \); \( k = 0, 1, 2, 3 \) and 4. With the conventions that \( a_0 \neq 0 \) for \( k = 1, 2, 3 \) and 4, \( a_{-1} = 0 \) for \( k = 0 \), and \( a_0 \neq a_i \) for all \( k \) (The last assumption is justified by the fact that the searcher needs to move from \( a_0 \) to a new point namely \( a_1 \), at the outset of his search). Then class \( Q_k \) consists of all search paths of the following type

\[
\begin{align*}
(2.1) \quad & \cdots <a_4 < a_2 < 0 = a_0 < a_1 < a_3 < a_5 < \cdots; \quad k = 0 \\
(2.2) \quad & \cdots <a_4 < a_3 < a_1 \leq 0 < a_0 < a_2 < a_4 < \cdots; \quad k = 1 \\
(2.3) \quad & \cdots <a_4 < a_2 \leq 0 < a_0 < a_1 < a_3 < a_5 < \cdots; \quad k = 2 \\
(2.4) \quad & \cdots <a_4 < a_3 \leq 0 \leq a_1 < a_0 < a_2 < a_4 < \cdots; \quad k = 3 \\
(2.5) \quad & \cdots <a_4 \leq 0 \leq a_2 \leq a_0 \leq a_1 \leq a_3 < a_5 < \cdots; \quad k = 4
\end{align*}
\]

and their duals which can be obtained by reversing the inequalities in (2.1) through (2.5) (See [2] for more details). For a search path \( A = \{a_i; i \geq 0\} \) form class \( Q_k \), the expected cost is denoted by \( D_k (A, F) \). As has been shown in [2] we have

\[
(2.6) \quad D_k (A, F) = M (F) + \Delta_k (A, F); \quad k = 0, 1, 2, 3 \text{ and } 4
\]

where \( M (F) = \int_{-\infty}^{\infty} |x| \, dF(x) \) (The first absolute moment of \( F \)).

\[
(2.7) \quad \Delta_0 (A, F) = 2 \sum_{i=1}^{\infty} |a_i| \{1 - \text{sign}(a_i) [F(a_i) - F(a_{i-1})]\}
\]

\[
(2.8) \quad \Delta_k (A, F) = -2 \left[ \int_{0}^{a_0} |x| \, dF(x) \right] \left[ -(-1)^k \, |a_0| + 2 \sum_{i=1}^{\infty} |a_i| \right]
\]

\[
\times \{1 - \text{sign}(a_i) [F(a_i) - F(a_{i-1})]\}; \quad k = 1, 2
\]

\[
(2.9) \quad \Delta_k (A, F) = -2 \left[ \int_{0}^{a_0} |x| \, dF(x) \right] \left[ -(-1)^k \, |a_0| - 4 \, |a_{k-2}| \right]
\]

\[
+ 2 \sum_{i=1}^{\infty} |a_i| \{1 - \text{sign}(a_i) [F(a_i) - F(a_{i-1})]\}; \quad k = 3, 4
\]
(see Balkhi [2], theorems 2.1, 2.3 and remark 2.4). An (O.S.P.) from class $Q_k$ is given formally by the following definition:

**Définition 2.1:** Let

\[(2.10) \quad m_k = \inf \{ D_k (A, F) : A \in Q_k \}; \quad k = 0, 1, 2, 3 \text{ and } 4 \]

If $A^* \in Q_k$ is such that $m_k = D (A^*, F)$, then $A^*$ is said to be an (O.S.P.) from class $Q_k$; $k = 0, 1, 2, 3$ and $4$.

The existence of (O.S.P.)'s in class $Q_0$ has been established in Beck [5] and Franck [10] by assuming different (but not equivalent) conditions on the underlying distribution $F$, that give necessary and sufficient conditions for such existence. For the (GLSP) considered here, Balkhi [2] proved the following two theorems.

**Théorème 2.1:** There exists a search path from class $Q_k$; $k = 0, 1, 3$ and $4$, with finite expected cost if and only if $M (F) < \infty$.

**Théorème 2.2:** Let $F^- (0)$, $F^+ (0)$ denote the left hand and right hand derivatives of $F$ at zero respectively. If $M (F) < \infty$, then there exists an (O.S.P.) from class $Q_k$ if

(i) For $k = 0$, 1, at least one of $F^- (0)$, $F^+ (0)$ is finite.

(ii) For $k = 2, 3$ and $4$, both $F^- (0)$, $F^+ (0)$ are finite.

Thus the existence of (O.S.P.)'s for $k = 0, 1$ ($k = 2, 3$ and $4$) is guaranteed under the finiteness of $M (F)$ and $F^- (0)$ or $F^+ (0)$ ($M (F)$, $F^- (0)$, and $F^+ (0)$). Under some special assumptions which include the above ones Fristedt and Heath [11] proved the following theorem.

**Théorème 2.3:** If $M (F) < \infty$, then there exists an (O.S.P.) from class $Q_0$ with constant speed equal to 1.

Theorem 2.3 does not have special assumptions concerning class $Q_0$ per se, so this theorem holds for any of the classes $Q_k$; $k = 0, 1, 2, 3$ and $4$. Thus, for all classes $Q_k$ we might consider the expected cost of the search to be either $D_k (A, F)$ or $T_k (A, F)$, where $T_k (A, F)$ denotes the expected searching time using the search path $A = \{ a_i : i \geq 0 \}$ from class $Q_k$ i.e.

\[(2.11) \quad D_k (A, F) = T_k (A, F) = M (F) + \Delta_k (A, F); \quad k = 0, 1, 2, 3$ and $4.\]

The following definition is often needed in the sequel.

**Définition 2.2:** If the target distribution $F$ is absolutely continuous with strictly positive density $f$, then $F$ is said to be regular.
Of special interest are symmetric target distribution \( i.e. \)
\[
F(-x) = 1 - F(x), \quad \forall x \in \mathbb{R}.
\]

For this type of distributions (\( i.e. \) symmetric) then more appropriate formulas, for theoretical and computational purposes, are available for the expected cost. To see this let \( A = \{ a_i; \ i \geq 0 \} \in Q_k; \ y_i = |a_i|, \ i \geq 0 \). If \( F \) is symmetric then
\[
1 - \text{sign}(a_i)[F(a_i) - F(a_{i-1})] = 2 - [F(y_i) - F(y_{i-1})];
\]
\[
i \geq k \quad \text{for } k = 1, 2, 3 \text{ and } 4 \quad \text{and } i \geq 1 \quad \text{for } k = 0.
\]

Let \( Y = \{ y_i; \ i \geq 0 \} \), then from (2.13) and our hypotheses we can easily see that \( \Delta_k(Y, F) = \Delta_k(A, F); \ k = 0, 1, 2, 3 \text{ and } 4 \), where
\[
\Delta_0(Y, F) = 2 \sum_{i=0}^{\infty} [1 - F(y_i)](y_i + y_{i+1}) = y_1 + 2 \sum_{i=1}^{\infty} [1 - F(y_i)](y_i + y_{i+1})
\]
\[
\Delta_k(Y, F) = -2 \int_{0}^{y_0} |x| \, dF(x)
\]
\[
- (-1)^k y_0 + 2 y_1 [1 - F((-1)^{k+1} y_0)]
\]
\[
+ 2 \sum_{i=1}^{\infty} [1 - F(y_i)](y_i + y_{i+1}); \quad k = 1, 2
\]
\[
\Delta_k(Y, F) = -2 \int_{0}^{y_0} |x| \, dF(x)
\]
\[
- (-1)^k y_0 - 2 y_{k-2} [1 + F(y_{k-2}) - F(y_{k-3})]
\]
\[
+ 2(k-3) y_1 [1 + F(y_0) - F(y_1)] + 2 y_{k-1} F(y_{k-2})
\]
\[
+ 2 \sum_{i=k-1}^{\infty} [1 - F(y_i)](y_i + y_{i+1}); \quad k = 3, 4
\]

Formulas (2.14) through (2.16) make it possible to disregard the signs of the entries of a search path \( A = \{ a_i; \ i \geq 0 \} \) by using the equivalent search path \( Y = \{ y_i; \ i \geq 0 \} \). This, in fact, results in more efficient computational algorithms that calculate the entries of (O.S.P.)’s and the corresponding optimal costs. Moreover, by using (2.14) and (2.15), Balkhi [1] proved the following interesting result (see [1] pp. 173-174).
THEOREM 2.4: If the underlying distribution $F$ is symmetric, and if $A$ is an (O.S.P.) from class $Q_k$, then

$$|a_{i+1}| > |a_i| \text{ for all } i \geq 0; \ k = 0, 1, 2.$$ 

Using similar techniques as those used in [1] we can easily show that this theorem holds, also, for $k = 3$ and 4 with $i \geq k - 2$. Thus we have

$$|a_{i+1}| > |a_i|;$$

$$i \geq 0 \text{ for } k = 0, 1 \text{ and } 2 \quad \text{and} \quad i \geq k - 2 \text{ for } k = 3 \text{ and } 4.$$ 

Thus for symmetric target distributions we can restrict our attention to the search paths $Y = \{y; i \geq 0\}$ for which

(2.17) \[
\begin{cases}
y_{i+1} > y_i; \\
i \geq 0 \text{ for } k = 0, 1 \text{ and } 2, \quad \text{and} \quad i \geq k - 2 \text{ for } k = 3 \text{ and } 4.
\end{cases}
\]

Remark 2.1. — There is a kind of scale invariance on the expected cost. For if $A = \{a_i; i \geq 0\}$ is a search path from class $Q_k; k = 0, 1, 2, 3$ and 4. And if we define $\lambda A = \{\lambda a_i; i \geq 0\}, F_\lambda(x) = F(x/\lambda)$, so that the support of $F_\lambda$ is $(\lambda c, \lambda d)$, then

(2.19) \[D_k(\lambda A, F_\lambda) = \lambda D_k(A, F); \quad k = 0, 1, 2, 3 \text{ and } 4\]

which can be easily seen from (2.6) through (2.9) (see also [13] remark 1.1).

Remark 2.1 means that the expected cost of the search does not depend on the type of distribution, but it depends also on its scale parameter. It is, therefore, meaningful to standardize the expected cost by other parameter of the same scale, say by $M(F)$. Relating to this fact Rousseeuw [13] has proved the following theorem.

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THEOREM 2.5: If the underlying distribution $F$ is symmetric and regular, and if $A$ is an (O.S.P.) from class $Q_0$, then

\[(2.18) \quad 2 < T_0(A, F)/M(F) < 4.591.\]

3. OPTIMAL SEARCH PATHS

(a) Critical search paths

As it can be seen from (2.6) through (2.9) the (GLSP) depends on two unknown factors. Those are the target distribution $F$, and the search path $A = \{a_i; i \geq 0\}$ used by the searcher. Let us assume, from now on, that the target distribution is known. Nevertheless we still face a difficult optimization problem. Because this problem has an infinite number of variables; that is $A = \{a_i; i \geq 0\}$. However, if we assume (from now on) that the target distribution $F$ is also regular and that $M(F)$ is finite. Then the structure of the (GLSP) becomes easy and even simple as we shall see below. But let us first give a pertinent definition and remark.

DEFINITION 3.1: If $A = \{a_i; i \geq 0\}$ is a search path from class $Q_k$ such that the derivative of $\Delta_k(A, F)$ with respect to $A$ does exist and all partial derivatives of $\Delta_k(A, F)$ with respect to the $a_i$'s vanish, then $A$ is said to be a critical search path (C.S.P.) from class $Q_k$, $k = 0, 1, 2, 3$ and $4$.

Remark 3.1. — We infer that if $\Delta_k(A, F)$ is differentiable on $Q_k$ then the set of critical search paths from $Q_k$ will contain all of the relative minimal and relative maximal search paths. Of course this set may also contain search paths at which $\Delta_k(A, F)$ does not have relative minimal or maximal search paths. In addition the function $\Delta_k(A, F)$ may have relative extremum at a search path from $Q_k$ at which the derivative of $\Delta_k(A, F)$ with respect to $A$ does not exist or $\Delta_k(A, F)$ may have a relative extremum at a search path which is not an interior point from $Q_k$. □

Now by the regularity condition on $F$ and the finiteness of $M(F)$, then Theorem 2.2 guarantees the existence of (O.S.P.)'s in each of the classes $Q_k; k = 0, 1, 2, 3$ and $4$. If $A = \{a_i; i \geq 0\}$ is a (C.S.P.) from class $Q_k$, then $(3.1)$ exists for all pertinent values of $i$ and $k$, and then

\[
\frac{\partial \Delta_k(A, F)}{\partial a_i} = 0
\]

\[(3.1) \quad i \geq 0 \quad \text{for} \quad k = 1, 2, 3 \text{ and } 4, \quad \text{and} \quad i \geq 1 \quad \text{for} \quad k = 0.
\]
Moreover, for the following tupled values of \( i \) and \( k \)
\[(3.2) \quad (k=0, 1 \text{ and } 2; \ i \geq 1), \quad (k=3, \ i \geq 2) \quad \text{and} \quad (k=4; \ i \geq 3).\]

Then relations (2.7) through (2.9) together with (3.1) give the following results.
\[(3.3) \quad \frac{\partial \Delta_k(A, F)}{\partial a_i} = 2 \text{sign}(a_i) \{1 - \text{sign}(a_i) [F(a_i) - F(a_{i-1})] - f(a_i) (|a_i| + |a_{i+1}|)\}
\]
\[(3.4) \quad |a_{i+1}| = \frac{1 - \text{sign}(a_i) [F(a_i) - F(a_{i-1})]}{f(a_i)} - |a_i|.
\]

And from our hypotheses we have
\[(3.5) \quad a_{i+1} = -\text{sign}(a_i) \cdot |a_{i+1}|.
\]

Using the same reasoning as applied for the tupled values of \( k \) and \( i \) in (3.2), the rest \( a_i \)'s of a (C.S.P.) \( A = \{a_i; \ i \geq 0\} \in Q_k \) are given by the following relations:
\[(3.6) \quad |a_1| = 1/2 f(a_0) - |a_0|; \quad k = 1
\]
\[(3.7) \quad |a_1| = |a_0| - 1/2 f(a_0); \quad k = 3
\]
\[(3.8) \quad |a_1| = 1/2 f(a_0) + |a_0|; \quad k = 2, 4
\]
\[(3.9) \quad |a_2| = \frac{1 + \text{sign}(a_1) [F(a_1) - F(a_0)]}{f(a_1)} + |a_1|; \quad k = 3
\]
\[(3.10) \quad |a_2| = |a_1| - \frac{1 - \text{sign}(a_1) [F(a_1) - F(a_0)]}{f(a_1)}; \quad k = 4
\]
\[(3.11) \quad |a_3| = \frac{1 + \text{sign}(a_2) [F(a_2) - F(a_1)]}{f(a_2)} + |a_2|; \quad k = 4.
\]

For the signs of these entries we recall, from the hypotheses, that: For \( k = 1, a_0 \) and \( a_1 \) have different signs. Whereas for \( k = 2, 3 \) and \( 4 \), all the \( a_i \)'s for which \( i \leq k - 1 \) have the same sign. Now, as it can be noticed from relations (3.4) through (3.11) and the signs of the \( a_i \)'s indicated above, then for \( k = 1, 2, 3 \) and \( 4 \) we have that \( a_1 \) is a function of \( a_0 \), and \( a_{i+1} \) is a function of \( a_{i-1} \) and \( a_i \) for all \( i \geq 1 \). Hence \( a_{i+1} \) is a function of \( a_0 \). Thus if we assume that \( a_0 = r \), then there exists a function \( \psi_i \) such that
\[(3.12) \quad a_i = \psi_i(r) \quad \text{for all} \ i \geq 0, \quad \text{and} \quad k = 1, 2, 3 \text{ and } 4
\]
where \( \psi_0 (r) = r \). But for the case \( k = 0 \) we have to take \( r = a_1 \) since then \( a_0 = 0 \). With the convention that \( \psi_0 (0) = 0 \) for \( k = 0 \), then a (C.S.P.) \( A = \{ a_i; i \geq 0 \} \) from class \( Q_k \); \( k = 0, 1, 2, 3 \) and \( 4 \) is of the form

\[
A = \{ \psi_i (r); i \geq 0 \}
\]

(3.13)

Therefore, if the set of (C.S.P.)’s from class \( Q_k \) is not empty (see Remark 4.1 in the next section) then we have

\[
\inf \{ \Delta_k (A, F); A = \{ a_i; i \geq 0 \} \in Q_k \} = \inf \{ \Delta_k (\{ \psi_i (r); i \geq 0 \}, F); r \in \mathbb{R} \}.
\]

(3.14)

Thus under regularity condition on \( F \), the (GLSP) problem has been reduced from a problem with an infinite number of variables \( \{ a_i; i \geq 0 \} \) to a problem with only one single variable, namely \( r = a_0 \) for \( k = 1, 2, 3 \) and \( 4 \), and \( r = a_1 \) for \( k = 0 \).

(b) Optimal search paths

Let us assume that the set of (C.S.P.)’s from class \( Q_k \) is not empty, and let

\[
\Delta_k^*(r^*, F) = \inf_{r \in \mathbb{R}} \{ \Delta_k (\{ \psi_i (r); i \geq 0 \}, F) \};
\]

(3.15)

\[
k = 0, 1, 2, 3 \) and \( 4 \).

We then can address ourselves to solving (3.15) under the side condition [recall the conditions (2.1) through (2.5)].

(3.16)

\[
| \psi_{i+2} (r) | > | \psi_i (r) | \quad \text{for all } i \geq k - 1; \quad k = 0, 1, 2, 3 \) and \( 4 \)
\]
at any distribution. And the side condition [recall (2.17)].

(3.17)

\[
| \psi_{i+1} (r) | > | \psi_i (r) |,
\]

\[
i \geq 0 \quad \text{for } k = 0, 1 \) and \( 2 \quad \text{and } i \geq k - 2 \quad \text{for } k = 3 \) and \( 4 \)
\]
at the symmetric distributions. Whenever these side conditions are not satisfied, we shall consider that the corresponding \( \Delta_k (\{ \psi_i (r); i \geq 0 \}, F) \) is not defined. The search path \( \{ \psi_i (r^*); i \geq 0 \} \) that defined by (3.15) and that satisfies these side conditions is an (O.S.P) from class \( Q_k \); \( k = 0, 1, 2, 3 \) and \( 4 \).

The procedure of finding an (O.S.P) from (3.15) would be as follows: For each \( r \in \mathbb{R} \) we construct all \( a_i = \psi_i (r) \) from the relevant relations of (3.4) through (3.11). And then we calculate the corresponding \( \Delta_k (\{ \psi_i (r) \}, F) \) from (2.7) through (2.9). From those values of \( r \) that satisfy the pertinent side condition, we choose the value \( r^* \) that satisfy (3.15). Another equivalent
procedure of finding an (O.S.P.) from (3.15) is as follows: From all (C.S.P.)'s of the form (3.13) we find the minimal search paths. Then we take the overall minimum of all minimizing search paths. However, there are some difficulties that arise when applying such procedures. One of the main difficulties for instance, is to consider all values of \( r \) from \( \mathbb{R} \). Another one is that; it is not known as to whether the relevant side conditions, indicated above, are fulfilled everywhere. A third one is that; though our optimization problem has been reduced from a problem with an infinite number of variables to a problem with only one single variable. But it is still one difficult variable. This is so since each (C.S.P.) has an infinite number of entries. It would be therefore, rather difficult to verify that a given (C.S.P.) is of minimal type. Unfortunately overcoming such difficulties is not always possible as we shall see in the next section. Nevertheless, the properties of (O.S.P.)'s which will be studied in the next section will provide us with valuable information that will, at least, be a helpful tool for verifying and facilitating the numerical calculations of (O.S.P.)'s.

4. PROPERTIES OF OPTIMAL SEARCH PATHS

Some properties of (O.S.P.)'s have already been established, and being held at any distribution \( F \) (see theorem 2.5 in [2] for the nonsymmetric distributions, and recall relation (2.17) for the symmetric ones). For regular distributions, however some other properties do, in fact, hold and are helpful in facilitating the solution of the (GLSP). In order to help the flow of our ideas we start with the following property of (O.S.P.)'s.

1. Nonminimality of some classes for certain type of distributions

Though the function \( \Delta_k(A, F) \) has an infinite number of variables, the structure of our problem makes it possible to take \( \Delta_k(A, F) \), with finite number of variables, as an approximation of its exact value. Such an approximation is justified by the fact that \(| a_i | \{ \text{1-sign}(a_i) [F(a_i) - F(a_{i-1})] \} \) approaches 0 as \( i \to \infty \) (recall that \( a_i \to -\infty \) and \( a_{i-1} \to \infty \) as \( i \to \infty \) or vice versa). Denote by \( n \) the number of entries from \( A = \{ a_i; i \geq 0 \} \) for which the indicated approximation is fulfilled for any desired level of precision. If \( A = \{ a_i; 0 \leq i \leq n \} \) is an (approximated) search path, then \( A \) can not be minimal unless the Hessian matrix evaluated at \( A \) is positive definite (see theorems 42.4, 42.5 in [3]). For \( k = 1, 2, 3 \) and 4, let \( \delta_i = \partial^2 \Delta_k(A, F)/\partial a_i^2; i \geq 0 \).
Simple calculations on (2.8) and (2.9) have shown that the Hessian is symmetric (provided that the derivative $f'$ of $f$ does exist and is continuous). And that the matrix $H$ has the following form:

$$H = \begin{bmatrix}
\delta_0 & 2f(a_0) & 0 & 0 & \ldots & 0 & 0 \\
2f(a_0) & \delta_1 & 2f(a_1) & 0 & \ldots & 0 & 0 \\
0 & 2f(a_1) & \delta_2 & 2f(a_2) & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & \delta_{n-1} & 2f(a_{n-1}) \\
0 & 0 & 0 & 0 & \ldots & 2f(a_{n-1}) & \delta_n
\end{bmatrix}$$

[For the case $k=0$ the resulting matrix $H$ has a similar form as (4.1) with replacement of $a_i$ by $a_{i+1}$ and $\delta_i$ by $\delta_{i+1}; i \geq 0$. But $H$ is positive definite if and only if the determinants of its principle submatrices are strictly positive. Thus $A = \{a_i; i \geq 0\}$ can not be a minimal search path from class $Q_k$ unless

(4.2) $\frac{\partial^2 \Delta_k (A, F)}{\partial r^2} > 0; \quad k = 0, 1, 2, 3$ and 4

And

(4.3) $\frac{\partial^2 \Delta_k}{\partial r^2} \frac{\partial^2 \Delta_k}{\partial a_i^2} - 4[f(r)]^2 > 0; \quad k = 1, 2, 3$ and 4

But when the derivative $f'$ of $f$ does exist, then (4.2) is equivalent to

(4.4) $h(r) = 2f(r) + \text{sign}(r) \frac{f'(r)}{f(r)} \{1 - \text{sign}(r) [F(r) - F(0)]\} < 0; \quad k = 0$

(4.5) $h(r) = 2f(r) + \text{sign}(r) \frac{f'(r)}{f(r)} < 0; \quad k = 1$ and 3

(4.6) $h(r) = 2f(r) - \text{sign}(r) \frac{f'(r)}{f(r)} < 0; \quad k = 2$ and 4.

Suppose now that the distribution $F$ is of the following type:

(4.7) "$F$ is regular and has unimodal density $f$ with the mode occuring at zero and the derivative $f'$ does exist and is continuous".

Then $f'(r)/f(r) > 0$ for $r < 0$ and $f'(r)/f(r) < 0$ for $r > 0$, which means that the necessary condition (4.6) can not hold for $k = 2, 4$. On the other hand,
simple calculations on (2.9) yield:

\[
\frac{\partial^2 \Delta_3(A, F)}{\partial a_1^2} = -4f(a_1) + 2(a_2 - a_1) f'(a_1) \text{sign}(a_1).
\]

From which we can easily see that \(\partial^2 \Delta_3/\partial a_1^2 < 0\) whenever \(F\) satisfies (4.7). But then (4.3) can not hold. For if \(\partial^2 \Delta_3/\partial r^2 \leq 0\) we are through, otherwise \(\partial^2 \Delta_3/\partial r^2 \partial^2 \Delta_3/\partial a_1^2 - 4[f(r)]^2 < 0\). Thus we have actually proved the following result.

**Theorem 4.1:** If \(F\) satisfies (4.7), then for \(k = 2, 3\) and 4, any critical search path is not of minimal type.

An illustration of Theorem 4.1 is given by the following example

**Example 4.1:** Suppose that the target position follows the Normal law

\[
F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-1/2 t^2} dt; \quad x \in \mathbb{R}.
\]

which is symmetric and unimodal with the mode occurring at zero. The optimal value \(\Delta^*_k(r^*, F)\) that has been obtained from the (C.S.P.)’s is found to be 2.11282145. However, some given search paths (Noncritical) for which \(a_{0} = r\) for \(k = 1, 2, 3\) and 4 such as

\[
\{a_i = 0.5[r^{i+1} + (i+1)r]; i \geq 0\},
\]

\[
\{a_i = r^{i+1}; i \geq 0, \text{ } |r| > 1\}
\]

(4.9)

\[
\{a_i = (i+1)r; i \geq 0\}, \quad \{a_i = r^{i+1} + ir; i \geq 0\}
\]

\[
\{a_i = r^{i+1} + ir; i \geq 0\}, \quad \{a_i = (i+1)r + r^i; i \geq 0\}
\]

\[
\{a_i = (i+1)r + ir; i \geq 0\}, \text{ etc.}
\]

And for which \(a_1 = r\) for \(k = 0\) such as

\[
\{a_i = ir; i \geq 1\}, \quad \{a_i = r^i; i \geq 1, \text{ } |r| > 1\}
\]

(4.10)

\[
\{a_i = r^i + (i-1)r; i \geq 1\}, \quad \{a_i = 0.5(r^i + ir); i \geq 1\}, \text{ etc.}
\]

have also been considered for comparison purposes (The value of \(\Delta_k\)’s at such search paths will be denoted by \(\Delta_k(r, F)\) so that those values can be distinguished from \(\Delta_k\left(\{\psi_i(r); i \geq 0\}, F\right)\) which we shall use for (C.S.P.)’s.) The minimal values of \(\Delta_k(r, F)\) at the special search paths defined by (4.9) have been found to be 1.969 376 523, 1.848 029 33, 2.024 573 89, 2.123 231 33, 1.993 869 08, 2.372 076 21, 2.073 795 00 respectively. One can easily see that...
the value of $\Delta_k^* (r^*, F)$ is greater than the minimal values of $\Delta_2 (r, F)$ at almost all special search paths defined by (4.9) giving thus an insight to theorem 4.1 for $k = 2$.

**Remark 4.1:** It has been found, by means of computers, that for the distribution (4.8), then

$$|a_1| < 0 \text{ for } k = 3 \text{ and } r \in \mathbb{R} \quad \text{and} \quad |a_2| < 0 \text{ for } k = 4 \text{ and } r \in \mathbb{R}$$

This means that the set of (C.S.P.)'s, from each of the classes $Q_3, Q_4$ and for the distribution (4.8), is empty, which seems to contradict the result of Theorem 2.2. However, by reasons mentioned in Remark 3.1, one may construct many noncritical search paths like those defined by (4.9) and (4.10) and then use the trial and error process to extract (O.S.P.)'s from them.

2. Bounds on $r$

As indicated above, solving (3.15) for all $r \in \mathbb{R}$ is not an easy task. However some useful bounds on the only characteristic variable $r$ are available. Since any (O.S.P.) is a minimal search path, some of these bounds come from the necessary condition (4.2) that have to hold for any minimal search path $A = \{ a_i; i \geq 0 \}$ from class $Q_k$. When the inequalities (4.4), (4.5) and (4.6) have solutions they would be of special importance for obtaining significant bounds on $r$. An illustration is given in the following example.

**Example 4.1 (Continued):** Considering again that the target distribution is given by (4.8). Then for $k = 1$, (4.5) gives.

\begin{equation}
(4.11) \quad h(r) = 2f(r) - r < 0
\end{equation}

which is equivalent to

\begin{equation}
(4.12) \quad r \in (-\infty, -\alpha) \cup (\alpha, \infty) \quad \text{where} \quad \alpha \approx 0.6471428.
\end{equation}

Obtaining thus a lower (upper) bound $\alpha (-\alpha)$ on $r$ when $r > 0 (r < 0)$.

However, the solution of each (4.4), (4.5) and (4.6) is highly dependent on the type of search (i.e. on $k$) and the type of target distribution $F$. For instance, equation (4.6) can not hold for any unimodal distribution with the mode occurring at zero as we have seen in the previous property.

Other bounds on $r$ may be obtained from the forms of $\Delta_k (A, F)$ given by (2.7), (2.8) and (2.9). To see this, let $\delta_k$ be the value of $\Delta_k (A, F)$ at a given search path such as those given by (4.9) and (4.10). And denote by $Q_k^n$ the
set of minimal search paths from class \( Q_k \), \( k = 0, 1, 2, 3 \) and 4. Then some other significant bounds on \( r \) are given by the following theorem.

**Theorem 4.2:** Let \( B_1 = \int_{-\infty}^{0} |x| \, dF(x) \), \( B_2 = \int_{0}^{\infty} |x| \, dF(x) \). If \( Q_k^{\infty} \) is not empty, then

\[
(4.13) \quad r_1 = -\left( \frac{1}{2} \delta_0 + B_1 \right) \leq r \leq \frac{1}{2} (\delta_0 + B_2) = r_2; \quad k = 0
\]

\[
(4.14) \quad r_1 = -(\delta_1 + 2 B_1) \leq r \leq \delta_1 + 2 B_2 = r_2; \quad k = 1
\]

\[
(4.15) \quad r_1 = -(\delta_k + 4 B_1) \leq r \leq \delta_k + 4 B_2 = r_2; \quad k = 2 \text{ and } 3
\]

\[
(4.16) \quad r_1 = -(\delta_4 + 6 B_1) \leq r \leq \delta_4 + 6 B_2 = r_2; \quad k = 4.
\]

**Proof:** Let \( M_k(r) \) be the subset from \( \mathbb{R} \) for which the resulting (C.S.P)'s are minimal search paths. Let also \( A_m = \{ a_i = \Psi_i(r); i \geq 0 \} \) be a minimal search path, and \( \Delta_k(A_m, F) \) be the corresponding value of \( \Delta_k \) at the search path \( A_m \). Since, \( Q_k^{\infty} \) is not empty so for each \( r \in M_k(r) \) we have

\[
(4.17) \quad \delta_k \geq \Delta_k(A_m, F); \quad k = 0, 1, 2, 3 \text{ and } 4.
\]

The proof of (4.14) is direct from (2.8) and (4.17) with \( k = 1 \). We shall now give the proof for \( k = 0 \) and \( k = 4 \). The proof for \( k = 2 \) and 3 can be done by similar fashion.

(i) \( k = 0 \): Let \( r \in M_0(r) \), then from (2.7) and (4.17) we have

\[
\delta_0 \geq \Delta_0(A_m, F) \geq 2 \left| r \right| \left\{ 1 - \text{sign}(r) [F(r) - F(0)] \right\} = 2 \left| r \right| - 2 \int_{0}^{r} \left| r \right| \, dF(x).
\]

which implies that \( \left| r \right| \leq (1/2) \delta_0 + \int_{0}^{r} \left| r \right| \, dF(x) \). Since the integrand on the right side of the last inequality is a nonnegative function so by [4] Lemma 3.8 we have

\[
\begin{cases}
\int_{r}^{0} \left| r \right| \, dF(x) \leq \int_{-\infty}^{0} \left| x \right| \, dF(x) \quad \text{for } r \leq 0, \\
\int_{0}^{r} \left| r \right| \, dF(x) \leq \int_{0}^{\infty} \left| x \right| \, dF(x) \quad \text{for } r \geq 0
\end{cases}
\]

which in turn implies (4.13).

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(ii) $k = 4$: Let $r \in M_4(r)$. Then from (2.9) and (4.17) with $k = 4$ we have

\[
\delta_4 \geq -2 \left| \int_0^r x \, dF(x) \right| - |r| - 4 |a_2| + 2 |a_1| \\
- 2 |a_1| \text{sign}(a_1)[F(a_1) - F(a_0)] \\
+ 2 |a_2| \{1 - \text{sign}(a_2)[F(a_2) - F(a_1)]\} \\
+ 2 |a_3| \{1 - \text{sign}(a_3)[F(a_3) - F(a_2)]\}.
\]

Since for $k = 4$, $|a_3| \geq |a_1| > |a_0| = |r| \geq |a_2|$ so we have

\[
\delta_4 \geq -2 \left| \int_0^r x \, dF(x) \right| - |r| + 2 |r| \\
- 2 |a_1| \text{sign}(a_1)[F(a_1) - F(a_0)] \\
- 2 |a_2| \{1 - \text{sign}(a_2)[F(a_2) - F(a_1)]\} \\
+ 2 |a_3| \{1 - \text{sign}(a_3)[F(a_3) - F(a_2)]\}
\]

or

\[
\delta_4 \geq -2 \left| \int_0^r x \, dF(x) \right| + |r| - 2 |a_1| \text{sign}(a_1)[F(a_1) - F(a_0)] \\
- 2 |a_3| \{1 - \text{sign}(a_3)[F(a_3) - F(a_2)]\}
\]

which implies

\[
|r| \leq 2 \left| \int_0^r x \, dF(x) \right| + 2 \left| \int_{a_0}^{a_1} a_1 \, dF(x) \right| + 2 \left| \int_{a_1}^{a_3} a_3 \, dF(x) \right| + \delta_4.
\]

But, for $k = 4$, the entries $a_0 = r$, $a_1$, $a_2$ and $a_3$ have the same sign, so by similar arguments as those used in proving (4.18) we can easily see that:

Each of $\int_0^r x \, dF(x)$, $\int_{a_0}^{a_1} a_1 \, dF(x)$ and $\int_{a_1}^{a_3} a_3 \, dF(x)$ is less than or equal
to \( \int_{-\infty}^{0} x \, dF(x) = B_1 \) for \( r \leq 0 \). And each of
\[
\int_{0}^{r} x \, dF(x), \quad \int_{a_0}^{a_1} a_1 \, dF(x) \quad \text{and} \quad \int_{a_1}^{a_3} a_3 \, dF(x)
\]
is less than or equal to \( \int_{0}^{\infty} x \, dF(x) = B_2 \) which in turn imply (4.16).

**Remark 4.2:** Examination of (2.14) through (2.16) show that, for symmetric distributions, the case \( r > 0 \) is equivalent to its dual \( r < 0 \) in the sense that both give the same value of \( \Delta_k (\{ \psi_i(r); i \geq 0 \}, F) \) (see also [13]). For nonsymmetric distributions, however, we have to solve (3.15) for the two cases \( r > 0 \) and \( r < 0 \). And then choose the one with the least expected cost. Moreover, the bounds on \( r \) given by Theorem 4.2 will be relaxed in case of symmetric distributions. This is so, since then \( B_1 = B_2 = (1/2) M(F) \). Thus for symmetric distributions we can content ourselves to the following bounds on \( r \):

(4.19) Either \( r_1 \leq r \leq 0 \), or \( 0 \leq r \leq r_2 \); \( k = 0, 1, 2, 3 \) and 4

where \( r_1 = -r_2 \), and

\[
\begin{align*}
\delta_0 + M(F) & \quad \text{for} \quad k = 0, \\
\delta_1 + M(F) & \quad \text{for} \quad k = 1 \\
\delta_{k+2} M(F) & \quad \text{for} \quad k = 2 \text{ and } 3 \\
\delta_{k+4} + 3 M(F) & \quad \text{for} \quad k = 4.
\end{align*}
\]

(4.20)

**Example 4.1 (continued):** For the distribution (4.8) and the fifth search path of (4.9) we obtain \( \delta_1 \approx 1.47, r_1 \approx -2.26, r_2 \approx 2.26 \). Thus for the distribution (4.8), relation (3.15) is equivalent to

(4.21)

\[
\Delta_{k}^{*} (r^*, F) = \inf_{r \in (-2.26, -0.647 142 8)} \{ \Delta_{k} (\{ \psi_i(r); i \geq 0 \}, F) \}
\]

and

(4.22)

\[
\Delta_{k}^{*} (r^*, F) = \inf_{r \in (0.647 142 8, 2.26)} \{ \Delta_{k} (\{ \psi_i(r); i \geq 0 \}, F) \}
\]

But (4.8) is symmetric, so by remark 4.2 we consider either (4.21) or (4.22).
3. Fixed points

We have mentioned in section 3(b) that the function $A_{fc}(\{\psi_i(r); i \geq 0\}, F)$ is not defined whenever the related side condition from (3.16) and (3.17) is not fulfilled. (recall that we are concerned with search paths of the form (3.13)). From our hypotheses, the relation

\[(4.23) \quad \psi_{i+1}(r) \neq \psi_i(r) \neq \psi_{i-1}(r)\]

for all pertinent $i$ and $k$ should also hold at any regular distribution. But it can happen that (4.23) does not hold everywhere for any distribution. Indeed if we assume that

\[(4.24) \quad \psi_{i+1}(r) = \psi_i(r) = \psi_{i-1}(r) = \gamma\]

for all pertinent $i$ and $k$, then equation (3.12) is equivalent to

\[(4.25) \quad \gamma = \psi_i(\gamma) (\Leftrightarrow \psi_i(r) = r)\]

for all pertinent $i$ and $k$.

In such cases, then by the bounds on $r$ indicated above and by Brouwer Fixed Point Theorem (see [3] Theorem 23.8] the continuous function $\psi_i(r)$ has at least one fixed point. In this case equation (3.4) is equivalent to

\[(4.26) \quad \frac{1}{f(\gamma)} - 2|\gamma| = 0; \quad i \text{ and } k \text{ are given by (3.2)}\]

which in turn gives the fixed points of $\psi_i(r)$ (if any) at any regular distribution $F$. Moreover, it is also possible to obtain some other kinds of fixed points for the function $|\psi_i(r)|$. For if we assume that

\[(4.27) \quad |\psi_{i+1}(r)| = |\psi_i(r)| = |\psi_{i-1}(r)| = \beta\]

for all pertinent $i$ and $k$.

Then (3.4) is equivalent to

\[(4.28) \quad \begin{cases} [1 + F(-\beta) - F(\beta)]/f(\pm \beta) - 2\beta = 0; \\ \beta \geq 0; \quad i \text{ and } k \text{ are given by (3.2)} \end{cases}\]

which gives the fixed points (if any) for the function $|\psi_i(r)|$ at any regular distribution $F$. When the $F$ is also symmetric, then by (2.13), relation (3.4) will have the form

\[(4.29) \quad y_{i+1} = \frac{2-[F(y_i) + F(y_{i-1})]}{f(y_i)}; \quad i \geq k; \quad k = 0, 1, 2, 3 \text{ and } 4.\]
Substituting (4.27) in (4.29) we obtain

\[ \begin{cases} [1 - F(\beta)]/f(\beta) - \beta = 0; \\ i \text{ and } k \text{ are given as in (4.29)} \end{cases} \]

which gives the fixed points (if any) of \(|\psi_i(r)|\) for regular and symmetric distributions. From the above discussion we observe that the existence of fixed points for the functions \(\psi_i(r), |\psi_i(r)|\) is not guaranteed. Because we cannot assure that each or both of (4.24) and (4.27) are really fulfilled for all pertinent \(i\) and \(k\). However when such points do exist the functions \(\psi_i(r)\) and \(|\psi_i(r)|; i \geq 0\) change their values very slowly near them. Therefore the search paths \(\{\psi_i(r); i \geq 0\}, \{|\psi_i(r); i \geq 0\}\), get trapped around these points. Then the side condition (3.16) [(3.17) for symmetric distributions] no longer holds which means that the corresponding \(\Delta_k(\{\psi_i(r); i \geq 0\}, F), (\Delta_k(\{|\psi_i(r); i \geq 0\}, F))\) is not defined.

Example 4.1 (continued): For the distribution (4.8), the solution of (4.30) is \(\beta \approx 0.7517915\). The corresponding \(\{\psi_i(r); i \geq 0\}\) for \(k = 0\) does in fact get trapped around this value of \(\beta\) (see [1] the table on pages 27-28 concerning with the case \(k = 0\) at the distribution (4.8)). This results in a gap on the plot of \(\Delta_0(\{|\psi_i(r); i \geq 0\}, F)\) as it can be seen from figure 2 below which shows the plot of \(\Delta_k(\{|\psi_i(r); i \geq 0\}, F), k = 0, 1\) and 2 as functions of \(r\), at the distribution \(F\) that is given by (4.8). Each point from the plot of \(\Delta_k(\{|\psi_i(r); i \geq 0\}, F)\), in this figure, corresponds to a critical search path from class \(Q_k; k = 0, 1\) and 2. (The set of critical search paths from each of \(Q_3\) and \(Q_4\), for the distribution (4.8), is empty as has indicated before.)

4. The (GLSP) as a function of \(r\) only

So far we have shown that the (GLSP) is completely characterized by the first entry \(r\). The question we address now is how the changes in \(r\) affect the values of the \(\Delta_k\)'s and the \(a_i\)'s i.e. what about the derivatives of the \(\Delta_k\)'s and the \(a_i\)'s with respect to \(r\) as the only significant variable. In fact we have

\[ \frac{d\Delta_k}{dr} = \sum_{i=0}^{\infty} \frac{\partial \Delta_k}{\partial a_i} \frac{da_i}{dr}; \quad k = 1, 2, 3 \text{ and } 4. \]

Since (3.1) holds at any critical search path, we obtain

\[ \frac{d^2 \Delta_k}{dr^2} = \sum_{i=0}^{\infty} \frac{\partial^2 \Delta_k}{\partial a_i^2} \left(\frac{da_i}{dr}\right)^2; \quad k = 1, 2, 3 \text{ and } 4 \]
(4.31) and (4.32) hold also for $k = 0$ with summations starting from $i = 1$. Let $D_i = da_i/dr$, then for $k = 1, 2, 3$ and 4 we clearly have $D_0 = 1$, and then from (3.6), (3.7) and (3.8) we obtain

$$D_1 = \frac{da_1}{dr} = \frac{(-1)^{k+1} \text{sign}(r)f'(r)}{2f^2(r)} + 1; \quad k = 1, 2, 3 \text{ and } 4$$

whereas $D_1 = 1$ for $k = 0$. 

Figure 2.
GENERALIZED OPTIMAL SEARCH PATHS FOR RANDOM VARIABLES

Since for \( i \geq 1 \) each \( a_{i+1} \) is a function of \( a_i \) and \( a_{i-1} \), we have

\[
D_{i+1} = \frac{da_{i+1}}{dr} = \frac{\partial a_{i+1}}{\partial a_i} \cdot \frac{da_i}{dr} + \frac{\partial a_{i+1}}{\partial a_{i-1}} \cdot \frac{da_{i-1}}{dr}
\]

\[
= \frac{\partial a_{i+1}}{\partial a_i} D_i + \frac{\partial a_{i+1}}{\partial a_{i-1}} D_{i-1}; \quad i \geq 1.
\]

With the convention that \( D_0 = 0 \) for \( k = 0 \), then simple calculations on (3.4) and (3.5) yield the following recursive formula.

\[
D_{i+1} = \left[ 2 + \text{sign}(a_i) \frac{f'(a_i)}{f(a_i)} (|a_i| + |a_{i+1}|) \right] D_i - \frac{f(a_{i-1})}{f(a_i)} D_{i-1}
\]

(4.34)

\[i \text{ and } k \text{ are given by (3.2).}\]

For the other values of \( i \) and \( k \) we may obtain \( D_{i+1} \) from (3.9) through (3.11). Now from (4.32), (4.33) and (4.34), \( D_i \) is a function of \( r \) for all \( i \geq 0 \). And both \( d \Delta_k/dr, d^2 \Delta_k/dr^2 \) could be expressed as functions of \( r \). These results may, in fact, facilitate the task of studying the convexity and concavity of the functions \( \Delta_k(\{\psi_i(r); i \geq 0\}, F); k = 0, 1, 2, 3 \) and \( 4 \). Moreover, the values of \( D_i \)'s as functions of \( r \) will provide us with good indications of how the changes in \( r \) will affect the entries \( a_i = \psi_i(r) \) as will be seen in the next example.

Example 4.1 (continued): Let us return to the distribution (4.8) and consider the case \( k = 1 \). The optimal value of \( \Delta_i(\{\psi_i(r); i \geq 0\}, F) \) has been found to occur at an extreme point for which we have obtained Computer Results (1) below concerning the optimal search path \( \{x(i); i \geq 0\} \) and the optimal value of \( r \) with the corresponding optimal value of \( \Delta_i(\{\psi_i(r); i \geq 0\}, F) \) together with the derivatives of \( x(i)'s \) with respect to \( r \).

Making an infinitesimal change in \( r \) gives Computer Results (2) with more of the entries \( x(i)'s \).

The infinitesimal changes in \( r \) have been continued to be made upon reaching 29 decimal digits giving us Computer Results (3) with about 20 of the entries \( x(i)'s \), upon reaching the system capacity.

One can easily see, from these results, how the changes in \( r \) affect significantly the entries of a (C.S.P.) especially the last ones.
MINIMUM VALUE OF DELTA(1) AS A FUNCTION OF R AT
NORMAL DISTRIBUTION START RO = .2500000000000000000000D+01
.273520710722492517D+01 .372588348920794711D+01 .460457032676041236D+01 .540398608439686088D+01
.614395778099737350D+01 .689701519515108318D+01 .111210411000573341D+02 .367287349999999915D+02

-- R MIN = 1.171183469499999582892114346D+01 --- MIN DELTA(1) = 1.14309704879697850D+01

THE INTEGRAL OF ABS(X)*DF(X) FROM 0 TO X0 IS ANS = .258848047691515D+00 IF = 0

DERIVATIVES OF X(I)'S (W.R.T.) R

576612968736077368D+01 .781086648443988746D+02 .212663080423648652D+04 .907175325788159298D+05
.536331679534951718D+07 .411617841232085908D+09 .494460305776306525D+11 -.167089042531448099D+26

Computer Results (1)

MINIMUM VALUE OF DELTA(1) AS A FUNCTION OF R AT
NORMAL DISTRIBUTION START RO = .15711834694173698250000000000D+01
.273520710674846096D+01 .372588348543465864D+01 .460457015104231702D+01 .540397858861002335D+01
.61439514161292364491D+01 .683574020217686371D+01 .74993648768177995D+01 .811602758813601743D+01
.871391014912029539D+01 .933660106048166987D+01 .2109240526519919D+02 .367287349999999919D+02

-- R MIN = 1.171183649173698249999999970D+01 --- MIN DELTA(1) = 1.14309704876551590D+01

THE INTEGRAL OF ABS(X)*DF(X) FROM 0 TO X0 IS ANS = .2588480476795603D+00 IF = 0

DERIVATIVES OF X(I)'S (W.R.T.) R

576612968612650895D+01 .781086647123870600D+02 .212663074643398112D+04 .907174525557497033D+05
.536338903291127185D+07 .410452862757316119D+09 .388926585546354830D+11 .442346814533079268D+13
.590195474436097331D+15 .909745017387603231D+17 .255014054225976485D+20 -.432787730246561105D+95

Computer Results (2)

MINIMUM VALUE OF DELTA(1) AS A FUNCTION OF R AT
NORMAL DISTRIBUTION START RO = .157118346941736982465319165665D+01
.273520710674846096D+01 .372588348543465820D+01 .460457015104231602D+01 .540397858860997820D+01
.614395141612928722D+01 .683574020199585647D+01 .74893648768177995D+01 .811602758813601743D+01
.871391014912029539D+01 .9266178492499770D+01 .983704806911862195D+01 .103687121329197060D+02
.10883279583185505D+02 .113891036129381916D+02 .137000266795300648D+02 .367287349999999919D+02

-- R MIN = 1.1711836491736982465319165635D+01 --- MIN DELTA(1) = 1.14309704876551589D+01

THE INTEGRAL OF ABS(X)*DF(X) FROM 0 TO X0 IS ANS = .2588480476795603D+00 IF = 0

DERIVATIVES OF X(I)'S (W.R.T.) R

576612968612650894D+01 .781086647123870593D+02 .212663074643598080D+04 .907174535374885579D+05
.5363389032909884246D+07 .410452862750036635D+09 .388926584934466641D+11 .442346814533079268D+13
.590183454198580020D+15 .907101227856345590D+17 .138249302400651113D+20 .309621214209567913D+22
.672609679726555356D+24 .160974138898564065D+27 .458477228620755781D+29 -.626248807280983071D+39

Computer Results (3)
5. ALGORITHM AND ILLUSTRATION

(a) Computational algorithm

We have pointed out that there are bounds on the main variable $r$. And that the functions $\left| \psi_i(r) \right|, \psi_i(r)$ may have some kinds of fixed points causing a gap in the graph of $\Delta_k \left( \{ \psi_i(r); i \geq 0 \}, F \right)$ (recall, Figure 2). In the case of no fixed points (i.e. the side conditions (3.16) or (3.17) are satisfied) then the function $\Delta_k \left( \{ \psi_i(r); i \geq 0 \}, F \right)$ would be of continuous type. For example the side conditions are always fulfilled for any of the special search paths (4.9). Note that, for these special search paths, the plot of $\Delta_1 (r, F)$ at the $F$ given by (4.8), as shown in [1] figure 26, is of continuous type.

If $\Delta_k \left( \{ \psi_i(r); i \geq 0 \}, F \right)$ is continuous and of convex or concave type, we may use the following algorithm for finding the optimal value $r^*$ of $r$ and the corresponding $\Delta_k^* (r^*, F)$ as defined by (3.15).

The algorithm

$r_1 =$ left bound of $r$, $r_2 =$ right bound of $r$.

$\varepsilon$ is an infinitely small positive quantity say $\varepsilon = 0.1 \times 10^{-10}$.

$I = 1$, and $N$ is the number of suitable iterations, say $N \approx 100$.

Step (1):

$$r_{11} = r_1 + (1/2) (r_2 - r_1 - \varepsilon)$$
$$r_{21} = r_{11} + \varepsilon$$
$$Q_1 = \Delta_k \left( \{ \psi_i(r_{11}); i \geq 0 \}, F \right)$$
$$Q_2 = \Delta_k \left( \{ \psi_i(r_{21}); i \geq 0 \}, F \right)$$
$$\delta = Q_1 - Q_2.$$

If $\delta$ is greater than zero go to step (2).
If $\delta$ equals to zero go to step (5).
If $\delta$ is less than zero go to step (3).

Step (2): $r_1 = r_{11}$ go to step (4).

Step (3): $r_2 = r_{21}$.

Step (4): If $I$ is greater than $N$ go to step (8) otherwise $I = I + 1$.
If $(r_2 - r_1)$ is greater then $2 \varepsilon$ go to step (1) otherwise go to step (5).

Step (5): If $Q_1$ is less than $Q_2$ go to step (6) otherwise go to step (7).

Step (6): The optimal values are $r^* = r_{11}, \Delta_k^* (r^*, F) = Q_1$ go to step (8).
Step (7): The optimal values are $r^* = r_{21}, \Delta_k^* (r^*, F) = Q_2$. 

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Step (8): Stop.

If, however, $\Delta_k(\{ \psi_i(r); \ i \geq 0 \}, F)$ is piecewise continuous so that its curve is constituted of several parts each of which is either convex or concave. Then we minimize on each part and take the overall minimum value of the minimums of those parts. In cases with gaps like figure 2 we have first to find the extreme points of these gaps (the points after which or before which the side conditions (3.16) or (3.17) start to be violated). Then we consider the left or right bounds of $r$ starting from these points. It is then to be noted that if $\Delta_k(\{ \psi_i(r); \ i \geq 0 \}, F)$ is concave or convex on the parts that result from the extreme points as it is the case in figure 2. Then one of the extreme points would be a strong candidate to represent the optimal solution as will be seen in the next example.

(b) Example (5.1)

By the result of Theorem 4.1, a family of distributions called the Bimodal Normal is to be considered. This family is characterized by the positive parameters $\mu$ and $\sigma$ so that their densities are given by

$$f(x) = \frac{1}{2 \sigma \sqrt{2 \pi}} \left[ e^{-\frac{(1/2)(x+\mu)/\sigma)^2}{2}} + e^{-\frac{(1/2)(x-\mu)/\sigma)^2}{2}} \right]; \ x \in \mathbb{R}$$

Each member of these densities is symmetric and have two modals occurring at $-\mu$, $\mu$. The results presented here are concerned with $\sigma=1$ (recall remark 2.1). The following formula for the $F$'s has been used for computations.

$$F_\mu(x) = \frac{1}{2} + \frac{1}{4} \left[ \text{ERF}((x+\mu)/\sqrt{2}) + \text{ERF}((x-\mu)/\sqrt{2}) \right]; \ x \in \mathbb{R}$$

because the error function $\text{ERF}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} \, dx$ does exist in the computer library. Table I contains the extremal values of $\Delta_k(\{ \psi_i(r); \ i \geq 0 \}, F)$ for $k = 0, 1, 2$ and 4, at different values of $\mu$ ($\mu = 1, 2, 3, 5, 7$ and 10). We note that, for $\mu \geq 2$, $k = 0, 1$ and 2 there are two extreme values of $r$, between which there is a gap. The optimal value of $\Delta_k(\{ \psi_i(r); \ i \geq 0 \}, F); \ k = 0, 1$ and 2, occurs at one of these values. When $\mu = 1$, however, there is only one (right) extreme point for each of $k = 0, 1$ and 2 at which $\Delta_k(\{ \psi_i(r); \ i \geq 0 \}, F)$ attains its optimal value. On the other hand, it has been found that $|a_2| < 0$ for $k = 4, \ \mu = 1$, which means that, for $\mu = 1$, the set of critical search...
<table>
<thead>
<tr>
<th>$\mu$ = 1</th>
<th>$\mu$ = 2</th>
<th>$\mu$ = 3</th>
<th>$\mu$ = 5</th>
<th>$\mu$ = 7</th>
<th>$\mu$ = 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta_1(\psi_1(r), F))</td>
<td>2.87127273</td>
<td>0.24252615</td>
<td>4.01255460</td>
<td>1.20034313</td>
<td>5.11436289</td>
</tr>
<tr>
<td>(\Delta_2(\psi_1(r), F))</td>
<td>1.34282437</td>
<td>4.35438795</td>
<td>0.99108155</td>
<td>6.27893356</td>
<td>0.28200713</td>
</tr>
<tr>
<td>(r)</td>
<td>1.29895724</td>
<td>0.47224569</td>
<td>2.36211261</td>
<td>1.56658200</td>
<td>3.44041114</td>
</tr>
<tr>
<td>(\Delta_4(\psi_1(r), F))</td>
<td>2.43603227</td>
<td>3.92741316</td>
<td>1.96960308</td>
<td>3.92740145</td>
<td>0.64565836</td>
</tr>
<tr>
<td>(\Delta_6(\psi_1(r), F))</td>
<td>EMPTY</td>
<td>0.84507105</td>
<td>1.56944457</td>
<td>1.74669804</td>
<td>2.96717147</td>
</tr>
<tr>
<td>(\Delta_8(\psi_1(r), F))</td>
<td>EMPTY</td>
<td>8.30154518</td>
<td>7.69095107</td>
<td>63.4327457</td>
<td>174601.077</td>
</tr>
</tbody>
</table>

*Table I*

*Extremal and Optimal values of \(\Delta_4(\psi_1(r); i \geq 0); K = 0, 1, 2, 4\)*

at the Bimodal Normal distribution with \(\mu = 1, 2, 3, 5, 7\) and \(10\) (*).  

---

(†) Except for \(\alpha, \beta\) and \(M(F)\) all other computations has been done in double precision on the CDC system in Brussels University. This system has the range of \(-10^{322}\) to \(10^{322}\) and the zero value for \(10^{-319}\). A double precision constants in this system are accurate up to 29 decimal digits.
## Table II

Optimal search paths with the corresponding optimal searching time $T_\mu^o (r^*, F)$, $K=0, 1, 2$ and 4, at the Bimodal Normal distribution with $\mu=1, 2, 3, 5, 7$ and 10.

<table>
<thead>
<tr>
<th>$r = X$</th>
<th>$\mu = 1$</th>
<th>$\mu = 2$</th>
<th>$\mu = 3$</th>
<th>$\mu = 5$</th>
<th>$\mu = 7$</th>
<th>$\mu = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>3.214655969691</td>
<td>4.09254094083</td>
<td>5.11722816163</td>
<td>7.252170504691</td>
<td>9.35507978548</td>
<td>12.467650287755</td>
</tr>
<tr>
<td>(2)</td>
<td>4.578116365823</td>
<td>5.54211348044</td>
<td>6.604853980157</td>
<td>8.770081640992</td>
<td>10.8950687797</td>
<td>14.03427712881</td>
</tr>
<tr>
<td>(3)</td>
<td>5.677195267114</td>
<td>6.680801794243</td>
<td>7.761684042148</td>
<td>9.944188494140</td>
<td>12.08291794395</td>
<td>15.2963224067</td>
</tr>
<tr>
<td>(5)</td>
<td>7.490148322959</td>
<td>8.533987361095</td>
<td>9.633861788931</td>
<td>11.83630675114</td>
<td>13.99280362461</td>
<td>17.17288295498</td>
</tr>
<tr>
<td>(7)</td>
<td>8.930619034936</td>
<td>9.996607893149</td>
<td>11.10793844002</td>
<td>13.24475116216</td>
<td>15.41583927176</td>
<td>18.6151810652</td>
</tr>
<tr>
<td>(10)</td>
<td>10.70609224498</td>
<td>11.79304538644</td>
<td>12.91619826807</td>
<td>15.08027508699</td>
<td>17.26648375693</td>
<td>20.48642986134</td>
</tr>
<tr>
<td>(14)</td>
<td>12.86082112568</td>
<td>14.12748603736</td>
<td>16.98009384417</td>
<td>22.6843424666</td>
<td>217.451972384</td>
<td>0.124 D + 263</td>
</tr>
<tr>
<td>(15)</td>
<td>53.30179037749</td>
<td>496.051340949</td>
<td>339.983422547</td>
<td>0.9162 D + 261</td>
<td>0.1028 D + 261</td>
<td>...........................</td>
</tr>
</tbody>
</table>

$T_\mu^o (r^*, F)$: 

<table>
<thead>
<tr>
<th>$r = X$</th>
<th>$T_\mu^o (r^*, F)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0)</td>
<td>4.487536669651</td>
</tr>
<tr>
<td>(1)</td>
<td>6.287272071844</td>
</tr>
<tr>
<td>(2)</td>
<td>8.140255601867</td>
</tr>
<tr>
<td>(3)</td>
<td>10.14365858018</td>
</tr>
<tr>
<td>(4)</td>
<td>12.25544474067</td>
</tr>
<tr>
<td>(5)</td>
<td>14.36476729814</td>
</tr>
<tr>
<td>(6)</td>
<td>16.47628213362</td>
</tr>
<tr>
<td>(7)</td>
<td>18.586000273145</td>
</tr>
<tr>
<td>(8)</td>
<td>20.69617389325</td>
</tr>
<tr>
<td>(9)</td>
<td>22.806232397171</td>
</tr>
<tr>
<td>(10)</td>
<td>24.916253239717</td>
</tr>
<tr>
<td>(11)</td>
<td>27.02628213362</td>
</tr>
<tr>
<td>(12)</td>
<td>29.13628213362</td>
</tr>
<tr>
<td>(13)</td>
<td>31.24628213362</td>
</tr>
<tr>
<td>(14)</td>
<td>33.35628213362</td>
</tr>
<tr>
<td>(15)</td>
<td>35.46628213362</td>
</tr>
</tbody>
</table>

\(r = X \) (0) = 4.012554601867, \(r = X \) (1) = 5.114362890815, \(r = X \) (2) = 6.60282133626, \(r = X \) (3) = 7.760000273145, \(r = X \) (4) = 8.749617389325, \(r = X \) (5) = 9.632523797171, \(r = X \) (6) = 10.94822432661, \(r = X \) (7) = 12.106723210079, \(r = X \) (8) = 13.24701782293, \(r = X \) (9) = 14.3965423466, \(r = X \) (10) = 15.547045468106.}

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| X (13) | 12.68263630811 | 14.2971182548 | 16.3099387317 | 61.00679178693 | 228.6439126.48 | 0.1102D + 263 |
| X (14) | 18.24658809394 | 5409.840101868 | 3752.03458.92 | 0.2831 D + 262 | 0.9191 D + 253 | ............................ |
| X (15) | 0.7492 D + 34 | 0.4954 D + 259 | 0.1071 D + 254 | ............................ | ............................ | ............................ |
| T_\alpha (r^*, F) | 2.509455508716 | 2.998062954314 | 3.28277143835 | 3.46992909527 | 2.54042389462 | 3.61618037871 |
| r = X (0) | 1.298957237647 | 2.362112607942 | 3.44011139417 | 5.53789749456 | 7.5997224835423 | 10.66230892911 |
| X (1) | 2.518761061530 | 3.700285548679 | 4.812362063684 | 6.98626047825 | 9.09996216863 | 12.22298313543 |
| X (2) | 4.076125395810 | 5.266377857421 | 6.398856477007 | 8.58791411980 | 10.71858525841 | 13.86593513699 |
| X (3) | 5.258850040400 | 6.452993643917 | 7.592145216363 | 9.792974586089 | 11.93843031567 | 15.1015647532 |
| X (4) | 6.263057390019 | 7.49525490104 | 8.602758705657 | 10.8133788044 | 12.96805347302 | 16.14311497876 |
| X (6) | 7.996837056064 | 9.167852310270 | 10.31674908506 | 12.4537596364 | 14.62344746399 | 17.81894612630 |
| X (7) | 8.639369907420 | 9.839672534178 | 10.99179867520 | 13.14049824480 | 15.3170368053 | 18.52124635524 |
| X (10) | 10.45205446328 | 11.65656340985 | 12.81539085482 | 14.9960190374 | 17.18134917086 | 20.40563513597 |
| X (11) | 11.0642148569 | 12.21177214511 | 13.37217006180 | 15.55382500234 | 17.74834979043 | 20.97812366362 |
| X (14) | 12.5653777384 | 13.78409841274 | 15.19573110789 | 106.9354597356 | 1185.56389053 | 2.431 D + 263 |
| X (15) | 14.32227530839 | 20.73148301570 | 723.7402716630 | 0.4629 D + 262 | 0.4894 D + 259 | ............................ |
| T_\alpha (r^*, F) | 3.602663214331 | 3.9865844886 | 3.64642268403 | 2.930208268070 | 2.56595606585 | 20.8899182088 |
| r = X (0) | 1.56944456730 | 2.18565711887 | 4.37132823399 | 6.136349731960 | 9.109824318074 | ............................ |
| X (1) | 2.941904273447 | 3.93172528129 | 5.898481584703 | 7.95619095549 | 10.97246969880 | ............................ |
| X (2) | 0.000000000424 | 1.25541490452 | 3.178922849853 | 5.25714503031 | 8.253366909598 | ............................ |
| X (3) | 5.432025432230 | 8.23379097466 | 11.19771600260 | 12.1834535590 | 15.1998386320 | ............................ |
| X (4) | 447.3859103095 | 106736.451212 | 2653959.0592 | 321055.653022 | 894255.236693 | ............................ |
| X (5) | 0.1603 D + 290 | 0.4445 D + 286 | 0.1533 D + 284 | 0.5830 D + 286 | 0.5339 D + 286(*) | ............................ |
| T_\alpha (r^*, F) | 8.857582009614 | 12.49372507934 | 9.991091059647 | 5.390653488692 | 2.760807975461 | ............................ |

(*) All such values in this table means that we have reached the computer ranges, therefore, such values should not be taken seriously.
paths from the class $Q_4$ is empty. The calculations showed that
$\Delta_k(\{\psi_i(r); i \geq 0\}, F); k=0, 1$ and $2$, decrease when $r$ varies from zero to the
left extreme (l.e.) point, and increase when $r$ takes on values greater than
the right extreme (r.e.) point, so that the optimal value of $\Delta_k$'s, $k=0, 1$ and $2$ occurs at the right extreme points. However, the situation for $k=4$ is quite
different from those of $k=0, 1$ and $2$. For $k=4$, $\mu \geq 2$ there are two extreme
points with a gap before the first, and a gap after the second so that the side
conditions (3.17) is fulfilled between these two extremes. It happens that the
optimal value of $\Delta_4(\{\psi_i(r); i \geq 0\}, F)$ for $\mu=2$ occurs at the second extreme,
whereas for $\mu > 2$ this optimal value occurs at a point that lies between the
indicated two extreme points. In all cases ($k=0, 1, 2$ and $4$), the existence of
a gap before or after an extreme point means that the side condition (3.17)
is violated, hence $\Delta_k(\{\psi_i(r); i \geq 0\}, F)$ is not defined. Some other significant
values in Table I has the following meaning: The value (values) of $\alpha$ indicates
the bounds on $r$ that can be obtained from (4.4) for $k=0$, (4.5) for $k=1, 3,$
and (4.6) for $k=2, 4$. The value of $\beta$ indicates the fixed points that have
been obtained from (4.30). Some other kinds of fixed points may be obtained
from (4.26). The $M(F)$ indicates the first absolute moment of $F$. Table II
contains some of the entries $x(i) = \beta_i, i \geq 0$ (for $k=0$) of the optimal
search path for each value of $\mu$ given in Table I and for each of the cases
$k=0, 1, 2$ and $4$. These entries has been calculated as far as the system
capacity. The fact that the optimal value of $\Delta_k(\{\psi_i(r); i \geq 0\}, F); k=0, 1$
and $2$, occurs at an extreme point was a very helpful tool in studying the
strong relations between $r$ and the entries $x(i); i \geq 1$ as these relations are
given by (4.32) and (4.34) (recall the last example in the previous section).
The entries $x(0), x(1), x(2), x(3), x(4), \ldots$, in Table II should be under-
stood, for $k=1$ for instance, as follows; $a_0=x(0), a_1=-x(1), a_2=x(2),$
$a_3=-x(3), a_4=x(4), \ldots \text{ with similar understanding for } k=0, 2$ and $4.$
Table II contains also the optimal searching time denoted by $T^*_k(r^*, F)$ for
$k=0, 1, 2$ and $4$. One can easily verify that $T^*(r^*, F)/M(F)$ satisfies
Theorem 2.5.

We would finally like to mention that the results in Table I and Table II
are only roughly correct due to many difficulties in the calculational system
such as accumulation errors, the bounds on the system ranges, the system
capacity, etc. Thus the large values of $x(i); i \geq 0$ in Table II would, in fact,
result in less precision than the small ones. Nevertheless, this will cause a
very slight change in the resulting $\Delta_k(\{\psi_i(r); i \geq 0\}, F)$. Because the term
$2-[F(x_i) - F(x_{i-1})]]$, for those large values of $x_i$, equals zero in the computer
digits. [Recall the values of $\Delta^*_k(r^*, F)$ in the three computer results concerning
the distribution (4.8), in the last example of Section 4.]
6. CONCLUSION

In this paper we have introduced analytical methods for constructing and studying some important properties of optimal search paths for the (GLSP) at the absolutely continuous class of target distributions that have strictly positive densities. We have shown, then, that the (GLSP) can be characterized by only a single variable instead of infinitely many. The techniques used in this study are those of standard calculus so that an optimal search path would, in general, be a critical one. It has also been shown that for three of the only five possible cases of search, and for the distributions of unimodal type with the mode occurring at zero, then these (C.S.P.)'s are not minimal (maximum, saddle, or extreme). We would finally, note that the results of Table I indicate that for the distributions (5.1), the class $Q_1$ is better than the class $Q_0$, and for most of the values of $\mu$ the classes $Q_2$, and $Q_4$ are better than the class $Q_0$ in the sense that they give less expected cost. Note also that some of the classes $Q_k$ is better than some others justifying, thus, the generalization of the linear search problem that has been introduced in [2]. Some other results concerning the search for a target located in the plane or on a line may be found in [14], and [15].

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REFERENCES


