RAIRO. RECHERCHE OPÉRATIONNELLE

CHARLES S. TAPIERO
DROR ZUCKERMAN

Optimum claims truncation of an insurance firm with a compound Poisson claim process : a diffusion approximation

RAIRO. Recherche opérationnelle, tome 14, nº 2 (1980), p. 129-136
<http://www.numdam.org/item?id=RO_1980__14_2_129_0>

R.A.I.R.O. Recherche opérationnelle/Operations Research
(vol. 14, n° 2, mai 1980, p. 129 à 136)

OPTIMUM CLAIMS TRUNCATION OF AN INSURANCE FIRM WITH A COMPOUND POISSON CLAIM PROCESS: A DIFFUSION APPROXIMATION (*)

by Charles S. TAPIERO and Dror ZUCKERMAN (1)

Abstract. — *This paper considers the problem of an insurance firm facing a compound Poisson claims restricted process where policies are defined by an investment barrier strategy as well as the choice of a loading factor. Lower truncation of the claim process, expressing insured persons participation in case of claims, is defined as a firm's decision variable and solved optimally. To obtain analytical results diffusion approximations of the process are made.*

Résumé. — *Cet article considère le problème d'une compagnie d'assurance face à une suite de dédommagement de fonds suivant un processus de Poisson composé avec une restriction. La politique de la firme consiste à déterminer une barrière au dessus de laquelle, toute accumulation de fonds est investie et un facteur déterminant les taux de paiements à la firme. La restriction du processus exprime un niveau au dessous duquel l'assuré assume la responsabilité des dommages. Cette variable est aussi considérée comme variable de décision et trouvée par la solution optimale d'un problème. Pour faciliter les calculs, une approximation de diffusion du processus a été faite.*

1. INTRODUCTION

In a previous paper (Tapiero-Zuckerman [3]) we have considered the optimum policies of an insurance firm facing a compound Poisson claim process where policies are defined by an investment barrier strategy as well as the choice of a loading factor. An essential premise was that claims were unrestricted. Yet, in practice, insurance firms tend to insure excessive claims by using re-insurance schemes (e. g. Borch [1] and Buhlmann [2]). In addition, firms may establish lower limits, below which claims are not honored. This is a common practice in the car insurance business where the first ξ dollars are deductible. The exact deductible amount ξ is often called the clients participation and it will be considered by us as a decision variable. Further, persons attitudes towards risk confirm the tendency of clients to be responsible for small damages and insured against larger ones. A consideration of such attitudes, appropriately defined, can

(*) Received Mai 1979.
(1) Hebrew University, School of Business Administration, Jérusalem.

be used by insurance firms in selecting "optimum" truncation policy of the claim distribution as well as in assessing the substitution effects with the premium loading factor, the investment policy, and the computable effects on a firm's long run average profit. The purpose of this paper is to study such effects by using a diffusion approximation of the firm's cash level process. In addition an example is resolved and discussed.

2. THE INSURANCE STOCHASTIC MODEL

We assume an insurance firm facing two processes; a premium and a claim process. The claim process is defined by a claim stream arriving according to a Poisson process with rate λ. Successive claim magnitudes $Y_1, Y_2, \ldots, Y_n, \ldots$, are assumed to be positive, independent and identically distributed random variables having a known distribution function $F(.)$. The claim process (without truncation) is therefore a Compound Poisson process with mean μ where:

$$\left.\begin{aligned} \mu &= \lambda\alpha, \\ \alpha &= \int_0^\infty u\, dF(u). \end{aligned}\right\} \tag{1}$$

The participation quantity, ξ, is considered by us as a decision variable.

The arrival claim rate λ is supposed to be a function of the truncation policy ξ and the loading factor π. It is reasonable to assume that $\lambda = \lambda(\pi, \xi)$ has the following properties:

$$\partial\lambda/\partial\pi \leqq 0 \qquad \text{and} \qquad \partial\lambda/\partial\xi \leqq 0. \tag{2}$$

Using well known properties of the Poisson process it can be seen that under a "ξ" truncation policy the actual claim process is also a compound Poisson process with a Poisson parameter $\tilde{\lambda}(\pi, \xi) = \lambda(\pi, \xi)[1 - F(\xi)]$.

Further the actual claim magnitude is distributed according to a distribution function $F_\xi(.)$, where

$$F_\xi(u) = \begin{cases} 0 & \text{if } u < 0, \\ \dfrac{F(u+\xi) - F(\xi)}{1 - F(\xi)} & \text{if } u \geqq 0. \end{cases} \tag{3}$$

Define the moments $\alpha_n(\xi)$ as follows:

$$\alpha_n(\xi) = \int_0^\infty u^n\, dF_\xi(u). \tag{4}$$

Premium rates when a "ξ" truncation policy is employed are assumed to be constant and given by

$$\beta(\pi, \xi)=\tilde{\lambda}(\pi, \xi)\,\alpha_1(\xi)(1+\pi). \tag{5}$$

The investment policy used by the firm is a barrier strategy. That is, management has in mind some target "Q" such that all incoming premiums above this level are invested in long term projects at a known rate of return r. The target Q defines therefore an investment barrier strategy. The purpose of the firm is to maximize long run average profits by selecting the appropriate policies; truncation ξ, loading factor π and investment barrier Q. In this paper we concentrate our attention on the truncation policy ξ.

The firm's profit is defined by its expected funds converted from liquid assets into investments less its costs.

Three types of cost can be distinguished:

(i) Claims processing costs; each claim has a fixed processing cost η.

(ii) An opportunity cost; holding cash at level x $(0<x<Q)$ incurs an opportunity cost of rx.

(iii) Borrowing cost; when the cash level x falls below zero $(x<0)$, a borrowing cost of $C(x)$, which may be nonlinear, is incurred.

Let $Z_\xi(t)$ be the cash level on hand (a stochastic process) at time t, when a "ξ" truncation policy is used. For given π and Q, the long run average profit can be expressed as follows:

$$\psi_{\pi,Q}(\xi)=\lim_{t\to\infty}\frac{E_{\pi,Q}\left\{I(t)-\int_0^t f(Z_\xi(u))\,du\right\}}{t}-\eta\tilde{\lambda}(\pi, \xi), \tag{6}$$

where $I(t)$ are the amount of liquid funds converted into investments in the time interval $[0, t]$ and the function $f(u)$ is defined by:

$$f(u)=\left\{\begin{array}{ll} ru & \text{if } u\geqq 0, \\ C(u) & \text{if } u<0. \end{array}\right\} \tag{7}$$

Without limitation of generality state that at the initial time, cash reserves are on the boundary Q:

$$Z_\xi(0)=Q. \tag{8}$$

Let T be the time spent on the boundary until a claim occurs, i. e.

$$T=\operatorname{Inf}\{t\geqq 0;\ Z_\xi(t)<Q\}. \tag{9}$$

Since claims arrive following a Poisson distribution, using a well known property of the Poisson process, it follows that;

$$E_{\pi,Q}(T)=1/\tilde{\lambda}(\pi,\,\xi). \tag{10}$$

Let S be the time at which we return to the boundary Q. That is,

$$S=\operatorname{Inf}\{\,t\geqq T;\ Z_\xi(t)=Q\,\}. \tag{11}$$

The strong Markov property implies

$$E_{\pi,Q}(S)=E_{\pi,Q}(T)+\int_0^\infty E_{\pi,Q}\,\{\operatorname{Inf}\{\,t\geqq 0;\ Z_\xi(t)=Q\,|\,Z_\xi(0)=Q-u\,\}\}\,dF_\xi(u). \tag{12}$$

We will refer to the time period $(0,\ S)$ as a cycle. It can easily be seen that at time S the process Z_ξ regenerates itself.

The average profit, using a renewal argument, is then simply expressed as the average cycle profit and is expressed as follows;

$$\psi_{\pi,Q}=\frac{\beta(\pi,\,\xi)\,E_{\pi,Q}(T)-r\,QE_{\pi,Q}(T)}{E_{\pi,Q}(S)}-\frac{E_{\pi,Q}\left[\displaystyle\int_T^S f(Z_\xi(u))\,du\right]}{E_{\pi,Q}(S)}-\eta\tilde{\lambda}(\pi,\,\xi). \tag{13}$$

Here, $\beta(\pi,\,\xi)\,E_{\pi,Q}(T)$ is the cycle premium stream converted into investments in $[0,\ T]$, $r\,QE_{\pi,Q}(T)$ is the opportunity cost over the time interval $(0,\ T)$ while the following term in equation (13) is the opportunity and borrowing cost in the subsequent time interval $(T,\ S)$ (where no funds are converted into investments). Finally, $\eta\tilde{\lambda}(\pi,\,\xi)$ is the expected processing cost per unit time.

An evaluation of $\psi_{\pi,Q}(\xi)$ and its maximization with respect to ξ will provide an optimum truncation policy. Of course, by maximizing with respect to π, Q and ξ simultaneously, the optimum insurance firm's policies are determined.

3. DIFFUSION APPROXIMATION

A simplification of the problem above can be reached by using a diffusion approximation. To obtain our approximation we replace the cash level process Z_ξ by a suitable diffusion process, say, X_ξ, defined over the set $(-\infty,\ Q)$. Such an approximation is reached as follows: Let $\Delta_h(Z_\xi(t))$ be an increment in the Z_ξ process accrued over the time interval $(t,\ t+h)$. Thus,

$$\Delta_h(Z_\xi(t))=Z_\xi(t+h)-Z_\xi(t), \tag{14}$$

and define the limits;

$$\mu(\xi)=\lim_{h\downarrow 0}\frac{1}{h}E_{\pi,Q}[\Delta_h(Z_\xi(t))\,|\,Z_\xi(t)],\tag{15}$$

and

$$\sigma^2(\xi)=\lim_{h\downarrow 0}\frac{1}{h}\operatorname{var}_{\pi,Q}[\Delta_h(Z_\xi(t))\,|\,Z_\xi(t)].\tag{16}$$

Using well known properties of the Compound Poisson process we have instead of (15) and (16);

$$\mu(\xi)=\beta(\pi,\ \xi)-\tilde{\lambda}(\pi,\ \xi)\,\alpha_1(\xi)=\tilde{\lambda}(\pi,\ \xi)\,\alpha_1(\xi)\,\pi\tag{17}$$

and

$$\sigma^2(\xi)=\tilde{\lambda}(\pi,\ \xi)\,\alpha_2(\xi).\tag{18}$$

The diffusion process X_ξ has a drift parameter $\mu(\xi)$, diffusion parameter $\sigma^2(\xi)$ and a reflecting barrier at $Q-$ the investment barrier. This process is used to approximate the expected cycle time and the cycle cost over the time interval (T, S). Let $T_{x,y}$ be the expected transition time between x and y, or

$$T_{x,y}=\operatorname{Inf}\{\,t\geqq 0;\ X_\xi(t)=y\,|\,X_\xi(0)=x\,\}.\tag{19}$$

Under a ξ-truncation policy, the expected cycle time can be expressed as follows,

$$E_{\pi,Q}(S)\approx E_{\pi,Q}(T)+\int_0^\infty E_{\pi,Q}(T_{Q-u,Q})\,dF_\xi(u).\tag{20}$$

The theory of diffusion processes implies that

$$E_{\pi,Q}(T_{x,y})=\frac{y-x}{\mu(\xi)}\qquad\text{for}\quad x<y<Q,\tag{21}$$

provided that $\mu(\xi)>0$.

Using equations (10), (20) and (21) it follows that

$$E_{\pi,Q}(S)=1/\tilde{\lambda}(\pi,\ \xi)+\int_0^\infty (u/\mu(\xi))\,dF_\xi(u)\ =1/\tilde{\lambda}(\pi,\ \xi)+\alpha_1(\xi)/\mu(\xi).\tag{22}$$

To obtain an approximate expected cost per cycle, we define

$$w(x)=E_{\pi,Q}\left[\int_0^{\tilde{S}} f(X_\xi(u))\,du\,\middle|\,X_\xi(0)=x\right],\qquad x\leqq Q,\tag{23}$$

where

$$\tilde{S}=\operatorname{Inf}\{\,t\geqq 0;\ X_\xi(t)=Q\,\}.\tag{24}$$

It can be seen that $w(x)$ satisfies the following partial differential equation;

$$\left.\begin{array}{c}\mu(\xi)\dfrac{\partial w(x)}{\partial x}+\dfrac{1}{2}\sigma^2(\xi)\dfrac{\partial^2 w(x)}{\partial x^2}+f(x)=0,\\ w(Q)=0,\end{array}\right\}\tag{25}$$

whose solution can be verified to be

$$w(x)=\int_x^Q\int_{-\infty}^v e^{-(2\mu(\xi)/\sigma^2(\xi))(v-y)}f(y)\,dy\,dv.\tag{26}$$

Now, define

$$W_Q(\xi)=\int_0^\infty w(Q-u)\,dF_\xi(u).\tag{27}$$

Using this expression, an approximate (long run) average profit, $\tilde{\psi}_{\pi,Q}(\xi)$ is given by:

$$\tilde{\psi}_{\pi,Q}(\xi)=\frac{\alpha_1(\xi)(1+\pi)-rQ/\tilde{\lambda}(\pi,\xi)-W_Q(\xi)}{\alpha_1(\xi)/\mu(\xi)+1/\tilde{\lambda}(\pi,\xi)}-\eta\tilde{\lambda}(\pi,\xi).\tag{28}$$

Using (17), equation (28) can be written as follows:

$$\tilde{\psi}_{\pi,Q}(\xi)=\tilde{\lambda}(\pi,\xi)\left\{\pi\alpha_1(\xi)-\frac{[\pi rQ/\tilde{\lambda}(\pi,\xi)+\pi W_Q(\xi)]}{1+\pi}-\eta\right\}.\tag{29}$$

Since η is a single claim processing cost, the term:

$$\varphi(\xi)=\pi\alpha_1(\xi)-\frac{[\pi rQ/\tilde{\lambda}(\pi,\xi)+\pi W_Q(\xi)]}{1+\pi},\tag{30}$$

is the revenue per claim (excluding the processing cost).

After some elementary manipulations we obtain that the optimal participation quantity ξ^* satisfies the following equation

$$\varphi(\xi^*)-\eta=-(\partial\varphi/\partial\xi)/(\partial\ln\tilde{\lambda}(\pi,\xi)/\partial\xi)\big|_{\xi=\xi^*}.\tag{31}$$

Clearly $\varphi(\xi^*)-\eta\geqq 0$ (otherwise, the firm would find it beneficial to determine an infinite participation quantity, i. e., not to operate at all). Recalling that $\partial\tilde{\lambda}(\pi,\xi)/\partial\xi\leqq 0$ and using (31) we obtain that the revenue *per claim* $\varphi(\xi)$ is an increasing function of ξ for values sufficiently close to ξ^*. An optimization of (29) with respect to Q and π lead to the following conditions ($\partial\tilde{\psi}/\partial Q=0$ and $\partial\tilde{\psi}/\partial\pi=0$);

$$-\partial W_Q(\xi)/\partial Q=r/\tilde{\lambda}(\pi,\xi)>0,\tag{32}$$

and

$$\tilde{\psi}_{\pi,Q}(\xi)\,\partial[\ln\tilde{\lambda}(\pi,\xi)]/\partial\pi+\tilde{\lambda}(\pi,\xi)\,\partial\varphi/\partial\pi=0. \tag{33}$$

More explicit results can be found by solving an example as will be done in the next section.

4. AN EXAMPLE

We consider a (piecewise) linear cost function

$$f(u)=\left\{\begin{array}{ll} ru & \text{if } u\geqq 0, \\ -cu & \text{if } u<0 \quad (c>0). \end{array}\right\} \tag{34}$$

Further, we let the claim distribution be exponential with parameter θ, so that

$$F(u)=[1-\exp(-\theta u)] \qquad \text{for} \quad u\geqq 0.$$

Thus

$$\tilde{\lambda}(\pi,\xi)=\lambda(\pi,\xi)[1-F(\xi)]=\lambda(\pi,\xi)\,e^{-\theta\xi}. \tag{35}$$

Because of the lack of memory property of the exponential distribution we have $F_{\xi}(u)=F(u)$ for every $\xi\geqq 0$ and $u\geqq 0$. Therefore $\alpha_1(\xi)=1/\theta$.

Using equations (17) and (18) we obtain:

$$\mu(\xi)=(\pi/\theta)\,\lambda(\pi,\xi)\,e^{-\theta\xi},$$

and

$$\sigma^2(\xi)=(2/\theta^2)\,\lambda(\pi,\xi)\,e^{-\theta\xi}$$

and therefore $2\mu(\xi)/\sigma^2(\xi)=\pi\theta$.

After some elementary manipulations, equations (26) and (34) become

$$w(x)=\frac{c+r}{(\pi\theta)^3}[e^{-\pi\theta x}-e^{-\pi\theta Q}]+\frac{r}{(\pi\theta)^2}\left[\frac{\pi\theta}{2}(Q^2-x^2)-(Q-x)\right] \tag{36}$$

and

$$W_Q(\xi)=(r/(\pi\theta)^2)[\pi Q-\pi/\theta-1/\theta]\ -\pi/(1+\pi)[(c+r)\,e^{-\pi\theta Q}/(\pi\theta)^3]. \tag{37}$$

It is important to note that $W_Q(\xi)$ is independent of ξ (because of the memoryless property). Let us abbreviate $W_Q=W_Q(\xi)$.

Using equation (29), the long run average profit can be expressed as follows:

$$\tilde{\psi}_{\pi,Q}(\xi)=\lambda(\pi,\xi)\,e^{-\theta\xi}\left\{\pi/\theta-\pi\,W_Q/(1+\pi)-\eta\right\}-r\,Q\,\pi/(1+\pi). \tag{38}$$

Clearly, $\lambda(\pi, \xi) e^{-\theta\xi}$ is monotonically decreasing in ξ. Therefore the optimal truncation policy is given by

$$\xi^* = \begin{cases} 0 & \text{if } \{\pi/\theta - \pi W_Q/(1+\pi) - \eta\} \geqq 0, \\ \infty, & \text{otherwise.} \end{cases} \tag{39}$$

In words, under the assumption that the claim magnitude is exponentially distributed, the optimal truncation policy is of bang bang type.

Finally using equation (32) and (33), the optimal investment barrier and the optimal loading factor can be obtained.

REFERENCES

1. K. Borch, The *Mathematical Theory of Insurance*, Lexington Books, Lexington Mass, 1974.
2. H. Buhlmann, *Mathematical Methods in Risk Theory*, Springer-Verlag, Bonn, 1970.
3. C. S. Tapiero and D. Zuckerman, *The Optimum Policies of an Insurance Firm with a Compound Poisson Claim Process*, Working paper Hebrew University, January, 1978.