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Optimal simultaneous replacement and maintenance of a machine with process discontinuities


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OPTIMAL SIMULTANEOUS REPLACEMENT
AND MAINTENANCE OF A MACHINE
WITH PROCESS DISCONTINUITIES

by Charles S. Tapiero (*)

Abstract. — In general, most pieces of equipment tend to become less efficient productively
as their age increases, until they reach a point where it is no longer deemed profitable to use
them. However, the obsolescence of a machine is usually limited to a particular production
process. This paper is concerned with productive obsolescence of machines, which is defined
as process discontinuities. These usually occur when, at some point in time \( t \), a machine
changes its economic function and is retired at another point in time \( T \). In this case the salva
gue value at \( T \) is controlled by the production processes during the time intervals \([0, t]\) and
\([t, T]\).

INTRODUCTION

An ever-present problem faced by almost all businessmen is that of maxi-
mizing the value of a machine or piece of equipment used in the operation of
their business. In general, most pieces of equipment tend to become less efficient
productively as their age increases until they reach a point where it is no
longer profitable to use them. However, the obsolescence of a machine is
usually limited to a particular production process. For example, passenger
trains can at some period \( T \) be transferred to some other function within the
same company. Similarly, commercial airlines can assign new planes to lucra-
tive passenger transportation, and assign the old planes to freight transpor-
tation. The plane itself would not be retired but simply put to some other
use. Retirement would come only when it is no longer deemed profitable to
maintain the plane.

This problem is defined here as process discontinuities, and usually occurs
when at some time \( t \) a machine changes function, and is retired only at another
point in time \( T \). In this case there is a salvage value at \( T \) controlled by the

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processes at which the machine was operating in the time intervals \([0, t_1]\) and \([t_1, T_1]\).

The purpose of this paper is to analyse simultaneous replacement and maintenance policies with process discontinuities. In a previous paper (4) we generalized the Thompson model (5) for simultaneous maintenance and retirement policies from one machine to \(n\) machines. Here we accept process changes for machines so that they are actually not retired but used in some other way. As a result, accumulation of machinery in a firm can be explained by replacement policies, obsolescence and technical progress. The replacement problem we are confronted with is:

Consider a machine put to some production process during an interval \([0, t_1]\). During this interval, the machine’s salvage value-maintenance relation is defined by a differential equation with specified obsolescence and effectiveness-maintenance functions. At \(t_1\) the machine is not retired but put to some other production process and retired at \(T_1\), \(T_1 > t_1\). During the time interval \([t_1, T_1]\), the salvage value resulting from this second process is defined by a differential equation with different obsolescence and effectiveness-maintenance functions. Find the optimal replacement times \(t_1, T_1\) as well as the optimal maintenance policies during both production processes.

**THE MODEL**

The simultaneous maintenance and replacement problem for one machine with no process discontinuities is (5) to determine a sequence of maintenance decisions, \(m(t)\) which will maximize the present value of the machine. This value equals the salvage value of the machine plus the anticipated stream of revenues minus the operating and maintenance costs, all discounted at the proper constant rate of interest. This is a constrained optimization problem in that the rate of change of the salvage value is a function of the deterioration of the machine and the maintenance policy employed. The model for determining the optimum maintenance policy and replacement time is a classic variational problem maximizing:

\[
V(T) = S(T) e^{-rT} + \int_0^T Q(t) e^{-rt} dt
\]

subject to:

\[
0 \leq m(t) \leq M
\]

\(Q(t) = p(t) S(t) - m(t)\)

and

\[
dS(t)/dt = -a(t) + f(t)m(t) \quad S(0) = K
\]
where

\[ T = \text{sale date of the machine} \]
\[ V(t) = \text{present value of the machine at time } t \]
\[ S(t) = \text{salvage value of the machine at time } t \]
\[ Q(t) = \text{net operating receipts at time } t \]
\[ r = \text{constant rate of interest} \]
\[ m(t) = \text{maintenance policy -- hours of maintenance at time } t \]
\[ f(t) = \text{maintenance effectiveness function a time } t -- S \text{ added to } S(t)/\text{hrs.} \]
\[ K = \text{cost of the machine} \]
\[ a(t) = \text{obsolescence function at } t -- S \text{ subtracted from } S(t) \]
\[ p(t) = \text{production rate at time } t; \text{ output value at } t/S(t) \]

\[ V(t) \text{ and } S(t) \text{ are state variables, while } m(t) \text{ is the control variable under management discretion.} \]

The resulting optimal maintenance policy for fixed \( p \) is [5]

\[
m(t) = \begin{cases} 
M & \text{if } f(t) > r/(p - (p - r) e^{-r(T-t)}) \\
\text{undetermined} & \text{if } \frac{\partial Q(t)}{\partial T} = 0 \\
0 & \text{if } f(t) < r
\end{cases}
\]

The implications of this maintenance policy are [5]

Spend money on maintenance at time \( t \) only if the machine is to be held for a sufficiently long time after \( t \) so that the maintenance expenditure will be recovered due to the increased profitability resulting from the maintenance before the machine is sold, p. 546.

The optimal retirement time \( T \) is found by maximizing (1) with respect to \( T \). Then we have:

\[
\frac{dV(T)}{dT} = 0 = S(T) e^{-rT} - rS(T) e^{-rT} + Q(T) e^{-rT} + \int_0^T \frac{\partial Q(t)}{\partial T} e^{-rT} dt
\]

Since in this case \( \frac{\partial Q(t)}{\partial T} = 0 \) [5] and if at \( T, m(T) = 0 \), we find \( T \) which satisfies the equation below

\[
S(T) = a(T)/(p - r)
\]

However, in reality, the retirement and maintenance policies are weighed against other machines that recently appeared on the market as well as other uses the original machine can be assigned to within the firm. This is the problem we are concerned with in this paper.

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Assume that at time $t_1$, the machine is transferred to a different process with different characteristics and is retired only at $T_1$. The problem is then:

Maximize

\[
V(T_1) = S(T_1) e^{-rt_1} + \int_0^{T_1} Q_1(t) e^{-rt} dt + \int_{t_1}^{T_1} Q_{11}(t) e^{-rt} dt
\]

subject to:

\[
\begin{align*}
Q_1(t) &= p_1(t)S(t) - m_1(t) & 0 \leq t < t_1 \\
Q_{11}(t) &= p_{11}(t)S(t) - m_{11}(t) & t_1 \leq t < T_1 \\
0 &\leq m_1(t) \leq M_1 \\
0 &\leq m_{11}(t) \leq M_{11}
\end{align*}
\]

\[
\frac{dS(t)}{dt} = -a_1(t) + f_1(t)m_1(t) & 0 \leq t < t_1 S(o) = k_1
\]

\[
\frac{dS(t)}{dt} = -a_{11}(t) + f_{11}(t)m_{11}(t) & t_1 \leq t < T_1 S(t_1) = ?
\]

This problem is a variational problem with discontinuities both in the cost and the system functions. However, $dS/dt$ and the cost function are continuous in the state $S(t)$ and the controls $m_1(t)$ and $m_{11}(t)$. Therefore, a solution based on the Maximum principle of Pontryiagin [2, 3] can be obtained. Such a solution will be obtained by introducing a variable $k_1(t, t_1)$.

\[
k_1(t, t_1) = \begin{cases} 
1 & 0 \leq t < t_1 \\
0 & \text{elsewhere}
\end{cases}
\]

$k_1(t, t_1)$ is a piecewise constant function whose levels (1 and 0) specify that it exists only during the first process of the machine. This variable is introduced so that the integrand and the systems equations will be defined over the whole life of the machine. Using $k_1$, and introducing the salvage value of the machine at $T_1$ in the integral, we obtain:

Maximize:

\[
V(T_1) = \int_0^{T_1} \left\{ [Q_1(t) + \dot{S}(t) - rS(t)] e^{-rt} k_1(t, t_1) \\
+ [Q_{11}(t) + \dot{S}(t) - rS(t)] e^{-rt}(1 - k_1(t, t_1)) \right\} dt
\]

subject to:

\[
\frac{dS(t)}{dt} = [-a_1(t) + f_1(t)m_1(t)]k_1(t, t_1) + [-a_{11}(t) \\
+ f_{11}(t)m_{11}(t)](1 - k_1(t, t_1)) & 0 \leq t < T_1 \quad \text{and} \quad S(o) = K_1
\]

and (8).
THE OPTIMAL MAINTENANCE POLICIES : ONE MACHINE

For (12), (13) and (8) a Hamiltonian $H$ can be defined by (we drop here all the time subscripts)

(14) \[ H = - \left\{ (Q_1 + \dot{S} - rS)k_1 + (Q_{11} + \dot{S} - rS)(1 - k_1) \right\} e^{-rt} \]
\[ + Z_1 \left\{ (-a_1 + f_{11} m_1)k_1 + (-a_{11} + f_{11} m_{11})(1 - k_1) \right\} \]
\[ Q_1 = p_1 S - m_1 \quad \text{and} \quad Q_{11} = p_{11} S - m_{11} \]

we observe that

(15) \[ \frac{\partial Z_1(t)}{\partial S} = e^{-rt}[(p_1 - r)k_1 + (p_{11} - r)(1 - k_1)] \]
\[ Z_1(T_1) = 0 \]

Thus for fixed production rates $p_1$ and $p_{11}$.

(16) \[ Z_1(t) = \frac{(p_1 - r)(1 - e^{-rt})}{r} + Z(o) \quad 0 \leq t < t_1 \]
(16 a) \[ Z_1(t) = \frac{(p_{11} - r)(e^{-rt} - 1 - e^{-rt})}{r} + Z(t_1) \quad t_1 \leq t < T_1 \]

Since $Z_1(T_1) = 0$, we have

(16 b) \[ Z(t_1) = \frac{p_{11} - r}{r}(e^{-rT_1} - e^{-rt_1}) \]
and

(16 c) \[ Z(o) = \frac{(p_1 - r)(e^{-rt_1} - 1)}{r} \]

Thus

(17) \[ Z_1(t) = \frac{(p_1 - r)(e^{-rt} - e^{-rt_1})}{r}k_1(t, t_1) \]
\[ + \frac{(p_{11} - r)(e^{-rT_1} - e^{-rt})(1 - k_1(t, t_1))}{r} \]

and

(18) \[ Z_1(t) = \frac{(p_1 - r)}{r}k_1 e^{-rt}(e^{-r(t_1 - t)} - 1) \]
\[ + \frac{(p_{11} - r)}{r}(1 - k_1) e^{-rt}(e^{-r(T_1 - t)} - 1) \]

Introducing (18) into the Hamiltonian (14) and grouping all terms $m_1$ and $m_{11}$, we find that the control variables $m_1$ and $m_{11}$ are linear. Therefore,
the optimal maintenance (controls) policies are bang-bang (1) and equal

\[ m_1(t) = \begin{cases} M_1 & \text{if } f_1(t) > r/[p_1 - (p_1 - r) e^{-r(t_1-t)}] \\ \text{undetermined} & \text{if } f_1(t) = \\ 0 & \text{if } f_1(t) < \end{cases} \]

(19)

\[ m_{11}(t) = \begin{cases} M_{11} & \text{if } f_{11}(t) > r/[p_{11} - (p_{11} - r) e^{-r(T_1-t)}] \\ \text{undetermined} & \text{if } f_{11}(t) = \\ 0 & \text{if } f_{11}(t) < \end{cases} \]

THE PROCESS REPLACEMENT AND RETIREMENT TIMES.

ONE MACHINE

Given the maintenance policy, the optimal process replacement time \( t_1 \) and retirement \( T_1 \) can be found by deriving (7) with respect to \( t_1 \) and \( T_1 \). Then since \( \partial Q_1/\partial t_1 = 0 \) and \( \partial Q_{11}/\partial T_1 = 0 \)

(20) \[ \frac{\partial V(T_1)}{\partial t_1} = [Q_1(t_1) - Q_{11}(t_1)] e^{-rt_1} + \int_{t_1}^{T_1} \frac{\partial Q_{11}}{\partial t_1} e^{-r^t} dt + \left[ \frac{\partial S(T_1)}{\partial T_1} \right] e^{-r^t} = 0 \]

and

(21) \[ \frac{\partial V(T_1)}{\partial T_1} = Q_{11}(T_1) e^{-rT_1} + \left[ \frac{\partial S(T_1)}{\partial T_1} \right] e^{-rT_1} - r e^{-rT_1} S(T_1) = 0 \]

Since \( \partial Q_{11}(t)/\partial t_1 = p_{11} \partial S(t)/\partial t_1 \), and from (10),

\[ \partial S(t)/\partial t_1 = \partial S(t_1)/\partial t_1 + a_{11}(t_1) - f_{11}(t_1) m_{11}(t_1) \]

Also, since, \( \partial S(t_1)/\partial t_1 = -a_1(t_1) + f_1(t_1) m_1(t_1) \)

and \( \partial S(T_1)/\partial T_1 = -a_{11}(T_1) + f_{11}(T_1) + m_{11}(T_1) \), we obtain for (20) :

(22) \[ \frac{\partial V(T_1)}{\partial t_1} = [p_1 S(t_1) - m_1(t_1) + p_{11} S(t_1) + m_{11}(t_1)] e^{-rt_1} + \frac{e^{-r(T_1-t_1)}}{r} [a_{11}(t_1) - a_1(t_1) - f_{11}(t_1) m_{11}(t_1) + f_1(t_1) m_1(t_1)] + e^{-r T_1} [a_1(t_1) + f_1(t_1) m_1(t_1) + a_{11}(t_1) - f_{11}(t_1) m_{11}(t_1)] = 0 \]

In (22), all terms are either expressed as specific values of the depreciation, effectiveness and maintenance functions or can be specifically described as

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a function of these known terms. Similarly for (21), we can show that:

\[
\frac{\partial V(T_1)}{\partial T_1} = (p_1 - r) \left[ K - \int_0^{T_1} a_1(t) \, dt + \int_0^{T_1} f_1(t)m_1(t) \, dt \right. \\
- \left. \int_0^{T_1} a_{11}(t) \, dt + \int_0^{T_1} f_{11}(t)m_{11}(t) \, dt \right] - a_{11}(T_1) \\
+ (f_{11}(T_1) - 1)m_{11}(T_1) = 0
\]

In (22) and (23) all terms are known except for \( t_x \) and \( T_x \). Furthermore, since the maintenance policies \( m_1 \) and \( m_{11} \) are functions of \( t_x \) and \( T_x \) respectively, computation of the integrals \( \int_0^{T_1} f_1(t)m_1(t) \, dt \) and \( \int_0^{T_1} f_{11}(t)m_{11}(t) \, dt \) create computational difficulties which can be resolved. Assume a certain value for \( T_x^* \), then compute the corresponding time \( t_x^* \). Given \( t_x^* \) and \( T_x^* \), the switching times for maintenance policies \( m_1(t) \) and \( m_{11}(t) \) can be computed according to (19). Therefore, the integrals of maintenance policies can also be computed and some number is obtained for \( \frac{\partial V(T_x^*)}{\partial T_x^*} \) in (23).

If \( \frac{\partial V(T_x^*)}{\partial T_x^*} > 0 \), then an increase in \( T_x^* \) to \( T_x^* = T_x^* + \Delta t \) will have this value decrease. If \( \frac{\partial V(T_x^*)}{\partial T_x^*} < 0 \), we decrease instead \( T_x^* \) to \( T_x^* = T_x^* - \Delta t \). This procedure is repeated until (23) is satisfied.

**CONCLUSION**

A machine operating in a certain production process is usually replaced because of deterioration, failure beyond repair or because of new competitive machines appearing in the marketplace. Also a machine can be «retired» from a production process not because of a failure to provide high quality products but merely because the opportunity cost of not buying the new machines is too high. However, in particular in big firms, a machine is not phased out of operation or sold, but is transferred to some other function within the firm. As an example, we have considered an airline buying new planes and phasing out older planes into other functions, such as freight transportation or an assignment to shorter and less glamorous routes. The question we have raised is then; given the characteristic forms of both production processes as well as the obsolescence functions in both processes, what should the switching-process time and the retirement time be. Simultaneously, we have provided conditions for the optimality of the maintenance policies to be applied on both production processes.

In our model \( a(t) \) reflects the depreciation rate resulting from technical progress. Thus, high values for \( a(t) \) will undermine the influences of the present machine and will favor the acquisition of new machines. However, technical
progress alone is not a sufficient criterion. In addition, the retirement time of a machine in a particular production process will also be determined by other competitive production processes within the firm.

REFERENCES