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# NOTES & DÉBATS

## THE IMPACT OF MODERN MATHEMATICS ON ANCIENT MATHEMATICS

Wilbur R. Knorr

Abstract. — In a hitherto unpublished lecture, delivered in Atlanta, 1975, W.R. Knorr reflects on historical method, its sensitivity to modern work, both in mathematics and in the philosophy of mathematics. Three examples taken from the work of Tannery, Hasse, Scholz and Becker and concerning the study of pre-Euclidean geometry are discussed: the mis-described discovery of irrational 'numbers', the alleged foundations crisis in the 5th century B.C. and the problem of constructibility.

Résumé. — L'impact des mathématiques modernes sur les mathématiques anciennes. — Dans une conférence prononcée en 1975 à Atlanta, et restée inédite, W.R. Knorr livre quelques réflexions sur la méthode historique, sa dépendance de travaux modernes, tant en mathématiques qu'en philosophie des mathématiques. Il s'appuie sur trois exemples tirés des travaux de Tannery, Hasse, Scholz et Becker sur la géométrie grecque pré-euclidienne: la découverte mal-nommée des 'nombres' irrationnels, la dite crise des fondements du V^e^ siècle avant J.C. et le problème de la constructibilité.

Edith Prentice Mendez found this lecture among Wilbur Knorr's papers after his death in March, 1997. Although Knorr probably never intended to publish it – and he surely would have attended to its occasional roughness – Ken Saito and I consider it an important methodological reflection on his just completed work on the early proportion theory,[1] but with much general interest. The three main examples he

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Ce texte inédit de Wilbur R. Knorr nous a été transmis par Henry R. Mendell et Ken Saito. Il a été transcrit par Ken Saito, introduit par Henry Mendell et annoté par tous deux.

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[1] Wilbur R. Knorr, *The Evolution of the Euclidean Elements*, Dordrecht: Reidel, 1975.

discusses, the theory of irrationals, the alleged foundations crisis in the fifth century and the problem of constructibility, remain important morality tales for contemporary researchers. Among specialists, the pendulum may have swung largely in the other direction, and for that reason, it is useful to quote a letter which warns against the opposed impediment to historical understanding. I thank Joseph Dauben for drawing it to my attention by sending me his transcription of it.

Wilbur Knorr to Joseph W. Dauben
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Whipple Museum, University of Cambridge
March 27, 1975.

"[...] Now, research in the ancient materials is something of an art, and I know that many scholars are by temperament unsuited for it, as they themselves would agree. Basically, the Greek record is fragmentary; we possess a few mathematical treatises virtually complete, others in part, others in random snippets preserved by accident in derivative works, plus a small para-mathematical literature, the logical writings of Plato and Aristotle, for instance. In this circumstance, literalism would be disastrous. For instance, most of the complete treatises which have survived expound a highly formal type of advanced geometry. Does this mean the Greeks were weak in the traditional areas of practical geometry and arithmetic? It goes against reason to believe so. But some scholars ... would have us draw such a conclusion. Rather, at every step one must make allowance for the selective survival of documents. The formal geometry survived because it was also philosophically interesting (from the axiomatic viewpoint) and because it merited study by serious practitioners of geometry. But easily duplicatable computations were hardly worth preserving via manuscript traditions. What mathematician has ever preserved his rough figures, once the final draft of his study has been completed? Occasionally, papyri containing everyday arithmetic and geometric problems survive. These are invariably schoolboys' exercises, amazing for the modesty of their mathematical content. Interestingly, computation throughout Greek antiquity – commercial arithmetic – was done by the Egyptian methods. But otherwise, we are left to surmise the nature of the whole from the upper most ten per cent. In this situation, a scholar with an imagination and a feeling for organizing incomplete evidence into rational frameworks can enjoy himself. But the end-products of such studies can never be much other than this or that degree of plausibility. I find this refreshing. But many find it appalling and seek the haven of documentary objectivity. I think that the student of mathematics from 1650 or so onward has the opposite problem of contending with more documentation than is manageable. Here, if ever one makes a general statement of fact, he must expect that in the materials he could not examine contrary patterns might emerge. But didn't Pascal develop this notion of the two types of reasoning? [... ]"

We have provided all footnotes and hence are responsible for any failure to capture Knorr's allusions. I have also checked the quotations and adjusted some (including a slight clarification of the status of one quotation) and did some other minor editing. As to the alluring title, fans of the novelist David Lodge will no doubt recall the hapless Persse Mc Garrigle and his "The Influence of T.S. Eliot on Shakespeare" in *Small World* (1984).

Knorr left many other important papers, which I hope to bring out in due time.

Henry Mendell

## TRANSCRIPT OF A LECTURE DELIVERED AT THE ANNUAL CONVENTION OF THE HISTORY OF SCIENCE SOCIETY, ATLANTA, DECEMBER 28, 1975.

Chairman: Prof. Joseph Dauben,
Lehman College, City University of New York[2]

When Joe suggested to me the possibility of speaking at this meeting, the topic then projected for the colloquium was nineteenth-century mathematics. I told him I was better prepared to speak on ancient mathematics than on 19th-century. But on thinking it over, I hit upon the idea of discussing the impact of modern mathematics on ancient mathematics.

Now, what ancient mathematics was and what ancient mathematicians did has not been influenced by more recent achievements, of course. But what we take ancient mathematics to have been is very strongly influenced by modern work, both in mathematics and in the philosophy of mathematics. It is this sensitivity of the historical criticism that I wish to examine, by way of a few examples from the study of pre-Euclidean geometry. – Afterwards, I will propose some general observations on historical method, based on these examples.

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"Why didn't the Greeks construct the irrational numbers?" This question was the subject of an article by Heinrich Scholz in 1928.[3] Scholz was examining a polemical statement by Oswald Spengler, to the effect that the Greeks, overburdened by a concrete and plastic intellectual outlook, thereby missed the mathematical abstraction accessible to us now through our algebraic conceptions. Scholz rightly branded the observation nonsense. The Greeks were not blind to an extension of the number-concept through some accidental failure of spirit. They rejected any such

[2] The other paper in the symposium was by Winifred Wisan on "Galileo's Mathematical Method: A Reassessment." They had both just arrived at the ill-fated New School of Liberal Arts, an honors division of Brooklyn College, whose mission was immediately modified by an open admissions policy and which was to suffer under the budget crunches of New York City in the late seventies. As a result, the positions of each were terminated.

[3] Heinrich Scholz, Warum haben die Griechen die Irrationalzahlen nicht aufgebaut, in Helmut Hasse und Heinrich Scholz, *Die Grundlagenkrisis der Griechischen Mathematik*, Berlin: Pan-Verlag, 1928, pp. 35–72.

extension on scientific and philosophical grounds: the *arithmos* must be whole-number; even the rational numbers, a necessary preliminary to irrational numbers, were excluded from the classical number theory; the problem of irrationals was thus resolved by Eudoxus in a geometric manner instead.

Scholz' assessment is sound. But what we should at once notice is that such a debate could not have arisen before the successful resolution of the problem of irrational numbers by Weierstrass and Dedekind in the 19th century. Before that time, the Euclid-editors – Barrow and de Morgan, for instance – had to answer the charges of obscurity and verbosity levelled against Euclid in his definition of proportion in Book V.[4] But already in Dedekind's time a reversal was taking place: critics like Lipschitz[5] now questioned whether Dedekind had added anything to the Euclidean theory. Somewhat later, Thomas Heath (1921; 1926) judged that "the definition of equal ratios [by Eudoxus and Euclid] corresponds exactly to the modern theory of irrationals due to Dedekind.[6] ... It is word for word the same as Weierstrass' definition of equal numbers.[7] So far from agreeing in the usual view that the Greeks saw in their rational no *number* ... it is clear from Euclid V. that they possessed a notion of number in all its generality as clearly defined, nay almost identical with, Weierstrass' conception of it." This latter judgment, in which Heath follows the view of Max Simon,[8] is undoubtedly overstated. Nevertheless,

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[4] Cf. Thomas L. Heath, *The Thirteen Books of Euclid's* Elements, translation with introduction and commentary, 3 vols., 2nd ed., Cambridge: Cambridge University Press, 1926, vol. 2, pp. 121–122.

[5] Knorr's source may be a letter from Richard Dedekind to Rudolf Lipschitz dated 6 July 1876, in which he quotes extensively from Lipschitz' previous letter to him. Lipschitz wonders if Dedekind's account of real number is merely Euclid, *Elements* V, def. 5, which he quotes in Latin, "rationem habere inter se magnitudines dicuntur, quae possunt multiplicatae sese mutuo superare [. . . ]" (cf. Richard Dedekind, *Gesammelte mathematische Werke*, ed. by Robert Fricke, Emmy Noether, and Öystein Ore, vol. 3, Braunschweig, 1932, pp. 469–470. Lipschitz' letters are now published, *Briefwechsel mit Cantor, Dedekind, Helmholtz, Kronecker, Weierstrass und anderen*, ed. by Winfried Scharlau, Braunschweig: Vieweg, 1986. For this letter of 6 July 1876, see pp. 70–73.

[6] Thomas L. Heath, *A History of Greek Mathematics*, Oxford: Clarendon Press, 1921, p. 327.

[7] Heath, *op. cit.*, 1926, (see note 4), vol. 2, p. 124.

[8] Maximilian Simon, *Euclid und die sechs planimetrischen Bücher*, Leipzig: Teubner, 1901, p. 110.

we meet later writers like A.E. Taylor,[9] who insist on finding traces of the modern real-number concept in obscure passages from ancient authors. When Plato is reported to have described how "the One equalizes the Great-and-Small",[10] this is read as the definition of an irrational number as the limit of an alternating rational sequence. Again, a puzzling, and likely corrupt, passage from Aristotle, that "number is also predicated of that which is not commensurable",[11] has recently been used to affirm the conception of irrational *numbers* in the early 4th century B.C. In letting such evidence over-ride the unanimous restriction to whole numbers in the pre-Diophantine literature on number theory, these writers clearly betray a distortion of critical viewpoint owing to their awareness of the modern real-number concept.

Thus, the successful "arithmetization of the continuum" in the 19th century has had perceptible effects on the interpretation of the ancients. In a positive way, it has drawn new attention to certain areas, here the Eudoxean proportion theory, until then not fully understood or appreciated. But once such a problem in mathematics has received a modern solution, this solution tends to be given an absolute status and

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[9] Alfred E. Taylor, *Plato: the Man and his Work*, 7th ed., London: Routledge, 1960, pp. 509–513.

[10] Cf. Taylor, *ibid.*, p. 512. Our primary source, Aristotle, *Metaphysics* M 8.1083$^{b}$23-32, N 3.1091$^{a}$23-5, attacks this view as part of Plato's position on number.

[11] The text would appear to be *Met.* D15.1021$^{a}$5-6. Of the three principal manuscripts (labelled E, J,A$^{b}$) used by W.D. Ross in *Aristotle's Metaphysics* (Oxford: Oxford University Press, 1924), E and J and Alexander of Aphrodisias have: *kata mê summetron de arithmon legetai* (or *legontai*), and so it is printed in every text before Ross, and which he translates in his first Oxford translation (1908), "but this relation may involve a 'non-commensurate number'." A$^{b}$ has instead: *kata mê summetron de arithmos ou legetai*, which Ross emends to: *kata mê summetrou de arithmos ou legetai* (number is not said of the non-commensurate). In general, where E, J, and Alexander agree against A$^{b}$, Ross sides with them against A$^{b}$ (cf. introduction to his text, p. clxi), but not always (cf. 1008$^{a}$25 and introduction p. clxii). All texts and most translations follow Ross (exceptions are translations by R. Hope and H. Apostle, who seem to translate unstated emendations along the lines of EJ). However, if E, J are in error, it still remains interesting that someone before Alexander (ca. 300 C.E.) wrote 'non-commensurable number', i.e. if they wrote it intentionally. It is possible that Knorr refers to Julius Stenzel, Zur Theorie des Logos bei Aristoteles, *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*, Abt. B, Bd. I, 1929, pp. 34–66, in particular pp. 57–60 (reprinted in *Kleine Schriften zur griechischen Philosophie*, Darmstadt: Wissenschaftliche Buchgesellschaft, 1956, pp. 188–219, in particular pp. 210–212). However, the reference may well be to a more recent (and less sophisticated) interpretation of the passage.

becomes a standard for judging prior work. Ancient work merits praise to the extent it is *like* the modern (recall Heath's phrases: "corresponds exactly to", "is word for word the same as", "is almost identical with") – but the ancients are also *blamed* for failing to institute the full modern approach. The ancients solved the problem of irrationals, to be sure, but as *magnitudes*, not *numbers* – why didn't they succeed in constructing the irrational *numbers*? In other words, there is a sense that the concept of number necessarily and inevitably generalizes from the whole numbers to the rational and irrational numbers. In modern eyes, a mathematical concept, like number, once seen in a certain way, is now viewed as necessarily of this character.

This use of modern concepts as a standard for judging ancient work accounts not only for the negative critiques of such as Spengler, but also the implausible distortions of interpretation one reads in Taylor. Apparently, one cannot be satisfied that a fully competent mathematical system could be different in any important respect from the related modern work.

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Scholz' examination of the Eudoxean study of irrationals was appended to a larger investigation into what he and Helmut Hasse called "the crisis of foundations in ancient mathematics".[12] Their joint article of 1928 was based on a set of courses on the work of Eudoxus, Weierstrass, Dedekind and Cantor. This notion of a "foundation crisis" had already appeared in Paul Tannery's study of Greek geometry in 1887.[13] Tannery *assumed* that the oldest Pythagorean geometry was built around the assumption of commensurability of all magnitudes. He concluded "the discovery of incommensurability by Pythagoras *must* thus have caused a veritable logical scandal, and to avoid it one *must* have tended to restrain the use of the principle of similarity as much as possible".[14] (Emphases mine.) The "scandal" was ultimately resolved through Eudoxus' theory of proportion; Tannery remarks "it is easy to separate [the embarrassment of

[12] See above note 3.

[13] Paul Tannery, *La géométrie grecque*, Paris: Gauthier-Villars, 1887.

[14] "La découverte de l'incommensurabilité par Pythagore dut donc causer, en Géométrie, un véritable scandale logique, et, pour y échapper, on dut tendre à restreindre autant que possible l'emploi du principe de similitude [. . . ]" (*ibid.*, p. 98).

its geometric form] from his theory – it sustains, without any disadvantage, comparison with modern expositions, so often defective".[15] The reference to "defective modern expositions" is especially interesting; for Tannery was well aware of the efforts in his own time to set the infinitesimal calculus on an adequate logical foundation.

Hasse and Scholz develop upon this interpretation. They remark of the irrationality of the square root of 2 that "the discovery of a case which cannot be comprehended in numbers, *must* naturally have shaken the idea of the 'arithmetica universalis' of the Pythagoreans."[16] (Emphasis mine.) Eudoxus' service in ending the crisis is explicitly likened to modern crises: "Just as in the past century and today, so also in the 2nd half of the 5th century, there was a severe foundations crisis."[17] The weak foundation of the limit-concept in infinitesimal calculus, they note, was remedied in the 19th century through the work of Abel, Cauchy, Weierstrass; the weak foundation of the set-concept was more recently remedied in the work of Hilbert, Brouwer; in antiquity it was the weak foundation of the ratio-concept, remedied by Eudoxus.

I cannot provide here a detailed study of the question of the ancient foundations crisis. For this, I defer to the discussion in my recent book on pre-Euclidean geometry[18] and to Hans Freudenthal's article of 1966.[19] But in labelling this "crisis" a "modern fiction", I ought to make one or two justifying remarks. First, there is no evidence of "restraint" in the use of proportions in geometry during the alleged crisis-period, say 450 to 350 B.C. The works of Hippocrates and Archytas, for instance, are indispensably based on such techniques. Second, on what grounds are we to believe that the discovery of incommensurability was a *challenge* or

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[15] "Il serait facile de l'en dégager, et elle soutiendrait alors sans aucun désavantage la comparaison avec les expositions modernes, si souvent défectueuses." (*ibid.*)

[16] "Diese Entdeckung eines Falles, der nachweislich nicht mit Zahlen zu erfassen war, mußte naturgemäß die Idee der Arithmetica universalis aufs schwerste erschüttern." (Hasse und Scholz, Die Grundlagenkrisis in der griechischen Mathematik, in Hasse und Scholz as cited in note 3, pp. 4–34).

[17] "Genau wie im vergangenen Jahrhundert und heute, so lag auch damals, in der zweiten Hälfte des 5. Jahrhunderts, eine schwere Grundlagenkrisis der Mathematik vor." (*ibid.*, p. 12).

[18] As cited in note 1.

[19] Hans Freudenthal, Y avait-il une crise des fondements des mathématiques dans l'antiquité?, *Bulletin de la Société mathématique de Belgique*, 18 (1966), pp. 43–55.

counter-example to naïve assumptions within the Pythagorean geometry? To be sure, this discovery was held to be significant: late writers suggest it was maintained as a secret of the school – but was it a challenge? Consider that the Pythagoreans based their natural philosophy on the conception of the world in terms of number and other mathematical categories, that is, in terms of certain abstract, rather than material principles. The discovery of incommensurability might well *support* this view: for this is a property of certain lines, for instance, the side and diameter of the square, which we can appreciate through a sequence of deductions to be necessarily true of these lines; yet no effort of practical measurement, no perception or procedure of an empirical character, could bring us to an awareness of this fact or of its certainty. Thus, the Pythagorean insistence on number as a fundamental principle could be *affirmed*; and we should note that the school never did relinquish its adherence to this principle.

When Tannery and Hasse and Scholz jump to the conclusion that the incommensurable was a counter-example to Pythagorean geometric method, they are thus already assuming the thesis of the foundations crisis. The logician and the philosopher, and following them, the historian might recognize that a certain result is paradoxical, and that it *ought* to provoke a crisis in the foundations of a given field of mathematics. But does the practising mathematician ever curtail his researches in accordance with such a challenge? In the 1820's Abel complained of the faulty state of the theory of infinite series–but his real complaint was that little was being done about it.[20] In the early 1920's Hermann Weyl who was occupied with constructing alternative models of the continuum in conformity with the intuitionist critique of logic and set-theory, wrote a paper on "the new foundations crisis in mathematics";[21] in it he criticized the mathematical profession for ignoring the implications of the Richard paradox in set-theory. Perhaps Hasse and Scholz, only a few years later studying the related ancient work, should have taken more seriously this

[20] Niels Abel, Recherches sur la série $1+m/1\,x+m(m-1)/(1\cdot 2)x^2+m(m-1)(m-2)/(1\cdot 2\cdot 3)x^3+$?‘, *Journal für die reine und angewandte Mathematik*, vol. 1, Berlin 1826 (reprinted in *Œuvres Complètes*, 2 vols., Christiana: De Grondahl & Son, 1881), vol. 1, chap. xiv, pp. 219–250.

[21] Hermann Weyl, Über die neue Grundlagenkrise der Mathematik, *Mathematische Zeitschrift*, 10 (1921), pp. 39–79 (reprinted in H. Weyl, *Gesammelte Abhandlungen*, ed. by K. Chandrasekharan, 4 vols., Berlin: Springer, 1968, vol. 2, pp. 143–180).

aspect of the "crises". They may have recognized, for instance, that Plato's strong words against the faulty mathematical procedures of his time betokened a similar insensitivity by geometers to some crisis which *should have been underway*;[22] however, that Plato was uniquely positioned in the Academy to encourage such mathematicians as Eudoxus to address the problems of the reorganization of geometry on a satisfactory logical foundation.

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Oskar Becker, more than any other modern scholar, has invigorated the study of ancient mathematics and logic. In a series of "Studies on Eudoxus" which appeared in the 1930's he investigated several logical problems in the Euclidean geometry.[23] His project was in large part inspired by the start made by Hasse and Scholz; with much greater detail, however, Becker sought to clarify the ancients' use of non-constructive assumptions, such as the principle of the excluded middle, the assumption of the existence of the fourth proportional, and others. But if Becker succeeded in showing that most of the Euclidean uses of the excluded middle can be brought into conformity with Brouwer's intuitionist criteria, for instance, we might well ask what is the significance of that? If, again, he has shown that the ancients' use of the fourth proportional – reducible to weaker constructive assumptions, but not done so by them – indicates their implicit acceptance of Dedekind's axiom of continuity, we might question whether Dedekind's axiom is indeed nothing more than the Platonic formula: "to that of which there is the greater and the smaller there is also the equal".[24] It would appear that Becker, like others mentioned earlier, is guilty of reading more formality into the ancient work than was actually

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[22] Plato, *Republic* vii 527AB.

[23] These are, from *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*, Abt. B: Eudoxos-Studien I: Eine voreudoxische Proportionenlehre und ihre Spuren bei Aristoteles und Euklid, 2 (1933), pp. 311–333; Eudoxos-Studien II: Warum haben die Griechen die Existenz der vierten Proportionale angenommen?, 2 (1933), pp. 369–387; Eudoxos-Studien III: Spuren eines Stetigkeitsaxioms in der Art des Dedekindschen zur Zeit des Eudoxos, 3 (1936), pp. 236–244; Eudoxos-Studien IV: Das Prinzip des ausgeschlossenen Dritten in der griechischen Mathematik, 3 (1936), pp. 370–388; Eudoxos-Studien V: Die eudoxische Lehre von den Ideen und den Farben, 3 (1936), pp. 389–410.

[24] Plato, *Parmenides* 161D, cf. Becker, Eudoxos-Studien III (see previous note).

there. But Becker's studies raise another more interesting problem about the objective of a historical analysis.

Becker was a proponent of the philosophy of Edmund Husserl. For several years in the 1920's he co-edited with Martin Heidegger and others the "Jahrbuch" of phenomenological research and philosophy. At this time Becker produced phenomenological analyses of subjects like mathematical existence and the logical foundations of geometry; his "Eudoxus-Studies" followed; and there after numerous books and articles on mathematical thought, both ancient and modern. Now, what did the commitment to the phenomenological outlook consist of? This philosophy was itself inspired by the mathematical work of the late 19th century. Husserl studied with Weierstrass and also with Kummer and Kronecker in Berlin in the 1880's. According to Becker, Husserl thus developed an ideal of mathematical exactitude as the appropriate standard for philosophy.[25] In an early work on the foundations of arithmetic (1887; 1891) Husserl[26] fused his mathematical training with the psychological principles of Brentano, with whom he studied at Vienna. This study of arithmetic, later renounced after Frege's withering attack on its "psychologism",[27] manifested important phenomenological features: the reduction of the meaning of such concepts as number and set to the activity of collecting and counting, for instance. Gradually, Husserl articulated the need for a precise description of the subjective processes necessarily involved in thought. Husserl hoped that, just as the new general theory of manifolds by virtue of its abstract character, could cover many different particular theories, so also a general phenomenological theory could

[25] Perhaps Knorr refers to the introduction to Oskar Becker, Beiträge zur phänomenologischen Begründung der Geometrie und ihrer physikalischen Anwendungen, *Jahrbuch für Philosophie und phänomenologische Forschung*, 6 (1923), pp. 385–560 (1–176), cf. pp. 385–388 (1–4), where Becker argues for the centrality of foundations for mathematics as central to Husserl's phenomenology. Cf. p. 386 (2): "Der Verfasser, dem wesentliche Teile jener Husserlschen Forschungen (in Vorlesungen, Übungen, Manuskripten, persönlichen Unterredungen) zur Verfügung gestellt wurden, setzte sich die Aufgabe, jene Begründung und Aufklärung in ihren Grundzügen zu leisten und damit eine Brücke von Phänomenologie zur heutigen Mathematik und Physik zu schlagen."

[26] Edmund Husserl, *Über den Begriff der Zahl, psychologische Analysen*, Habilitationschrift, Universität Halle-Wittenberg, 1887, and his revised expansion, *Philosophie der Arithmetik*: *Psychologische und logische Untersuchungen*, Halle-Saale: Pfeffer, 1891.

[27] Gottlob Frege, review of Husserl's book (1891), in *Zeitschrift für Philosophie und philosophische Kritik*, 103 (1894), pp. 313–332.

provide the absolute basis for comprehending all thought. Ultimately, Husserl espoused a full transcendental philosophy, faced with many of the problems which had challenged such earlier rationalists as Plato and Descartes.

The phenomenologist thus seeks an absolute description of the thought-processes necessary for deductive thinking, as in mathematics and logic; he considers that these processes, being absolute, do not depend for their understanding on a consideration of the particular individuals who happen to do the thinking or the particular circumstances, cultural or otherwise, within which the thinking occurs. What, then, are the objectives of *historical* analyses within this philosophy? Theodore Kisiel, writing of Husserl's investigation of the origins of geometry, answers thus: "the more basic considerations of the birth and becoming of science lie on the *a priori* level of meaning rather than the empirical level of facts. Hence as Husserl sees it, the 'essential history' ... transcends the 'noisy events' of daily concerns ... it is a history which can be traced even when the facts are no longer accessible [for instance, a study of the history of geometry is designed] to gain some insight into what the original but now submerged sense *must* have been when it *first* emerged."[28] (Emphases his.)

These remarks seem to clarify the purposes of such historical analyses as Becker's. For, his interest is in explicating the necessary logical relations among the several assumptions, some explicit, some implicit, in the ancient geometry; ostensible defects in Euclid may be removed by implicit axioms, "traces" of which may then be sought in other authors. These objectives seem consistent with the phenomenological search for how the historical development *must* have happened.

My uneasiness with this approach of Becker's is that it runs the risk of anachronism. The ancient mathematician was not working within the context of any such complete systems – either mathematical or philosophical – of the type presumed in Becker's philosophical analysis. Becker takes ideas and methods which he knows as necessary; he may thus read into past work what was not there, or at best only barely or intuitively perceived; conversely, he may miss the significance of other aspects which his

[28] Theodore Kisiel, Husserl on the History of Science, in Kockelmans (Joseph) and Kisiel (Theodore), eds., *Phenomenology and the Natural Sciences*, Evanston: Northwestern University Press, 1970, pp. 68–92, see pp. 69–70 for the quotation.

own background leads him to view as accidental or superfluous.

An instance of the strengths and weaknesses of Becker's method may be seen in his reconstruction of a pre-Eudoxean proportion theory.[29] Although Eudoxus resolved the problem of proportions of incommensurable magnitudes, might not prior attempts have been made along alternative lines? A passage from Aristotle's *Topics* appears to indicate as much; "some things in geometry are difficult to prove, because of the lack of a definition" – a theorem is cited: "that when a rectangle is cut parallel to its side, the areas are in the same ratio as bases" – and the necessary definition is given: "magnitudes have the same ratio when their antanaireses are the same".[30] Becker was able to work out from this that all the theorems of the Euclidean proportion theory (Book V) *could* be proved via such a notion: using, instead of Eudoxus' definition, a test for equal ratio based on the Euclidean division algorithm (anthyphairesis = antanaireses). Further passages, as well as the internal structure of the *Elements*, seem to point to Plato's contemporary Theaetetus as the originator of this alternative theory.

Becker's thesis has several strengths and has thereby attracted numerous adherents. But notice that the starting-point of Becker's study coincides with his own philosophical objective of delineating the steps in the development of logic. For, on what grounds, other than mere assumption, are we to accept that Theaetetus, or any predecessor of Eudoxus, took up the systematic study of the logical problems of proportion theory?

Suppose we reverse Becker's approach. What do we know about Theaetetus? Among other things, that he initiated the classification of irrational lines: the expanded theory of these survives as Euclid's Book X. Now, in Book X the division algorithm, anthyphairesis, is used to determine whether or not given lines possess a common measure, that is, whether or not they are commensurable. If we peruse the theorems in the book, we find that several results from proportion theory and the theory of similar rectangles are required, perhaps a dozen theorems in all. The theorem named by Aristotle, for instance, is one of them: it is required for all but the first 18 of the 115 theorems in the book. One

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29 Cf. Oskar Becker, Eudoxos-Studien I (note 23).

30 *Topics* VIII.3 158b29–35.

theorem whose anthyphairetic proof Becker found problematic is deceptively simple: "if $A : B = A : C$, then $B = C$"[31] – it is just the sort of mathematical fact which would pass notice until detected by a geometer specifically interested in foundations.

These facts suggest to me a pattern significantly different from that argued by Becker: that Theaetetus initiated the classification of irrationals and required a set of theorems on proportions to prove certain results about these irrationals – such as, that a line is designatable as a binomial irrational in one way only; that Theaetetus chose the division algorithm as the means of proving these theorems on proportions; and that later geometers, among them Eudoxus, in the course of extending this theory of irrationals and its use of proportions, detected those difficulties which then motivated the quest for a new definition of proportion and a revised theory based on it.

I will not here go into further details or seek to provide textual and technical supports beyond this (These are given in my book, mentioned earlier). For now, it should be clear that Becker's analysis might as well lead to the view of Theaetetus not as a predecessor of Eudoxus in the study of foundations, but rather as a mathematician interested in the geometric construction and description of irrational lines.

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The basic issue, then, is this: what is a historical analysis intended to do? Here, I believe, each scholar is entitled to his own answer. But for me, a coherent philosophical synthesis, such as Becker and others seek, is only part of the objective. What I expect in addition is that a historical analysis conform to what it meant then and there to *be* a mathematician and to do mathematics. Of course, our ability to know these things is also restricted and thus becomes a task for historical analysis. But certainly, a purely logical or philosophical investigation will not suffice. We will find that wider considerations are necessary: certainly of the individual mathematician and the placement of his work in the setting of the labors of his contemporaries, but also frequently of other relevant cultural factors – which, depending on the culture, may include religious, social, educational or political elements.

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[31] *Elements* V-9; Becker, as cited in note 23, p. 320.

Without doubt, the philosophical reviews of ancient logic and mathematics done by Tannery, Hasse, Scholz, Becker and others, have enriched our appreciation with new hypotheses and new insights into the meaning of the concepts and methods employed. But as I pointed out, a cultural dimension ought to be present in the analysis also – serving to subject the philosophical analysis to a wider documentary test, thereby to reduce the risk of anachronism, and also to keep alive the possibility of alternative views.

This last point is important. Whether or not you accept *my* view on the impact of the discovery of incommensurability on the Pythagorean philosophy or *my* view on Theaetetus' use of the division algorithm in proportion theory, you will accept, I believe, that my views are at least as tenable as the views they seek to modify. I believe the generalist in the history of mathematics especially should be made aware of this: he should recognize the degree of dependence which many historical treatments have on philosophical preconceptions. Above all, he should recognize the pervasiveness of what we could call a Platonic bias in the writing and research of the history of mathematics – that is, a conviction of the absoluteness and culture non-dependence of the concepts and truths of mathematics.

What are the *facts* about the history of ancient mathematics? In part, they are the research opinions of the very few specialists in this field – I have indicated how much these can be influenced by modern ideas and by philosophical assumptions. But the *facts* are also what is to be found in the survey histories – such as those by Heath, van der Waerden, Cantor, Hofmann,[32] and many others – and the dictionary and encyclopedia articles, and so on – the materials which the non-specialist is likely to consult – first and perhaps only.

In works of this general category it is a *fact*, for instance, that Greek mathematics from about 450 to 350 B.C. was in the throes of a paralyzing crisis of foundations. As I and others have insisted, this is

[32] Thomas L. Heath, *A History of Greek Mathematics*, Oxford: Clarendon Press, 1921 (*op. cit.* note 6); Bartel L. van der Waerden, *Ontwakende Wetenschap*, Groningen: Noordhoff, 1950, trans. by Arnold Dresden as *Science Awakening*, Groningen: Noordhoff, 1954; Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, 4 vols., 4th ed., Leipzig: Teubner, 1922; and Joseph E. Hofmann, *Geschichte der Mathematik*, 3 vols., Berlin: DeGruyter, 1963.

at best a highly misleading – if not an entirely false – description of pre-Euclidean geometry, strongly reflective of the concern over foundations among logicians in the early part of this century.

We do well to beware that many similar misconceptions derived from an insufficiently guarded application of modern mathematical conceptions may permeate what we commonly accept to be ancient mathematics.

In short, whenever we read about ancient mathematics, we ought to keep two warnings in mind: *cave modernum* and *cave Platonicum*.