Revue d'histoire des mathématiques,
1 (1995), p. 3–38.

NATURPHILOSOPHIE AND ITS ROLE IN RIEMANN'S MATHEMATICS

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ABSTRACT. — This paper sets out to examine some of Riemann's papers and notes left by him, in the light of the "philosophical" standpoint expounded in his writings on *Naturphilosophie*. There is some evidence that many of Riemann's works, including his *Habilitationsvortrag* of 1854 on the foundations of geometry, may have sprung from his attempts to find a unified explanation for natural phenomena, on the basis of his model of the ether.

Keywords: ether theory, complex function theory, Riemannian differential geometry.

RÉSUMÉ. — LE RÔLE DE LA *NATURPHILOSOPHIE* DANS LES TRAVAUX MATHÉMATIQUES DE RIEMANN. Dans cet article, nous proposons une lecture de certains mémoires et notes de Riemann à la lumière du point de vue «philosophique» qu'il a développé dans ses écrits sur la *Naturphilosophie*. Il apparaît que l'origine de nombreux travaux de Riemann, y compris l'*Habilitationsvortrag* de 1854 sur les fondements de la géométrie, peut être trouvée dans sa tentative d'explication unitaire des phénomènes naturels sur la base de son modèle de l'éther.

INTRODUCTION

Riemann's writings on *Naturphilosophie*[1] can be regarded as the result of his attempt to find a unified, mathematical explanation of various physical phenomena such as gravitation, electricity, magnetism and light. They also allow us to include some of his better known papers — such as his *Inauguraldissertation* [1851], his *Habilitationsvortrag* [1854b] and other papers on physical subjects as well — in a wide-ranging research program.

[1] Heinrich Weber gathered these with others manuscripts of Riemann on philosophical subjects, such as psychology, metaphysics and gnosiology, and published them in Riemann's collected works under the title *Fragmente philosophischen Inhalts* [Riemann 1876a].

(*) Texte reçu le 21 mars 1994, révisé le 17 octobre 1994.
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As Klein once said, Riemann's work was characterized by his continual attempt to put "*in mathematical form a unified formulation of the laws which lie at the basis of all natural phenomena*" [1894, p. 484]. Klein did not hesitate to claim that "*the origins of Riemann's pure mathematical developments*" lay in this research which, in Riemann's words, was at a certain stage his own "*main work*".

Searching for a mathematical description of the known physical phenomena, Riemann thought of space as pervaded by substance (*Stoff* [2]), and in a section of his *Fragmente* he considered the state of a single particle of substance and analysed locally the space around it [Riemann 1853].

This passage from "local" to "global" constitutes the basic method used by Riemann in some of his most important works in geometry as well as in analysis and physics. In analytical terms this corresponds to the analytical continuation of a complex function. This is "*a well known theorem*" [Riemann 1857b, p. 88] which is at the basis of the "*new method*" he set up in his thesis. This "*method*" [Riemann 1851, p. 37–39] could be applied to Abelian functions, as he did in [1857b], and also "*in its essential lines*" to "*every function which satisfies a linear differential equation with algebraic coefficients*" [1857a, p. 67]. Accordingly, in this paper he studied the transcendental functions defined by the hypergeometric differential equation "*almost without calculations*" [*Werke*, p. 85] and "*and in their totality*" on the complex sphere.

The same point of view inspired his *Habilitationsvortrag* where he defined metrics on manifolds by using the linear element ds. In particular, Riemann stated that "*questions about the immeasurably large are idle questions for the explanation of Nature. But the situation is quite different with questions about the immeasurably small*" [Riemann 1854b/1979, p. 151]. As Riemann explained in the introduction to the first course he gave in Göttingen as a *Privatdozent*, the laws for all space could be deduced by integrating partial differential equations expressing some "*elementary*" principles valid for infinitely small portions of space.[3]

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[2] Instead of this, in his later lectures on gravitation, electricity and magnetism Riemann preferred to use the term ether.

[3] "*Wahre Elementargesetze können nur im Unendlichkleinen, nur für Raum — und Zeitpunkte stattfinden. Solche Gesetze aber werden im Allgemeinen partielle Differentialgleichungen sein, und die Ableitung der Gesetze für ausgedehnte Körper und Zeiträume aus ihnen erfordert die Integration derselben. Es sind also Methode*

Such a research method had already been announced by Riemann himself in a rather cryptic way in 1850. When lecturing at the *Pädagogische Seminar* he noticed that it was possible to formulate a mathematical theory by moving from elementary principles toward general laws valid in all of a given continuous space without distinguishing between gravity, electricity, magnetism and equilibrium of heat.[4]

As Klein pointed out, the method of studying functions on the basis of their behaviour in the infinitely small had a physical counterpart in the concept of a line of force. Moreover, Klein suggested a kind of dualism between Riemann's mathematical thought and Faraday's concept of action by contact, writing that: "*If I may dare to proceed with so forceful the analogy, then I shall say that Riemann in the field of mathematics and Faraday in the field of physics are parallel*" [Klein 1894, p. 484].

Supporting Klein's point of view, in *Raum Zeit Materie* Weyl stated that the passage from Euclidean to Riemannian geometry "*is founded in principle on the same idea as that which led from physics based on action at a distance to physics based on infinitely near action*" [1919a/1922, p. 91]. In fact, according to Weyl:

"The principle of gaining knowledge of the external world from the behaviour of its infinitesimal parts *is the mainspring of the theory of knowledge in infinitesimal physics as in Riemann's geometry, and, indeed, the mainspring of all the eminent work of Riemann, in particular, that dealing with the theory of complex functions*" [1919a/1922, p. 92].

1. ON THE SOURCE OF RIEMANN'S ANALYTICAL WORK

Riemann introduced his ideas on complex function theory in his 1851 paper which concluded his studies at Göttingen. Riemann's starting point

nöthig, durch welche man aus den Gesetzen im Unendlichkleinen diese Gesetze im Endlichen ableitet, und zwar in aller Strenge ableitet, ohne sich Vernachlässigungen zu erlauben. Denn nur dann kann man sie an der Erfahrung prüfen" [Riemann 1869, p. 4].

4 "*So z.B. lässt sich eine vollkommen in sich abgeschlossene mathematische Theorie zusammenstellen, welche von den für die einzelnen Punkte geltenden Elementargesetzen bis zu den Vorgängen in dem uns wirklich gegebenen continuirlich erfüllten Raume fortschreitet, ohne zu scheiden, ob es sich um die Schwerkraft, oder die Electricität, oder den Magnetismus, oder das Gleichgewicht der Wärme handelt*" (in [Dedekind 1876, p. 545]).

was given by the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{1.1}$$

which have to be satisfied by the function $w = u + iv$ of a variable $z = x + iy$. From (1.1) he deduced the equations $\Delta u = 0$, $\Delta v = 0$ which are the basis for investigating the properties of the functions u and v [Riemann 1851, p. 7].

As Prym was to write to Klein after Riemann's death,[5] since his student days Riemann had attributed great importance to equations (1.1) for the continuation of a function from one complex domain to another. According to him, equations (1.1) explain why correct results can be obtained even when working with divergent series, as Euler repeatedly did.

It is a well known fact that Riemann's complex function theory is deeply connected with potential theory in two dimensions — a theory he was well acquainted with. Indeed, as a student Riemann had followed Weber's lectures in 1849 and the following year he participated in the physics seminar jointly founded and led by Gauss and Weber. Gauss himself had developed the theory of the Laplace equation in a paper of 1839. He had determined the potential function in different cases and, in particular, he had studied the problem of the distribution of masses or electric charges on a closed surface S, assuming the potential to be constant on S.

From a mathematical point of view, this reduced the problem to minimizing the following integral

$$J = \int_V |\operatorname{grad} u|^2 \, \mathrm{d}v.$$

[5] "*Nach einer Mittheilung, die mir Riemann in Frühjahre 1865 während meines Pisaner Aufenthalts machte, ist derselbe zu einer Theorie der Functionen einer verändlichen complexen Grösse durch die Beobachtung gekommen, dass Beziehungen zwischen Functionen, die durch Entwicklung der betreffenden Functionen in Reihen erhalten worden, bestehen bleiben, auch wenn man über die Convergenzgebiete der darstellenden Reihen hinausging, und dass man in vielen Fällen richtige Resultate erhält, wenn man, wie Euler z.B. es wiederholt getan, mit divergenten Reihen operiert. Er frug sich dann, was denn eigentlich die Function aus dem einen Gebiete in das andere fortgesetzt, und gelangte zu der Einsicht, dass dies die partielle Differentialgleichung thue. Dirichlet, mit dem er den Gegenstand besprach, stimmte dieser Ansicht vollständig bei; es fällt also diese Idee wohl noch in die Studienjahre Riemanns, vor die Auffassung seiner Inauguraldissertation*". This letter from February 6, 1882 is kept in Klein's *Nachlass* [11, 383].

Since $J > 0$, "*a [homogeneous] distribution must necessarily exist, so that the integral J has a minimum*", Gauss wrote [1839, p. 233]. This argument was used systematically by Dirichlet in his lectures on the forces which are inversely proportional to the square of distance.

Dirichlet [1876, p. 127] faced the problem of proving that a function with continuous first partial derivatives on a given bounded domain, which satisfies the Laplace equation within the domain and has given values on the boundary, always exist. Dirichlet's existence proof of the solution of the "Dirichlet problem" was based on the fact that the minimum for the integral J existed ("Dirichlet principle").

According to Riemann, potential theory as developed by Gauss and Dirichlet was well suited to a particular geometrical object, the "Riemann surface", he had introduced in order to study multi-valued functions such as algebraic functions and their integrals. Riemann required that the surface associated to a function be composed of as many sheets as were the branches of the function, connected in such a way to preserve continuity and to yield a single-valued function on the surface. In this way, he attained an abstract conception of the space of complex variables by means of a geometrical formulation which his contemporaries were to find very hard to understand. Referring to a conversation he had with Prym in 1874, Klein reported that Prym "*told me that* Riemann's surfaces originally are not necessarily many-sheeted surfaces over the plane, but that, on the contrary, complex functions of positions can be studied on arbitrarily given curved surfaces in exactly the same way as on the surfaces over the plane" [Klein 1882/1893, p. x].

Riemann made the surface simply connected with suitable transversal cuts (*Querschnitte*) and analysed the behaviour of the function in the neighbourhood of the singularities — poles and branch points. Then, thanks to the Dirichlet principle, Riemann stated and proved a fundamental existence theorem for a function with given singularities and boundary conditions [1851, p. 34–35]. This is the global theorem which, in Riemann's words, opens the way to the the study of complex functions independently of their analytical expressions [Riemann 1851, p. 35].

Many of the ideas of this paper, as Klein first emphasized, were inspired by physical topics. As Riemann told Betti (see [Bottazzini 1985, p. 559]), the idea of transversal cut on a surface struck him after a long discussion with Gauss on a mathematical-physical problem. In Brill and Noether's opinion, the origin of the concepts of Riemann surface and transversal cut could be found in an unpublished note [Riemann 1876c] on a problem of electrostatic or thermal equilibrium on the surface of a cylinder with transversal cuts.[6]

Apparently Riemann sought to evaluate the distribution of the static electricity on the surface of a cylinder, due to constant forces along the directrices of the cylinder. Laplace had already observed that, when the interacting masses are placed on an infinite cylinder and the forces are constant along straight lines parallel to the directrices of the cylinder, the evaluation of the potential of the bodies can be seen as a plane problem. To this end, it is sufficient to replace every directrix by its intersection point with a plane orthogonal to the directrix. Thus, the differential equation involved reduces to the Laplace equation $\Delta u = 0$ in two variables and the sought for potential function $\sum m/r$ reduces to the "logarithmic" potential $\sum m \log(r)$.

Accordingly, Riemann stated that solving this problem was equivalent to finding a function u having given boundary values and satisfying the differential equation $\Delta u = 0$ within a surface S, which he supposed to be plane, simply connected, simple sheeted and bounded by n arbitrarily given curves [Riemann 1876c, p. 440]. In turn, this problem could be reduced to the easier one of determining a function $\zeta = \xi + i\eta$ of the complex variable $z = x + iy$ which is finite and continuous within S and takes real values on the boundary, where it becomes infinite of the first order in just one point for each curve of the boundary.

Under the condition that ζ goes from $-\infty$ to $+\infty$ when any of the boundary curves is travelled in a positive sense, "*one can easily*" show that ζ takes every real value exactly once on each curve of the boundary whereas within S it takes every complex value n times (with $\operatorname{Im}\zeta > 0$). Thus, one gets a conformal map of S onto a n-sheeted surface T over the upper half plane, whose boundary lines on the n sheets of the surface coincide with the real axis. As within T there are $(2n - 2)$ branch points [Riemann 1857b, p. 113], the problem reduces to the determination of

[6] As Weber remarked, apart from some hints by Riemann, this note reduces to sheets with calculations. "*Wir sind geneigt, die*[*se*] *Note als eine der frühesten Arbeiten Riemann's, oder doch ihren Gedankengang als den Ausgangspunkt für Riemann's Arbeiten über Functionentheorie zu bezeichnen*" [Brill and Noether 1894, p. 259].

a function of ζ branched like T, such that its real part u is continuous within T and takes arbitrarily given values on the n lines of the boundary.

By means of both the Green function and the Riemann-Schwarz principle of symmetry, Riemann showed how the problem could be solved by following the same procedure he used in his paper on Abelian functions [Riemann 1857b, pp. 113, 119] and determining the Green function by means of an Abelian integral of the third kind. Next he discussed the particular case in which the boundary of S is given by n circles. As Brill and Noether [1894, p. 258] remarked, in this way Riemann established a close connection between the theory of Abelian integrals, the conformal mapping problem and the fundamental existence theorem Riemann stated in his 1851 paper.

On the contrary, according to Klein the motivation for Abelian function theory, which was to be developed by Riemann [1857b], lay in Riemann's researches in conductors and galvanic currents. In Klein's opinion, Riemann's fundamental existence theorem of a harmonic function u could be obtained from the following "*thought experiment*" (*Gedankenexperiment*): let a n-sheeted, closed Riemann surface over the complex plane be a uniform conductor with the two poles of a galvanic battery at the points A_1 and A_2; on these assumptions a current is created, whose potential u is defined and single-valued on the surface, it satisfies the equation $\Delta u = 0$, and in A_1 and A_2 becomes infinite when r_1 and r_2 go to infinity as $\log r_1$ and $-\log r_2$ respectively [Klein 1926, p. 260–261]. Klein took this as the starting-point of his presentation of Riemann's theory of algebraic functions and their integrals, claiming that:

"*I have no doubt that he* [*Riemann*] *started from precisely those physical problems, and then, in order to give what was physically evident the support of mathematical reasoning, he afterwards substituted Dirichlet's Principle*" [Klein 1882/1893, p. x].

After referring to "*the conditions under which Riemann worked in Göttingen*" as well as to his writings on *Naturphilosophie*, Klein concluded that "*anyone who clearly understands*" this "*will, I think, share my opinion*". In Klein's view, Riemann's "*general problem*" (*allgemeine Fragestellung*) was the following: "*to study the streamings in the first place and thence to work out the theory of certain analytical functions*" [Klein 1882/1893, p. 22]. Admittedly, Klein added that his presentation "*by no means include*[*d*] *the whole of what* [*Riemann*] *intended in the theory of functions*" and recognized that, even in the case of algebraic functions and their integrals, his point of view was "*necessarily very subjective*".

In spite of Klein's hope, his interpretation of the origins of Riemann's theory met strong criticism. In his review of Klein's booklet, M. Noether [1882] openly questioned Klein's historical reconstruction. This was also rejected by Prym[7] and Betti did the same in a letter to Klein from March 22, 1882. Answering a precise question asked by Klein, he stated that:

"*Riemann ne m'a jamais dit avoir développé la théorie des courants stationnaires dans un fluide incompressible, dans un espace quelconque à trois dimensions et à courbure quelconque, quoiqu'il se soit entretenu avec moi plusieurs fois sur les travaux de Mr. Helmholtz*" [Klein *Nachlass* 8, 86].[8]

Some days later Klein addressed Bianchi [*Opere* 11, pp. 116–117], a former student of Betti, who had spent a period in München with Klein. In his reply Bianchi reported that, according to Betti's record, Riemann never made any connection between electric currents and closed surfaces "*free in space*". Instead, "[*Betti*] *can ensure you with all certainty that once Riemann told him that he* [*Riemann*] *had been led to his analytical theory and his way of thinking* [Anschauungsweise] *after dealing with questions and problems of physics*" [Klein *Nachlass* 8, 100].

[7] Contrary to Klein's opinion, in the letter mentioned above (see footnote 5) Prym stated that: "*Ich halte es daher auch für sehr wahrscheinlich, dass ähnliche Ideen, wie Sie sie entwickeln, von Riemann verfolgt worden sind, aber erst nachdem die Theorie der Abelschen Functionen vollendet war. Sie dagegen scheinen der Ansicht zu sein, dass Riemann von dem allgemeinen, auf beliebig beschaffene Flächen bezüglichen Falle zu dem Falle der Abelschen Functionen als einem speciellen Hinabgestiegen sei. Ich meine, der umgekehrte Weg, den ja auch Sie, wenn auch nicht in Ihrer Abhandlung, so doch in Ihren Studien eingeschlagen haben, sein der natürliche und entspreche am meisten den Gesetzen der geschichtlichen Entwicklung.*"

[8] French revised [Eds.].

2. RIEMANN'S *NEUE MATHEMATISCHE PRINCIPIEN DER NATURPHILOSOPHIE*

After the completion of his thesis, Riemann himself gave a hint of his research projects by writing in an undated note that his "*main work*" involved "*a new interpretation of the known laws of nature — whereby the use of experimental data concerning the interaction between heat, light, magnetism, and electricity would make possible an investigation of their interrelationship. I was led to this primarily through the study of the works of Newton, Euler and, on the other side, Herbart*".[9]

As for the latter, it is worth noting here that Riemann could agree "*almost completely*" with the results of Herbart's early research while he rejected his philosophy "*at an essential point*" which involved *Naturphilosophie.* In particular, Herbart's psychology[10] inspired both Riemann's model of the substance (or ether) and his principles of *Naturphilosophie.*

Herbart had defined the "*psychic act*" (or representation) as an act of self-preservation with which the *ego* opposed the perturbations coming from the external world. He imagined a continuous flow of representations going from the *ego* to the conscious and back and studied the connections between different representations in mechanical terms as compositions of forces.

Riemann's more coherent attempt to give a systematic presentation of his ideas on the propagation of physical phenomena like gravitation

[9] Riemann wrote in fact: "*Meine Hauptarbeit betrifft eine neue Auffassung der bekannten Naturgesetze - Ausdruck derselben mittelst anderer Grundbegriffe — wodurch die Benutzung der experimentellen Data über die Wechselwirkung zwischen Wärme, Licht, Magnetismus und Electricität zur Erforschung ihres Zusammenhangs möglich wurde. Ich wurde dazu hauptsächlich durch das Studium der Werke Newton's, Euler's und — anderseits — Herbart's geführt*" [1876a, p. 507].

[10] "*Was* [*Herbart*] *betrifft* — Riemann wrote — *so konnte ich mich den frühesten Untersuchungen Herbart's, deren Resultate in seinen Promotions — und Habilitationsthesen (vom 22. u. 23. Oktober 1802) ausgesprochen sind, fast völlig anschliessen, musste aber von dem späteren Gange seiner Speculation in einem wesentlichen Punkte abweichen, wodurch eine Verschiedenheit in Bezug auf seine Naturphilosophie und diejenigen Sätze der Psychologie, welche deren Verbindung mit der Naturphilosophie betreffen, bedingt ist*" [1876a, pp. 507–508]. Riemann summarized his philosophical views by saying: "*Der Verfasser ist Herbartianer in Psychologie und Erkenntnisstheorie (Methodologie und Eidologie), Herbart's Naturphilosophie und den darauf bezüglichen metaphysischen Disciplinen (Ontologie und Synechologie) kann er meistens nicht sich anschliessen*" [1876a, p. 508].

and light was made in March 1853 in a paper which, echoing Newton's *Principia*, he did not hesitate to entitle *Neue mathematische Principien der Naturphilosophie* (New mathematical principles of natural philosophy). Indeed, Riemann himself regarded his writings on *Naturphilosophie* as fundamental and intended to publish them, as he wrote to his brother Wilhelm in December 1853 (see [Dedekind 1876, p. 547]).

In his paper Riemann began by claiming that the basis of the general laws of motion for ponderable bodies, which are posed at the beginning of Newton's *Principia*, lies in the internal state of the bodies [1853, p. 528]. Led by the analogy with Herbart's psychology Riemann made the hypothesis that the universe (*Weltraum*) was filled with a substance (*Stoff*) flowing continually through atoms and there disappearing from the material world (*Körperwelt*) [1853, p. 529].[11]

On the basis of this rather obscure idea Riemann tried to build a mathematical model of the space surrounding two interacting particles of substance. He introduced a cartesian coordinate system and considered a single particle of substance as concentrated at the point $O(x_1, x_2, x_3)$ at the time t and at the point $O'(x'_1, x'_2, x'_3)$ at the time t', where x'_1, x'_2, x'_3 are functions of x_1, x_2, x_3. Then, "*according to a well known theorem*", the two homogeneous differential forms

$$\mathrm{d}s^2 = dx_1^2 + dx_2^2 + dx_3^2 \quad \text{and} \quad \mathrm{d}s'^2 = \mathrm{d}x_1'^2 + \mathrm{d}x_2'^2 + \mathrm{d}x_3'^2$$

could be expressed in the following way

$$\mathrm{d}s'^2 = G_1^2\,\mathrm{d}s_1^2 + G_2^2\,\mathrm{d}s_2^2 + G_3^2\,\mathrm{d}s_3^2,$$
$$\mathrm{d}s^2 = \mathrm{d}s_1^2 + \mathrm{d}s_2^2 + \mathrm{d}s_3^2,$$

where $\mathrm{d}s_1$, $\mathrm{d}s_2$, $\mathrm{d}s_3$ was an appropriate new basis [Riemann 1853, p. 530].[12] Riemann called the quantities $G_1 - 1, G_2 - 1, G_3 - 1$ the main dilatations (*Hauptdilatationen*) of the particle at O and denoted them by $\lambda_1, \lambda_2, \lambda_3$.

[11] In a footnote Riemann added that: "*In jedes ponderable Atom tritt in jedem Augenblick eine bestimmte, der Gravitationskraft proportionale Stoffmenge ein und verschwindet dort. Es ist die Consequenz der auf Herbart'schem Boden stehenden Psychologie, dass nicht der Seele, sondern jeder einzelnen in uns gebildeten Vorstellung Substantialität zukomme*" [1853, p. 529].

[12] The conditions according to which the equation $\mathrm{d}s'^2 = \mathrm{d}s^2$ is satisfied are the Lamé equations. They were first published by Lamé [1859, pp. 99, 101].

Riemann's result can be interpreted in terms of the classical theory of elasticity. Supposing, as Riemann did, that the ether is an elastic, homogeneous, isotropic medium, then one can consider an elastic deformation changing $P(x_1, x_2, x_3)$ and $Q(x_1+\mathrm{d}x_1, x_2+\mathrm{d}x_2, x_3+\mathrm{d}x_3)$ to $P'(x'_1, x'_2, x'_3)$ and $Q'(x'_1 + \mathrm{d}x'_1, x'_2 + \mathrm{d}x'_2, x'_3 + \mathrm{d}x'_3)$ respectively. Under these assumptions, it is possible to compare the distance $\mathrm{d}s$ between P and Q with the corresponding distance $\mathrm{d}s'$ after the deformation.[13]

If $x'_i = x_i + u_i$ (where u_i is the displacement of P due to the deformation) one has

$$\mathrm{d}x'_i = \mathrm{d}x_i + \mathrm{d}u_i = \mathrm{d}x_i + \sum_{k=1}^{3} \frac{\partial u_i}{\partial x_k}\,\mathrm{d}x_k.$$

Then

$$\mathrm{d}s'^2 = \mathrm{d}s^2 + \sum_{k,\ell=1}^{3} \frac{\partial u_\ell}{\partial x_k}\,\mathrm{d}x_k\,\mathrm{d}x_\ell + \sum_{k,\ell,i=1}^{3} \frac{\partial u_\ell}{\partial x_i}\,\frac{\partial u_\ell}{\partial x_k}\,\mathrm{d}x_i\,\mathrm{d}x_k.$$

Now it is possible to calculate the variation $\delta(\mathrm{d}s^2) = \mathrm{d}s'^2 - \mathrm{d}s^2$

$$\delta(\mathrm{d}s^2) = \sum_{k,i=1}^{3} e_{ik}\,\mathrm{d}x_i\,\mathrm{d}x_k, \tag{2.1}$$

where

$$e_{ik} = \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \sum_{\ell=1}^{3} \frac{\partial u_\ell}{\partial x_i}\,\frac{\partial u_\ell}{\partial x_k}.$$

This quantity defines the deformation and coincides with the strain tensor in elasticity theory. If the deformation is supposed infinitely small, the strain tensor becomes $e_{ik} = \dfrac{\partial u_i}{\partial x_k} + \dfrac{\partial u_k}{\partial x_i}$. Since it is symmetric, it is possible to find a basis such that $e_{ik} = 0$ if $i \neq k$.

Riemann compared $\mathrm{d}s'^2$ and $\mathrm{d}s^2$ by a direct calculation on the same basis $\mathrm{d}s_1, \mathrm{d}s_2, \mathrm{d}s_3$, with respect to which both $\mathrm{d}s'^2$ and $\mathrm{d}s^2$ were orthogonal. He found that

$$\delta(\mathrm{d}s^2) = \mathrm{d}s'^2 - \mathrm{d}s^2 = (G_1^2 - 1)\,\mathrm{d}s_1^2 + (G_2^2 - 1)\,\mathrm{d}s_2^2 + (G_3^2 - 1)\,\mathrm{d}s_3^2. \tag{2.2}$$

[13] See for instance Brillouin [1938, ch. X].

In this case for the components of the strain tensor one has

$$e_{11} = G_1^2 - 1, \quad e_{22} = G_2^2 - 1, \quad e_{33} = G_3^2 - 1, \quad e_{ik} = 0 \quad \text{if} \quad i \neq k.$$

Since the *linear extensions* after the deformation with respect to the axes x_1, x_2, x_3 are defined as $\Delta\ell_i = \sqrt{1 + e_{ii}} - 1$, for $i = 1, 2, 3$, then

$$\Delta\ell_1 = G_1 - 1, \quad \Delta\ell_2 = G_2 - 1, \quad \Delta\ell_3 = G_3 - 1.$$

They coincide with Riemann's *Hauptdilatationen* $\lambda_1, \lambda_2, \lambda_3$, which therefore can be studied in the conceptual framework given by the theory of elasticity.

Riemann then supposed that the variation $\delta(\mathrm{d}s^2)$ produced a force able to modify the particle in such a way that the particle itself, opposing this deformation, would propagate the physical forces through the space.[14] In order to describe the reaction of the particle, Riemann introduced the hypothesis that the dilatations were infinitely small, and so the produced forces were linear functions of $\lambda_1, \lambda_2, \lambda_3$. By assuming the homogeneity of the substance, the moment of force (*Kraftmoment*) was given by

$$\tfrac{1}{2}\,\delta\big[a(\lambda_1 + \lambda_2 + \lambda_3)^2 + b(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\big]$$

where a, b were constants.

Riemann regarded this moment as a resultant of the forces which tend to lengthen or shorten the line elements ending at O [1853, p. 531]. He formulated the following "*law of effects*" (*Wirkungsgesetz*): if $\mathrm{d}V$ and $\mathrm{d}V'$ represent the volumes of the particle at time t and t' respectively, then the force arising from two different states (*Stoffzustände*) of the substance in different times t, t' can be described in the following way

$$a\,\frac{\mathrm{d}V - \mathrm{d}V'}{\mathrm{d}V} + b\,\frac{\mathrm{d}s - \mathrm{d}s'}{\mathrm{d}s}\cdot \tag{2.3}$$

Moreover, since the two addenda in (2.3) are independent of each other, the two actions produce different effects, whose development follows different laws. One could observe that if the *Hauptdilatationen* $\lambda_1, \lambda_2, \lambda_3$, of the particle of ether were equal to zero, the distance between two infinitely close points of ether remains constant after the deformation

[14] See footnote 15 below.

at issue. In this case no physical phenomenon could occur in the space surrounding the particle, and the *Kraftmoment* given by (2.3) vanishes.

Riemann calculated the variation of ds during the time dt and found that the quantity $\delta\,\mathrm{d}s/\mathrm{d}t$ was equal to

$$\int_{-\infty}^{t} \frac{\mathrm{d}V' - \mathrm{d}V}{\mathrm{d}V}\,\psi(t-t')\,\delta t' + \int_{-\infty}^{t} \frac{\mathrm{d}s' - \mathrm{d}s}{\mathrm{d}s}\,\varphi(t-t')\,\delta t'.$$

But how can we choose the functions ψ and φ in order that gravity, heat and light propagate through space? This was the question Riemann left open [1853, p. 532]. In this connection he limited himself to state that "*the effects of ponderable matter on ponderable matter were attractive and repulsive forces inversely proportional to the square of the distance*" or "*light and radiant heat*". They could be explained by assuming that every particle of the homogeneous substance filling space has a direct effect only on its neighbourhood.[15] The mathematical law (2.3) according to which this happens, can be divided into:

"1) *the resistance with which a particle opposes a change of its volume, and*

2) *the resistance with which a physical line element opposes a change of length.*

Gravity and electric attraction and repulsion are founded on the first part, light, heat propagation, electrodynamic and magnetic attraction and repulsion on the second part".[16]

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[15] "*Es kann also die Wirkung der allgemeinen Gravitation auf ein ponderables Atom durch den Druck des raumerfüllenden Stoffes in der unmittelbaren Umgebung desselben ausgedrückt und von demselben abhängig gedacht werden. Aus unserer Hypothese folgt nothwendig, dass der raumerfüllende Stoff die Schwingungen fortpflanzen muss, welche wir als Licht und Wärme wahrnehmen*" [Riemann 1853, p. 529].

[16] "*Beide Classen von Erscheinungen lassen sich erklären, wenn man annimt, dass den ganzen unendlichen Raum ein gleichartiger Stoff erfüllt, und jedes Stofftheilchen unmittelbar nur auf seine Umgebung einwirkt.*

Das mathematische Gesetz, nach welchem dies geschieht, kann zerfällt gedacht werden

1) in den Widerstand, mit welchem ein Stofftheilchen einer Volumänderung, und

2) in den Widerstand, mit welchem ein physisches Linienelement einer Längenänderung widerstrebt.

Auf dem ersten Theil beruht die Gravitation und die electrostatische Anziehung und Abstossung, auf dem zweiten die Fortpflanzung des Lichts und der Wärme und die electrodynamische oder magnetische Anziehung und Abstossung" [Riemann 1853, p. 532].

In *Gravitation und Licht* which constitutes the last section of his *Fragmente* on *Naturphilosophie* [1876a, pp. 532–538], Riemann presented a mathematically more sophisticated attempt at a unified explanation of both gravitation and light. Facing the problem of explaining Newton's theory of gravitation, Riemann supposed that a determinate cause (*bestimmte Ursache*) existed at every point of space. In virtue of this cause the point was able to interact with other ponderable bodies by means of a force, which was inversely proportional to the square of their distance. He assumed that this cause should be sought in "*the form of motion of a substance spread continuously through the entire infinite space. ... This substance can therefore be conceived as a physical space whose points move in geometrical space*".[17] By means of this assumption, it was possible to explain the propagation of phenomena through space and then light and gravitation had to be explained by means of the motion of this substance.[18]

The further development he drew from this hypothesis divided in two parts:

(a) to find the mathematical laws of motion of this substance which can be used for explaining the phenomena;

(b) to clarify the causes (*Ursachen*) by means of which one could explain these motions.

In commenting upon the second point, which was metaphysical in character, Riemann observed that, contrary to Newton's own opinion, his law of attraction had not been thought to need any further explanation for long time. Riemann was pleased to add in a footnote the celebrated passage of Newton's third letter to Bentley where Newton stated that believing in action at a distance "*without the mediation of anything else*" was "*so great an absurdity*" [1876a, p. 534].

In this connection, it is worth noting that in the *General Scholium*

[17] "*Die nach Grösse und Richtung bestimmte Ursache ... suche ich in der Bewegungsform eines durch den ganzen unendlichen Raum stetig verbreiteten Stoff. ... Dieser Stoff kann also vorgestellt werden als ein physischer Raum, dessen Punkte sich in dem geometrischen bewegen*" [Riemann 1876a, p. 533].

[18] "*Nach dieser Annahme müssen alle von ponderablen Körpern durch den leeren Raum auf ponderable Körper ausgeübte Wirkungen durch diesen Stoff fortgepflanzt werden. ... Diese beiden Erscheinungen, Gravitation und Lichtbewegung durch den leeren Raum, aber sind die einzigen, welche bloss aus Bewegungen dieses Stoffes erklärt werden müssen*" [Riemann 1876a, p. 533].

at the end of the third edition of the *Principia*, Newton himself had spoken about "*a certain most subtle spirit which pervades and lies hid in all gross bodies*", through which one could explain both the attraction of the "*particles of bodies*" and the action of the electric bodies and light [1726/1934, p. 547]. The hypothesis about ether had also been advanced by Euler, "*in his magnificent attempt*" — as A. Speiser defined it [1927, p. 106] — to draw up a unified theory of gravitation, light, electricity and magnetism. Euler's related papers, together with his *Lettres à une princesse d'Allemagne*, are included in the 1838 Brussels edition of his works. Both these papers and the *Lettres* are likely to have been the source of inspiration to which Riemann referred in his undated manuscript note.[19]

From the mathematical point of view, Riemann supposed that the real motion of the substance v was composed of two motions u and w, that is $v = u + w$, which were responsible for the propagation of gravitation and light respectively. Then, he proposed to find the "*laws of motion of substance in empty space*" [1876a, p. 537]. To this end he identified the forces with the motion caused by them and observed that the gravitational force u, deduced from the potential V by means of the relation $u = \operatorname{grad} V$, had to satisfy both the conditions of being a closed form and the equation $\Delta V = -4\pi\rho$ as well [Riemann 1876a, pp. 534–535].

The differential equations which characterized the propagation of light were a continuity equation

$$\operatorname{div} w = 0$$

and a wave equation for transwerse oscillations in velocity w

$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + \frac{\partial^2 w}{\partial x_3^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2},$$

where c was the speed of the light.

As N. Wise has remarked, "*combining the two motions for gravity and light produced a well-behaved velocity function, which confirmed the possibility of uniting the two processes*" [1981, p. 290]. As a consequence of the conditions valid for u and w, Riemann found for the motion of the

[19] See above footnote 9.

substance in empty space, v, the following equations

$$\operatorname{div} v = 0, \tag{2.4}$$

$$\begin{cases} \Big[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\Big)\Big]\Big(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}\Big) = 0, \\ \Big[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\Big)\Big]\Big(\frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3}\Big) = 0, \\ \Big[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\Big)\Big]\Big(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\Big) = 0. \end{cases} \tag{2.5}$$

According to Riemann, these equations show how the motion of a single particle of the substance depends only on the motions of the particles around it.[20] Equation (2.4) proves that the density is not changed during the motion of the substance while the condition expressed by the equations (2.5) coincides with the condition that

$$\Big[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\Big)\Big](v_1\,\mathrm{d}x_1 + v_2\,\mathrm{d}x_2 + v_3\,\mathrm{d}x_3)$$

is an exact differential $\mathrm{d}W$.

Starting from a set of relations which were valid in the infinitely small, Riemann succeeded in formulating a theory valid in finite regions. In fact, the solution of the differential system (2.4, 2.5) determined the motion of the ether propagating light and gravity through the space. According to his student Schering, by means of his model of the ether Riemann hoped to eliminate from the laws of interaction those "*specifications which refer to action at a distance*" as they "*always depend on the properties of the surrounding space*" [1866/1991, p. 838].

3. PHYSICAL RESEARCHES

In 1854 during his Easter holidays, while preparing his *Habilitationsvortrag* on the principles of geometry, Riemann met the experimental physicist Kohlrausch, who told him about an "*unexplored phenomenon*" concerning the electric residuum in Leyden jars. Riemann intended to explain

[20] "*Diese Gleichungen zeigen, dass die Bewegung eines Stoffpunktes nur abhängt von den Bewegungen in den angrenzenden Raum- und Zeittheilen, und ihre (vollständigen) Ursachen in den Einwirkungen der Umgebung gesucht werden können*" [Riemann 1876a, p. 537].

Kohlrausch's experiment, as he wrote to his brother, by means of his research on the connections between electricity, gravity, light and magnetism. In such a way, Riemann hoped that his "*main work could achieved a favourable reception*".[21]

According to Kohlrausch's experiment, in a Leyden jar which had been charged, then discharged and left isolated for some time, a residual charge appeared. Kohlrausch [1854, pp. 61–62] divided the total charge in the jar into two parts: the free charge, L, which could be discharged, and the charge r that appeared (at least in part) only after the exhaustion of the free charge (r was called *das Residuum* or *der Rückstand*). The "*hidden*" residuum was a quantity of r that could no longer be traced in the jar, while the "reappearing" residuum was the portion of residuum which came back again after the discharge. Kohlrausch observed the "*reappearing*" residuum by means of the *Sinuselektrometer*, an ingenious instrument invented and built by Kohlrausch himself [1853].

In September 1854, lecturing at a meeting of the German "*natural philosophers*", Riemann tried to give an explanation of Kohlrausch's experiment. To this end, he assigned to every ponderable body a certain conductivity coefficient β, observing that in nature neither complete insulators nor perfect conductors exist.[22] If u is the potential and ρ the charge density, he deduced for the components of the electromotive force at the point (x, y, z) at time t the following expressions

$$\Big(-\frac{\partial u}{\partial x} - \beta^2 \frac{\partial \rho}{\partial x}, -\frac{\partial u}{\partial y} - \beta^2 \frac{\partial \rho}{\partial y}, -\frac{\partial u}{\partial z} - \beta^2 \frac{\partial \rho}{\partial z}\Big). \tag{3.1}$$

Quantity (3.1) has to be proportional to the components of the current intensity vector (ξ, η, ζ), and therefore the last expression is equal to

[21] "*Kohlrausch hatte nun einige Zeit vorher sehr genaue Messungen über eine bis dahin unerforschte Erscheinung (den electrischen Rückstand in der Leidener Flasche) gemacht und veröffentlicht und ich hatte durch meine allgemeinen Untersuchungen über den Zusammenhang zwischen Electricität, Licht und Magnetismus die Erklärung davon gefunden. ... Mir ist diese Sache deshalb wichtig, weil es das erste Mal ist, wo ich meine Arbeiten auf eine vorher noch nicht bekannte Erscheinung anwenden konnte, und ich hoffe, dass die Veröffentlichung dieser Arbeit dazu beitragen wird, meiner grösseren Arbeit eine günstige Aufnahme zu verschaffen*" (in [Dedekind 1876, pp. 548–549]).

[22] Laplace in his *Mécanique céleste* had already associated to each medium a gravitational coefficient α, and had deduced the attraction law between the two masses m_1 and m_2: $G(m_1 m_2/r^2)\exp(-\alpha r)$, where G is the constant of universal gravitation and r the distance between m_1 and m_2 [*Œuvres* 5, pp. 445–452].

$(\alpha\xi, \alpha\eta, \alpha\zeta)$, where α is a constant depending on the nature of the body at issue. By means of both the principle of conservation of charge

$$\frac{\partial \rho}{\partial t} + \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0,$$

and the formula

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\rho,$$

Riemann deduced

$$\alpha \frac{\partial \rho}{\partial t} + \rho - \beta^2 \Big(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2} \Big) = 0. \tag{3.2}$$

He then compared "*the consequences of the law in certain particular cases with experience*" [Riemann 1854c, p. 51] and found well known expressions in the case when the bodies were metallic and perfect conductors. The law (3.2) was also in accordance with the experimental data of Kohlrausch about the free charge and the "reappearing" residuum.

In the last part of this paper, Riemann presented the two theories on the nature of electric force, both discussed at that time — the unitary conception and the dualistic one. According to the dualistic conception, two opposite electricities existed in a body, a positive electricity and a negative one. Riemann rejected this theory since, on Weber's advice, he had submitted it to calculations "*without obtaining any satisfactory results*" [Riemann 1854c, p. 54]. Then Riemann chose Franklin's unitary theory that attaches a certain quantity of electricity to every body and defines a positive or a negative electric state of a body as an excess or a defect of this electricity respectively.

Riemann intended to publish an extended version of his results in *Annalen der Physik und Chemie* but then, as Dedekind [1876, p. 550] and Weber (in [Riemann 1876d, p. 367]) have reported, he changed his mind because of the substantial changes requested by Kohlrausch, the editor of the review.[23] Eventually his original manuscript was published by Weber [Riemann 1876d] and it includes a section crossed out by Riemann

[23] Instead of it, Riemann published in Kohlrausch's *Annalen* his 1855 paper on Nobili's rings.

himself.[24] In this section, he developed a physical explanation of the electromotive force and of electric propagation through a body on the basis of the model of ether exposed in the *Fragmente*. Thus, instead of Franklin's unitary theory of electricity, Riemann assumed that the electric current was caused by a reaction the body opposed to its own electric state. This reaction was proportional to the charge density ρ multiplied by the coefficient β^2, and it decreased or increased the electric density according as the body contained positive or negative electricity. Therefore, the transmission of electricity could not be instantaneous but, Riemann continued, the electricity moves "*against ponderable bodies*" with a speed which equals the electromotive force deriving from the potential u. Moreover, this law of motion must be changed in order to show its connection with heat and magnetism.[25]

In a note dated July 1835 Gauss had already suggested a new theory of electrodynamics. According to Gauss, two elements of electricity attract and repulse each other with a force depending on their moving state.[26] He gave for the force between two electric particles ε and ε' placed in (x, y, z) and in (x', y', z') respectively, the following formula

$$F = \frac{\varepsilon\varepsilon'}{r^2}\left[1 + \frac{1}{c^2}\left(v^2 - \frac{3}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right)\right], \tag{3.3}$$

where r is the distance from ε to ε', c is a constant velocity,[27] and v is the velocity of $\varepsilon(x, y, z)$ with respect to $\varepsilon'(x', y', z')$. As Maxwell

---

[24] "*Dieser ganze Artikel* — Weber observed in a footnote — *ist im Manuscript durchgestrichen, wahrscheinlich nur aus dem Grunde, weil der Verfasser durch die Eigenthümlichkeit der hier vorgetragenen Auffassung, welche auf das Innigste mit seinen naturphilosophischen Principien zusammenhängt, bei den Physikern damals Anstoss zu erregen befürchtete*" [Riemann 1876d, p. 371].

[25] "*Die Electricität bewegt sich gegen die ponderabeln Körper mit einer Geschwindigkeit, welche in jedem Augenblicke der aus diesen Ursachen hervorgehenden electromotorischen Kraft gleich ist* [the cause is the potential u which satisfies the equation $\Delta u = -\rho$].

Uebrigens müssen diese Bewegungsgesetze der Electricität, wenn deren Verhältniss zu Wärme und Magnetismus in Rechnung gezogen werden soll, vorbemerktermassen selbst noch abgeändert und umgeformt werden, und dann wird eine veränderte Auffassung dieser Erscheinungen nöthig" [Riemann, 1876d, p. 371].

[26] "*Zwei Elemente von Electricität in gegenseitiger Bewegung ziehen einander an oder stossen einander ab nicht eben so als wenn sie in gegenseitiger Ruhe sind*" [Gauss 1835, p. 616].

[27] Within the limit of observation, this is the velocity of light.

[1873, p. 484] showed, the formula (3.3) is inconsistent with the principle of the conservation of energy.

In 1845, in a letter to Weber, Gauss [1845, p. 629] supposed that electricity propagated from one point to another *not instantaneously*, but in time, as in the case of light. Gauss never published his ideas about electrodynamics during his lifetime, and W. Weber's theory of electrodynamics, published in his celebrated *Elektrodynamische Maassbestimmungen* (1846), was the first result of this kind known to the scientific world.

According to Weber, the force between the two particles ε and ε' was given by

$$F = \frac{\varepsilon\varepsilon'}{r^2}\left[1 + \frac{1}{c^2}\left(r\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right)\right]. \tag{3.4}$$

The expressions (3.3) and (3.4) lead to the same result for mechanical force between two electric currents, and this result coincides with that of Ampère's. Maxwell remarked that the formula (3.4) satisfied the principle of the conservation of energy on assuming for the potential energy

$$P = \frac{\varepsilon\varepsilon'}{r^2}\left[1 - \frac{1}{2c^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right]. \tag{3.5}$$

But, as he pointed out, "*an indefinite amount of work cannot be generated by a particle moving in a periodic manner under the action of the force assumed by Weber*" [Maxwell 1873, p. 484]. So Weber's formula must be rejected too.

As a follower of Gauss and Weber, Riemann himself tried to formulate a new theory of electrodynamics in his 1858 paper *Ein Beitrag zur Elektrodynamik*. He supposed that electric phenomena travel with the velocity of light and that the differential equations for electric force are the same of those valid for light and heat propagation [Riemann 1858, p. 288].

Riemann considered the electrodynamic system of two conductors C and C' moving one with respect to the other, and the galvanic currents running through them. He studied the interaction of two particles ε and ε', the first in C and the latter in C', with coordinates (x, y, z) and (x', y', z') respectively. "*With admirable directness* — Rosenfeld has remarked — *he* [*wrote*] *down the generalised Poisson equation* [*for the potential*] *involving the operator now called 'D'Alembertian' and the expression for its solution in the form of retarded potential*" [1956, p. 139].

Riemann indeed gave the following equation for the potential function U

$$\frac{\partial^2 U}{\partial t^2} - \alpha^2\Big(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\Big) + 4\alpha^2\pi\rho = 0, \tag{3.6}$$

where ρ was the charge density in the point (x, y, z) and α the velocity of propagation of electricity. According to (3.6), the potential U travels with a velocity α and it reaches a point distant r from (x, y, z) after the time r/α. As Reiff and Sommerfeld [1902, p. 46] wrote, the interest in this work is based on this result which makes Riemann appear as a "precursor" of Maxwell.

Riemann supposed that the current was due to the positive and negative electricities running through the wire, and that the two sums $\sum \varepsilon f(x, y, z)$, $\sum \varepsilon' f(x', y', z')$ extended to all the charges were negligible with respect to the same sums extended only to the positive or negative charges. On these assumptions, Riemann found that the potential function at the point (x, y, z) due to the point (x', y', z') was $f(t - r/\alpha)/r$, if the charge $-f(t)$ was placed in the point in question at time t, and r was the distance between ε and ε'. Moreover, if the coordinates of ε at the time t were (x_t, y_t, z_t), and those of ε' at time t' were $(x'_{t'}, y'_{t'}, z'_{t'})$, he set

$$F(t, t') = \big[(x_t - x'_{t'})^2 + (y_t - y'_{t'})^2 + (z_t - z'_{t'})^2\big]^{-\frac{1}{2}},$$

and deduced the potential expressing the total effect of the conductor C on the conductor C' from the time 0 to the time t

$$P = -\int_0^t \sum\sum \varepsilon\varepsilon' F\Big(t - \frac{r}{\alpha}, \tau\Big)\, d\tau. \tag{3.7}$$

Riemann tried to obtain the electrodynamic potential derived from the Weber's law from (3.7). But, as Clausius [1868] was to point out, Riemann committed a mistake in the permutation of two integrations. According to Weber (see [Riemann 1858, p. 293]), the mistake was discovered by Riemann himself and this was the reason that convinced him to withdraw the paper.

In Betti's opinion however Riemann did not publish this paper because it was in contrast with what he had stated in [Riemann 1854c, p. 54].[28]

[28] "*Questo concetto della corrente elettrica, tutto ideale* — Betti observed — *è poco in armonia con ciò che si conosce di essa, e pare che Riemann non ne fosse soddisfatto*" [1868, p. 242].

Betti himself suggested a new theory of electrodynamics. He supposed the closed circuits to consist of polarized elements which acted on one another as little magnets. According to Betti, the polarized elements interacted not instantaneously but only after a time proportional to the distance between the elements.[29]

Maxwell was aware of Riemann's and Betti's electrodynamic researches, and even he could not agree with "*these eminent men*". In their theories, indeed, the action travelled in a manner similar to that of light but, Maxwell asked:

"*If something is transmitted from one particle to another at a distance, what is its condition after it has left the one particle and before it has reached the other?*" [1873/1954, II, p. 493].

In his 1858 paper Riemann did not mention any medium which propagated the electric phenomena. He tried however to describe the ether surrounding two interacting electric particles in his lectures on electricity, gravity and magnetism, held in 1861 and published by Hattendorff in 1876. Here, Riemann tackled the electrodynamic problem of two conductors moving one with respect to the other by using the calculus of variations. He deduced from the principle of conservation of energy the "*extended theorem of Lagrange*"

$$\delta \int_0^t (T - D + S)\,\mathrm{d}t = 0, \tag{3.8}$$

where T is the kinetic energy, S the potential depending only on the coordinates (electrostatic potential) and D the potential depending on the velocities too (electrodynamic potential) [Riemann, 1876e, p. 316].

If ε and ε' were two electric particles in C and in C' at the points (x, y, z) and (x', y', z') respectively, S and D were expressed by the following equations

$$D = \frac{\varepsilon\varepsilon'}{r}\,\frac{u^2}{2c^2}, \quad S = -\frac{\varepsilon\varepsilon'}{r},$$

where r was the distance between ε and ε', u the velocity of one particle with respect to the other, and c the velocity of light.

[29] Clausius also criticized some parts of Betti's calculations.

By solving the Euler-Lagrange equations deduced from (3.8), Riemann obtained the following expression for the force acting on ε and ε'

$$
\begin{cases}
F_x = \dfrac{\varepsilon\varepsilon'}{r^2}\Big(1+\dfrac{u^2}{2c^2}\Big)\dfrac{\partial r}{\partial x} + \dfrac{\varepsilon\varepsilon'}{c^2}\dfrac{\mathrm{d}}{\mathrm{d}t}\Big(\dfrac{x-x'}{r}\Big), \\
F_y = \dfrac{\varepsilon\varepsilon'}{r^2}\Big(1+\dfrac{u^2}{2c^2}\Big)\dfrac{\partial r}{\partial y} + \dfrac{\varepsilon\varepsilon'}{c^2}\dfrac{\mathrm{d}}{\mathrm{d}t}\Big(\dfrac{y-y'}{r}\Big), \\
F_z = \dfrac{\varepsilon\varepsilon'}{r^2}\Big(1+\dfrac{u^2}{2c^2}\Big)\dfrac{\partial r}{\partial z} + \dfrac{\varepsilon\varepsilon'}{c^2}\dfrac{\mathrm{d}}{\mathrm{d}t}\Big(\dfrac{z-z'}{r}\Big).
\end{cases} \tag{3.9}
$$

If D is Weber's electrodynamic potential, the interaction between ε and ε' deduced from (3.8) is given by Weber's law (3.4) (see [Reiff and Sommerfeld 1902, pp. 48–49]).

By considering the effect of all the particles of the conductor C' on the particle ε, Riemann set $S = \varepsilon V$ and found for the potential function V the relation[30]

$$
\frac{\partial V}{\partial t} = \operatorname{div} u = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}, \tag{3.10}
$$

where $u_1 = \sum_{\varepsilon'} \frac{\varepsilon'}{r}\frac{\mathrm{d}x'}{\mathrm{d}t}$, $u_2 = \sum_{\varepsilon'} \frac{\varepsilon'}{r}\frac{\mathrm{d}y'}{\mathrm{d}t}$, $u_3 = \sum_{\varepsilon'} \frac{\varepsilon'}{r}\frac{\mathrm{d}z'}{\mathrm{d}t}$.

According to Riemann, "*from this differential equation it would be possible to derive the meaning of the functions V, u_1, u_2, u_3. One could assume that electricity is propagated by an ether. By means of equation* [(3.10)] *V may be regarded as the density and u_1, u_2, u_3 as* [*the components of*] *the intensity of the flux of this ether*".[31] By this identification, the principle of conservation of charge held for the ether too.

Riemann explained his model of ether in his 1853 paper on the principles of *Naturphilosophie*. However, he was not able to incorporate electrostatic and electromagnetic effects into it. In his 1861 lectures he assumed that the ether satisfied physical properties which guaranteed electrical propagation. As N. Wise has remarked: "*a stable gradient in density would*

[30] For a discussion of this point see Wise [1981, p. 290].

[31] "*Auf Grund dieser Differentialgleichung könnte man über die Bedeutung der Functionen V, u_1, u_2, u_3 eine Annahme machen. Man kann annehmen, die elektrische Wirkung werde durch einen Aether vermittelt. Vermöge der Gleichung* [(3.10)] *liessen sich dann V als die Dichtigkeit, u_1, u_2, u_3 als die Stromintensitäten dieses Aethers ansehen*" [Riemann 1876e, p. 330].

then correspond to electrostatic force, whereas a time rate of change in density, and associated fluxes, would correspond to electrodynamic effects" [1981, p. 290]. Therefore, the hypothesis (in [Riemann 1858]) that the same differential equation held for light and electric propagation, made the ether responsible for the transmission of both the phenomena. Referring to his 1858 paper Riemann wrote to his sister that, notwithstanding the fact that Gauss had formulated another theory of the connection between electricity and light, "*I am sure that my theory is right and that in a couple of years it will be generally recognized as such*".[32]

4. ON THE FOUNDATION OF GEOMETRY

The attempt undertaken by Riemann in the *Fragmente* at formulating a theory which could explain the transmission of phenomena is in our opinion among the basic motivations for the study of manifolds presented by Riemann in his 1854 *Habilitationsvortrag.*

As Dedekind [1876, p. 547] reported, even though Riemann should have thought about this subject for long time, the question concerning the principles of geometry that Gauss choose as the subject of Riemann's *Probevorlesung* was certainly the least prepared of those Riemann had proposed, as he wrote to his brother on December 28, 1853. In this letter Riemann stated that after the completion of his *Habilitationsschrift* [Riemann 1854a] he once more took up his "*other research*" on *Naturphilosophie*, which was almost ready for printing.[33] Some months later he confessed to his brother to having been "*so fully immersed*" in his research on *Naturphilosophie* that he was not able to rid himself of it when the subject

[32] "*Ich bin aber völlig überzeugt, dass die meinige die richtige ist und in ein paar Jahren allgemein als solche anerkannt werden wird*" (in [Dedekind 1876, p. 553]).

[33] "*Meine andere Untersuchung über den Zusammenhang zwischen Electricität, Galvanismus, Licht und Schwere hatte ich gleich nach Beendigung meiner Habilitationsschrift wieder aufgenommen und bin mit ihr so weit gekommen, dass ich sie in dieser Form unbedenklich veröffentlichen kann. Es ist mir dabei aber zugleich immer gewisser geworden, dass Gauss seit mehreren Jahren auch daran arbeitet, und einigen Freunden, u. A. Weber, die Sache unter dem Siegel der Verschwiegenheit mitgetheilt hat, — Dir kann ich dies wohl schreiben, ohne dass es mir als Anmaassung ausgelegt wird — ich hoffe, dass es nun für mich noch nicht zu spät ist und es anerkannt werden wird, dass ich die Sachen vollkommen selbständig gefunden habe*" (in [Dedekind 1876, p. 547]).

of his lecture (*Probevorlesung*) was proposed to him (see [Dedekind 1876, pp. 547–548]). H. Weyl quoted this passage from Riemann's letter for supporting his claim that Riemann's research "*on the connection between light, electricity, magnetism and gravity*" was not "*objectively in any relation*" to the content of his lecture [1919b/1991, p. 741].

Contrary to Weyl's claim, in our opinion these letters suggest that Riemann's 1854 lecture was deeply connected with his research in mathematical physics. In fact, one can argue that in this lecture Riemann tried to generalize the ideas stated in his 1853 paper on *Naturphilosophie* to n-dimensional manifolds, extending the "local" investigation of particles of ether to the "global" analysis of space. In this connection the reference Riemann made to Gauss and Herbart is also worth noting [1854b, p. 273].

From this point of view one can explain the seeming anomalies with respect to contemporary research in foundations of geometry and in non-Euclidean geometry, often mentioned by historians of mathematics. To be sure, in his lecture Riemann only made a cryptic allusion in passing to the possibility of the elliptic geometry. Nor did he refer to the work of Bolyai and Lobachevsky, even though a paper of the latter had appeared in 1837 in Crelle's *Journal* and his *Geometrische Untersuchungen* had been published in Berlin in 1840. However one can argue that Riemann probably knew Lobachevsky's geometry, at least in the version that appeared in that *Journal.* In fact, the relevant volume of Crelle's *Journal* included both Lobachevsky's paper and Dirichlet's 1837 paper on the representation of "arbitrary" functions in series of Legendre polynomials. Thinking of Riemann's paper [1854a], it seems very likely that he had this volume in his hands when he worked at his *Habilitationsschrift.* Had Riemann limited himself to reading the first page of Lobachevsky's paper, he could have discovered that: "*rien n'autorise, si ce ne sont les observations directes, de supposer dans un triangle rectiligne la somme des angles égale à deux angles droits, et que la géométrie n'en peut pas moins exister, sinon dans la nature, au moins dans l'analyse, lorsqu'on admet l'hypothèse de la somme des angles moindre que la demi-circonférence du cercle*" [Lobachevsky 1837, p. 295].

After summarizing his main results concerning the geometry of a non-Euclidean rectilinear triangle, Lobachevsky concluded that:

"L'hypothèse de la somme des angles d'un triangle moindre que deux angles droits ne peut avoir d'application que dans l'analyse, puisque les mesures directes ne nous montrent pas dans la somme des angles d'un triangle la moindre déviation de deux angles droits. *J'ai prouvé ailleurs, en m'appuyant sur quelques observations astronomiques, que dans un triangle dont les côtés sont de la même grandeur à peu près que la distance de la terre au soleil, la somme des angles ne peut jamais différer de deux angles droits d'une quantité qui puisse surpasser* $0'', 0003$ *en secondes sexagésimales. Or cette différence doit être d'autant moindre que les côtés d'un triangle sont plus petits*" [1837, pp. 302–303].

In addition to the results of the geodetical observation that Gauss had reported at the end of his 1827 paper, this could have given Riemann further suggestions for the remarks on the "*empirical certainty*" of geometry which can be read in the concluding lines of the introduction to [Riemann 1854b]. According to Riemann, Euclidean geometry was not a "*necessity*" but merely an "*empirical certainty*" and the facts on which it was based were only "*hypotheses*", no matter how high their probability within the limits of observation [Riemann 1854b, p. 273]. As he wrote in a manuscript, "*the word hypothesis now has a slightly different meaning than in Newton. Today by hypothesis we tend to mean everything which is mentally added to phenomena*" [Riemann 1876a, p. 525].

As Gauss had done for the surfaces in his 1827 paper, Riemann too attributed a crucial importance to the definition of the linear element ds as starting point for the study of manifolds. With this in mind, we can appreciate better the suggestions he could have drawn from the following passage one can read in Lobachevsky's paper:

"*La géométrie imaginaire est conçue sur un plan plus général que la géométrie usitée qui n'en est qu'un cas particulier, et qui en dérive dans la supposition des lignes extrêmement petites; de sorte que cette dernière géométrie n'est sous ce rapport qu'une géométrie différentielle.*

Les valeurs des éléments différentiels des lignes courbes, des surfaces, et du volume des corps sont les mêmes dans la géométrie imaginaire et dans la géométrie usitée" [1837, p. 302].

Whatever the case may have been, apparently Riemann had no real interest in the problem of the foundations of geometry as such, in research concerning the axioms of geometry (and in particular that of parallels). "*However interesting it may be to consider the possibility of this approach* *to geometry, actually realising it would prove utterly sterile since in this way we could never find new theorems*", he wrote in a note kept in his *Nachlass.*[34]

In our opinion, Riemann's *Habilitationsvortrag* can be interpreted as a moment of both *naturphilosophische* and mathematical construction. In an attempt to overcome "*the shortcomings of the concepts*" and to favour progress "*in recognizing the connection of things*" [Riemann 1854b, p. 286], he supplied a generalization to n-dimensional manifolds of what he had elaborated concerning 3-dimensional space and the laws of propagation of the phenomena as well.

Riemann defined a metric in the whole space by associating to the n-dimensional manifold V^n the fundamental form

$$\Phi = \sum_{i,j=1}^{n} g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j \quad (g_{ij} \text{ positive definite matrix}).$$

In this way, it was always possible to compare distances between different points and to study deformations of an elastic fluid filling the whole universe. Riemann tried to calculate how much V^n, with the fundamental form Φ, differed from the Euclidean n-dimensional space E^n. More generally, he proposed to find necessary and sufficient conditions such that the forms Φ and $\Phi' = \sum_{i,j=1}^{n} g'_{ij}\, \mathrm{d}x'^i\, \mathrm{d}x'^j$ could transform into each other. This was equivalent to determine n new coordinates such that the differential system

$$\sum_{i,j=1}^{n} g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j = \sum_{i,j=1}^{n} g'_{ij}\, \mathrm{d}x'^i\, \mathrm{d}x'^j \tag{4.1}$$

was satisfied. System (4.1) was the generalization to a n-dimensional manifold of the expression

$$\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 = \mathrm{d}x_1'^2 + \mathrm{d}x_2'^2 + \mathrm{d}x_3'^2,$$

which had been introduced in his 1853 paper on *Naturphilosophie* (see § 2 above). If the solution of (4.1) existed, it involved $\frac{1}{2}n(n+1)$ arbitrary constants. As n equations were satisfied by the new coordinates so $\frac{1}{2}n(n-1)$

[34] "*Wenn es aber auch interessant ist, die Möglichkeit dieser Behandlungsweise der Geometrie einzusehen so würde doch die Ausführung derselben äusserst unfruchtbar sein, denn wir würden dadurch keine neuen Sätze finden können*" (in [Scholz 1982, p. 229]).

"*functions of position*" had to be deduced "*by the nature of the manifold to be represented*" [Riemann 1854b/1979, p. 143]. In order to characterize these functions, Riemann introduced a system of normal coordinates, which we denote $x^1, \dots, x^n$, and affirmed that if $\mathrm{d}s_0^2 = \sum_{i=1}^{n} (\mathrm{d}x^i)^2$ then there existed numbers $c_{ij,k\ell}$ such that Φ became

$$\Phi = \mathrm{d}s_0^2 + \sum_{ij,k\ell} c_{ij,k\ell}(x^k\,\mathrm{d}x^i - x^i\,\mathrm{d}x^k)(x^\ell\,\mathrm{d}x^j - x^j\,\mathrm{d}x^\ell).$$

If $\Phi = \mathrm{d}s_0^2$ the manifold was called plane.

In 1861, Riemann once more considered this problem, when trying to answer a question proposed by the Paris Academy on heat conduction in homogeneous solid bodies. According to Weber, Riemann did not win this prize since he did not completely explain the ways according to which he found his results (see [Riemann 1861, p. 391]).

Riemann sought to answer the following general question: "*quales esse debeant proprietates corporis motum caloris determinantes et distributio caloris, ut detur systema linearum quae semper isothermae maneant*". In the second part of the paper he studied the particular case of a homogeneous body. Riemann stated that the problem proposed by the Academy was equivalent to finding necessary and sufficient conditions according to which

$$\sum_{i,j=1}^{n} g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \sum_{i,j=1}^{n} a_{ij}\,\mathrm{d}x'^i\,\mathrm{d}x'^j$$

where a_{ij} are given constants. Since both the forms were positive definite, the problem was to determine a new coordinate system in which Φ could be expressed as the Euclidean form $\sum_{i=1}^{n}(\mathrm{d}x'^i)^2$. To this end, Riemann introduced the quantity $(ij, k\ell)$ which corresponded — but for the inessential factor 2 — to the curvature tensor

$$R_{ij\ell k} = \frac{1}{2}\Big[\frac{\partial^2 g_{ik}}{\partial x^j \partial x^\ell} + \frac{\partial^2 g_{j\ell}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{i\ell}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^\ell}\Big] + \sum_{\alpha,\beta=1}^{n} g^{\alpha\beta}[j\ell;\alpha][ik;\beta] - [i\ell;\alpha][jk;\beta] \tag{4.2}$$

where

$$[ij;k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

is the Christoffel symbol of first kind. He found that the necessary condition such that the system

$$\sum_{i,j=1}^{n} g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \sum_{i=1}^{n}(\mathrm{d}x'^i)^2 \tag{4.3}$$

is satisfied, is given by $R_{ij\ell k} = 0$.

If we suppose, as Riemann had done in his 1853 paper, that an ethereal medium fills all the universe, then we can associate to space the fundamental forms

$$\Phi = \sum_{i,j=1}^{n} \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \quad \text{and} \quad \Phi' = \sum_{i,j=1}^{n} g'_{ij}\,\mathrm{d}x'^i\,\mathrm{d}x'^j$$

at time t and t' respectively.

Now, let us consider an elastic deformation changing $P(x_1, x_2, \ldots, x_n)$ and $Q(x_1 + \mathrm{d}x_1, x_2 + \mathrm{d}x_2, \ldots, x_n + \mathrm{d}x_n)$ in $P'(x'_1, x'_2, \ldots, x'_n)$ and $Q'(x'_1 + \mathrm{d}x'_1, x'_2 + \mathrm{d}x'_2, \ldots, x'_n + \mathrm{d}x'_n)$, and the distance $\mathrm{d}s$ between P and Q in the distance $\mathrm{d}s'$ between P' and Q'. On these assumptions

$$\mathrm{d}s^2 = \sum_{i,j=1}^{n} \delta_{ij}(P)\,\mathrm{d}x^i\,\mathrm{d}x^j = \sum_{i=1}^{n}(\mathrm{d}x^i)^2,$$

$$\mathrm{d}s'^2 = \sum_{i,j=1}^{n} g'_{ij}(P')\,\mathrm{d}x'^i\,\mathrm{d}x'^j.$$

Let $a_i^\ell = \partial x'^\ell/\partial x^i$, then

$$\mathrm{d}s'^2 = \sum_{i,j=1}^{n} g'_{ij}(P') a_i^\ell a_j^k\,\mathrm{d}x^i\,\mathrm{d}x^j = \sum_{i,j=1}^{n} h_{ij}(P')\,\mathrm{d}x^i\,\mathrm{d}x^j.$$

Since the system of coordinates is the same for $\mathrm{d}s^2$ and $\mathrm{d}s'^2$, we can evaluate the virtual displacement

$$\delta(\mathrm{d}s^2) = \mathrm{d}s'^2 - \mathrm{d}s^2 = \sum_{i,j=1}^{n} e_{ij}(P')\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{4.4}$$

where $e_{ij} = h_{ij} - \delta_{ij}$. The variation (4.4) is the generalization to any curvilinear n-coordinate system of the expression (2.2) given by Riemann [1853, p. 530] in cartesian, orthogonal 3-coordinates.

By extending the same procedure to any particle of ether one obtains the quantities e_{ij} (the *strain tensor* in elasticity theory) defined on the whole space. Moreover, according to Riemann's assumption that an ether particle propagates physical phenomena reacting to the deformation described by e_{ij}, one can suppose that the transmission of forces modifies the fundamental form Φ. From this point of view (4.4) can be interpreted as a deep connection between physical phenomena and the variation of the fundamental form. No variation occurs if $e_{ij} = 0$: in this case no physical force is propagated through space. The results obtained by Riemann in his 1861 paper show that, if a solution of the system

$$\sum_{i,j=1}^{n} \delta_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j = \sum_{i,j=1}^{n} g'_{ij}\, \mathrm{d}x'^i\, \mathrm{d}x'^j \tag{4.5}$$

exists, then the curvature tensor $R'_{ij\ell k}$ is equal to zero. On the contrary, if the strain tensor is different from zero, then the fundamental form has been modified and the curvature tensor does not vanish. Thus force and curvature appear to be deeply connected, space being the responsible for the transmission of phenomena by means of a variation of its fundamental form. In other words, a force is always coupled to a change in the curvature of space. This suggests a physical model of the space independent of assumptions about the existence of the ether.

K. Pearson, the editor of Clifford's book [1885], emphasized this point asking the question "*whether physicists might not find it simpler to assume that space is capable of a varying curvature, and of a resistance to that variation, than to suppose the existence of a subtle medium pervading an invariable homaloidal* [*Euclidean*] *space*" (in [Clifford 1885, p. 203]). According to Clifford, "*this variation of the curvature of space is what really happens in that phenomenon which we call the* motion of matter*, whether ponderable or ethereal*" and "*in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity*" [Clifford 1876, p. 22].

Having this in mind and thinking of the research program on *Naturphilosophie* which Riemann was pursuing at that time, we can try to understand the cryptic and prophetic conclusion he drew at the end of the lecture:

"*Now it seems that the empirical notions on which the metric determinations of space are based, the concept of a solid body and that of a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry*" [Riemann 1854b/1979, p. 152].

Thus, "*the question of the validity of the hypotheses of geometry in the infinitely small*" was linked to the determination of "*metric relations of space*". This was a question which, in Riemann's view, might "*still be ranked as a part of the study of space*".

In this connection Riemann contrasted discrete manifolds with continuous manifolds: "*Either the reality underlying space*" is given by a discrete manifold which has the principle of its metrical relations in itself or "*the basis for the metric relations must be sought outside it* [*space*]*, in binding forces acting upon it*". What has one to understand by this "*reality*"? Did this involve Riemann's speculations on ether which, as he wrote in the manuscript *Gravitation und Licht* [*Werke*, p. 532 *sq.*], could be conceived as "*a substance spread continuously through the entire infinite space*", "*as a physical space whose points move in geometrical space*"? Riemann left open the questions he raised concerning the nature of space. According to his concluding remarks, an answer "*can be found only by starting from that conception of phenomena which has hitherto been approved by experience, for which Newton laid the foundation, and gradually modifying it under the compulsion of facts which cannot be explained by it*" [Riemann 1854b/1979, pp. 152–153]. "*This leads us away into the domain of another science, the realm of physics*", he concluded.

According to Weyl [1919a/1922, p. 97], the "*full purport*" of Riemann's concluding remarks "*was not grasped by his contemporaries*" with the exception of "*a solitary echo*" in Clifford's writings. In *Raum Zeit Materie* Weyl interpreted Riemann's words in terms of Einstein's theory of relativity: "*Only now that Einstein has removed the scales from our eyes by the magic light of his theory of gravitation do we see what these words actually mean*". In this respect we cannot do better than referring to his book (see in particular [Weyl 1919a/1922, pp. 96–102]). From a historical point of view, however, Riemann's statements seem to be better explained in the light of his own attempts "*to unify nature on the basis of a geometrically conceived system of continuous dynamic processes in ether*"; "*the first attempt at a mathematically founded unified field theory, much in the spirit of Einstein's later attempts*", as N. Wise [1981, p. 289] defined Riemann's research on *Naturphilosophie*.

Acknowledgments

During the past years, sections of this paper have been presented at various meetings. In particular, we would like to thank the colleagues attending the Workshop on the history of mathematics held at the Mittag-Leffler Institute (Djursholm, 1991), the Annual Göttingen Workshop (1992) and the Oberwolfach meeting (1994) for their useful comments. We are also grateful to the anonymous referees for their helpful remarks and to Jeremy Gray for checking the English. We also wish to thank the Niedersächsischen Staat-und Universitätsbibliothek Göttingen, Handschriftenabteilung, for the kind permission of publishing excerpts from the letters kept in Klein's *Nachlass*.

BIBLIOGRAPHY

Betti (E.)

[*Opere*] *Opere matematiche* (Accademia dei Lincei, ed.), 2 vols., Milano: Hoepli, 1903–1913.

[1868] Sopra la elettrodinamica, *Opere* 2, pp. 242–245.

Bianchi (L.)

[*Opere*] *Opere* (Unione Matematica Italiana, ed.), 11 vols., Roma: Cremonese, 1952–1959.

Bottazzini (U.)

[1985] Funzioni complesse e varietà multidimensionali: globale e locale nella matematica di B. Riemann, *Scienza e filosofia*: *Saggi in onore di L. Geymonat* (C. Mangione, ed.), Milano: Garzanti, 1985, pp. 554–573.

[1986] *The higher calculus*: *a history of real and complex analysis from Euler to Weierstrass*, New York: Springer, 1986.

[1994] Geometry and "Metaphysics of space" in Gauss and Riemann, *Romanticism in science* (S. Poggi and M. Bossi, eds.), Dordrecht: Reidel, 1994, pp. 15–29.

Bottazzini (U.) and Tazzioli (R.)

[1991] Etere, spazio, geometria: riflessioni sui frammenti filosofici di Riemann, *Conoscenza e matematica* (L. Magnani, ed.), Milano: Marcos y Marcos, 1991, pp. 383–409.

Brill (A.) and Noether (M.)

[1894] Die Entwicklung der Theorie der algebraischen Funktionen in älterer und neuerer Zeit, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 3 (1894), pp. 107-565.

Brillouin (L.)

[1938] *Les Tenseurs en mécanique et en élasticité*, Paris: Masson, 1938.

Clausius (R.)

[1868] Von Gauss angeregte neue Auffassung der elektrodynamischen Erscheinungen, *Annalen der Physik und Chemie*, 135 (1868), p. 606.

Clifford (W.K.)

[*Papers*] *Mathematical papers* (R. Tucker, ed.), London: Macmillan, 1882.

[1876] On the space-theory of matter, *Proceedings of the Cambridge philosophical Society*, 2 (1876), pp. 157–58 (read Feb. 21, 1870); *Papers*, pp. 21–22.

[1885] *The common sense of the exact sciences* (K. Pearson, ed.), London: Macmillan, 1885.

Dedekind (R.)

[1876] Bernhard Riemann's Lebenslauf, in [Riemann *Werke*, pp. 541–558].

Dirichlet (P.G. Lejeune-)

[*Werke*] *Werke* (L. Kronecker and L. Fuchs, eds.), 2 vols., Berlin: Mayer & Müller, 1889–1897.

[1837] Sur les séries dont le terme général dépend de deux angles, et qui servent à exprimer des fonctions arbitraires entre des limites données, *Journal für die reine und angewandte Mathematik*, 17 (1837), pp. 35–56; *Werke* 1, pp. 273–306.

[1876] *Vorlesungen über die im umgekehrten Verhältniss des Quadrats der Entfernung wirkenden Kräfte* (F. Grube, ed.), Leipzig: Teubner, 1876.

Faraday (M.)

[1851] On the possible relation of gravity to electricity, *Experimental researches in electricity*, t. III, London, 1885, pp. 161–168.

[1857] On the conservation of force, *Experimental researches in chemistry and physics*, London, 1859, pp. 443–463.

Farwell (R.) and Knee (Ch.)

[1990] The missing link: Riemann's 'Commentatio', differential geometry and tensor analysis, *Historia mathematica*, 17 (1990), pp. 223–255.

Gauss (C.F.)

[*Werke*] *Werke* (Königlische Gesellschaft der Wissenschaften zu Göttingen, ed.), 12 vols., Leipzig-Berlin, 1863–1933.

[1813] Theorem über die Anziehung, *Werke* 11, p. 71.

[1827] Disquisitiones generales circa superficies curvas, *Commentationes societatis regiae scientiarum Gottingensis recentiores*, 4 (1828); *Werke* 4, pp. 217–258. French transl. 1855, repr. Blanchard, 1967. English transl. 1902, repr. in *Astérisque* 62 (1979).

[1835] Grundgesetz für alle Wechselwirkungen galvanischer Ströme, *Werke* 5, pp. 616–620.

[1839] Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse wirkenden Anziehungs-und Abstossungs Kräfte, *Resultate aus der Beobachtungen des magnetischen Vereins im Jahre 1839* (K.F. Gauss and W. Weber, eds.), Leipzig, 1840; *Werke* 5, pp. 194–242.

[1845] Brief an Weber, *Werke* 5, pp. 627–629.

Klein (F.)

[*Ges. math. Abh.*] *Gesammelte mathematische Abhandlungen* (Fricke and others, eds.), 3 vols., Berlin: Springer, 1921–1923.

[1882] *Über Riemanns Theorie der algebraischen Funktionen und ihrer Integrale*, Leipzig: Teubner, 1882; *Ges. math. Abh.* 3, pp. 499–573. Engl. transl. 1893; repr., Dover, 1963.

[1894] Riemann und seine Bedeutung für die Entwicklung der modernen Mathematik, *Jber. d. Dt. Math.-Verein.*, 4 (1894–1895), pp. 71-87; *Ges. math. Abh.* 3, pp. 482–497.

[1926] *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert* (R. Courant and O. Neugebauer eds.), t. I, Berlin: Springer, 1926. Engl. transl., Brooklin: Math. Sci. Press, 1979.

[*Nachlass*] Niedersächsischen Staat-und Universitätsbibliothek Göttingen, Handschriftenabteilung, Prinzenstr. 1, D-3400 Göttingen.

KOHLRAUSCH (R.)

[1853] Das Sinuselektrometer, *Ann. Phys. Chem.*, 88 (1853), p. 497.

[1854] Theorie des elektrischen Rückstandes in der Leidener Flasche, *Ibid.*, 91 (1854), pp. 56–82.

LAMÉ (G.)

[1859] *Leçons sur les coordonnées curvilignes et leurs applications*, Paris: Mallet-Bachelier, 1859.

LANDAU (L.D.) and LIFSHITZ (E.M.)

[1951] *The classical theory of fields*, Oxford: Pergamon Press, 1951, repr. 1979.

[1959] *Théorie de l'élasticité*, Oxford: Pergamon Press, 1959.

LAPLACE (P.S.)

[*Œuvres*] *Œuvres complètes*, 14 vols., Paris: Gauthier-Villars, 1878–1912.

LOBACHEVSKY (N.)

[1837] Géométrie imaginaire, *J. reine angew. Math.*, 17 (1837), pp. 295–320.

MAXWELL (J.C.)

[1856] On Faraday's lines of force, *Transactions of the Cambridge philosophical Society*, 10 (1856), p. 27.

[1873] *A treatise on electricity and magnetism*, 2 vols., Oxford: Clarendon Press, 1873; 3rd ed. 1891; repr., New York: Dover, 1954.

NEWTON (I.)

[1726] *Philosophiae Naturalis Principia Mathematica*, 3rd ed., London 1726; Engl. transl. by A. Motte revised by F. Cajori, Berkeley and Los Angeles: University of California Press, 1934; 5th ed., 1962.

NOETHER (M.)

[1882] Zu F. Klein's Schrift 'Ueber Riemann's Theorie der algebraischen Functionen', *Zeitschrift für die Mathematik und Physik* 27 (1882), pp. 201–206.

PORTNOY (E.)

[1982] Riemann's contribution to differential geometry, *Hist. math.*, 9 (1982), pp. 1–18.

REIFF (R.) and SOMMERFELD (A.)

[1902] Elektrizität und Optik, *Encyklopädie der mathematischen Wissenschaften, Physik*, V, 2, Leipzig: Teubner, pp. 3–62.

RIEMANN (B.)

[*Werke*] *Gesammelte mathematische Werke und wissenschaftlicher Nachlass* (H. Weber, ed.), Leipzig: Teubner, 1876; 2nd ed. 1892; repr. New York: Springer, 1991. French edition, Paris 1898, repr. Blanchard, 1968. (All references refer to the 1991 edition.)

[1851] Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen complexen Grösse, *Inauguraldissertation*, Göttingen, 1851; *Werke*, pp. 3–48.

[1853] Neue mathematische Principien der Naturphilosophie, *Werke*, pp. 528–532.

[1854a] Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe, *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, 13 (1868); *Werke*, pp. 227–271.

[1854b] Über die Hypothesen, welche der Geometrie zu Grunde liegen, *Ibid.*, 13 (1868); *Werke*, pp. 272–287. English transl. in [Spivak 1970/1979, pp. 135–153].

[1854c] Über die Gesetze der Vertheilung von Spannungselectricität in ponderablen Körpern, wenn diese nicht als vollkommene Leiter oder Nichtleiter, sondern als dem Enthalten von Spannungselectricität mit endlicher Kraft widerstrebend betrachtet werden, *Amtlicher Bericht über die 31. Versammlung Deutscher Naturforscher und Aerzte zu Göttingen im September 1854* (1854); *Werke*, pp. 49–54.

[1855] Zur Theorie der Nobili'schen Farbenringe, *Ann. Phys. Chem.*, 95 (1855); *Werke*, pp. 55–66.

[1857a] Beiträge zur Theorie der durch die Gauss'sche Reihe $F(\alpha, \beta, \gamma, x)$ darstellbaren Functionen, *Abh. K. Ges. Wiss. Göttingen*, 7 (1857); *Werke*, pp. 67–83.

[1857b] Theorie der Abelschen Funktionen, *J. reine angew. Math.*, 54 (1857); *Werke*, pp. 88–144.

[1858] Ein Beitrag zur Elektrodynamic, *Ann. Phys. Chem.*, 131 (1858); *Werke*, pp. 288-293.

[1861] Commentatio mathematica, qua respondere tentatur quaestioni ab Ill$^{\text{ma}}$ Academia Parisiensi propositæ: ' Trouver quel doit être l'état calorifique d'un corps solide homogène indéfini pour qu'un système de courbes isothermes, à un instant donné, restent isothermes après un temps quelconque, de telle sorte que la température d'un point puisse s'exprimer en fonction du temps et de deux autres variables indépendantes', *Werke*, pp. 391–404.

[1869] *Partielle Differentialgleichungen und deren Anwendung auf physikalische Fragen* (K. Hattendorff, ed.), Braunschweig: F. Vieweg, 1869.

[1876a] Fragmente philosophischen Inhalts, *Werke*, pp. 507–538.

[1876b] Über das Potential eines Ringes, *Werke*, pp. 431–436.

[1876c] Gleichgewicht der Electricität auf Cylindern mit kreisförmigem Querschnitt und parallelen Axen, *Werke*, pp. 440–444.

[1876d] Neue Theorie des Rückstandes in electrischen Bindungsapparaten, *Werke*, pp. 367–378.

[1876e] *Schwere, Elektricität, Magnetismus. Nach den Vorlesungen von B. Riemann* (K. Hattendorff, ed.), Hannover: C. Rümpler, 1876.

[1902] *Nachträge* (M. Noether and W. Wirtinger, eds.), 1902; *Werke*, viii + 116 p. (separate numeration following p. 558).

ROSENFELD (L.)

[*Papers*] *Selected papers of Léon Rosenfeld* (R.S. Cohen and J.J. Stachel, eds.), Boston Studies in the Philosophy of Science 21, Dordrecht/Boston: Reidel Publishing Company, 1979.

[1956] The velocity of light and the evolution of electrodynamics, *Suppl. Nuovo Cimento* 4 (1956), pp. 1630–1669; *Papers*, pp. 134–177.

SCHAFFNER (K.F.)

[1972] *Nineteenth-century aether theories*, Oxford: Pergamon Press, 1972.

SCHERING (E.)

[*Werke*] *Gesammelte mathematische Werke* (R. Haussner and K. Schering, eds.), 2 vols., Berlin: Mayer and Müller, 1902–1909.

[1866] Zum Gedächtniss an B. Riemann, *Werke* 2, pp. 367–383; repr. in [Riemann, *Werke* 1991, pp. 828–847].

Scholz (E.)
[1982] Riemanns frühe Notizen zur Mannigfaltigkeitsbegriff und zu den Grundlagen der Geometrie, *Archive for history of exact sciences*, 27 (1982), pp. 213–232.
Speiser (A.)
[1927] Naturphilosophische Untersuchungen von Euler und Riemann, *J. reine angew. Math.*, 157 (1927), pp. 105–114.
Spivak (M.)
[1970] *A comprehensive introduction to differential geometry*, vol. II, 1st ed. 1970; 2nd ed., Houston: Publish or Perish, 1979.
Weber (W.)
[1846] Elektrodynamische Maassbestimmungen, *Abhandlungen bei Begründung der Königlischen Sächsichen Gesellschaft der Wissenschaften am Tage des 200 jährigen Geburtstages Leibnizens* (1846), pp. 209–378.
Weyl (H.)
[1913] *Die Idee der Riemannschen Fläche*, Leipzig: Teubner, 1913; repr. 1964.
[1919a] *Raum Zeit Materie*, Berlin: Springer, 1919. Engl. transl. 1922; repr. Dover, 1952. French transl. 1922, repr. Blanchard 1958.
[1919b] Bernhard Riemann, *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (H. Weyl ed.), Berlin, 1919. Weyl's *Vorwort* and *Erläuterungen* are reprinted in Riemann [*Werke*, 1991, pp. 740–768].
[1923] *Mathematische Analyse des Raumproblems*, Berlin: Springer, 1923.
Whittaker (E.)
[1951] *A history of the theories of aether and electricity*, vol. I, New York: Harper, 1951; repr. Dover 1991.
Wise (N.M.)
[1981] German concepts of force, energy, and the electromagnetic ether: 1845–1880, *Conception of ether. Studies in the history of ether theory*, 1740–1900 (G.N. Cantor and M.J. Hodge, eds.), Cambridge: Cambridge University Press, 1981, pp. 269–307.