OLIVIER MATHIEU

On the Dimension of Some Modular Irreducible Representations of the Symmetric Group

Les rencontres physiciens-mathématiciens de Strasbourg - RCP25, 1995, tome 47
« Conférences de M. Audin, D. Bernard, A. Bilal, B. Enriquez, E. Frenkel, F. Golse, M. Katz, R. Lawrence, O. Mathieu, P. Von Moerbeke, V. Ovsienko, N. Reshetikhin, S. Theisen », , exp. nº 7, p. 183-191

<http://www.numdam.org/item?id=RCP25_1995__47__183_0>

On the dimension of some modular irreducible representations of the symmetric group.

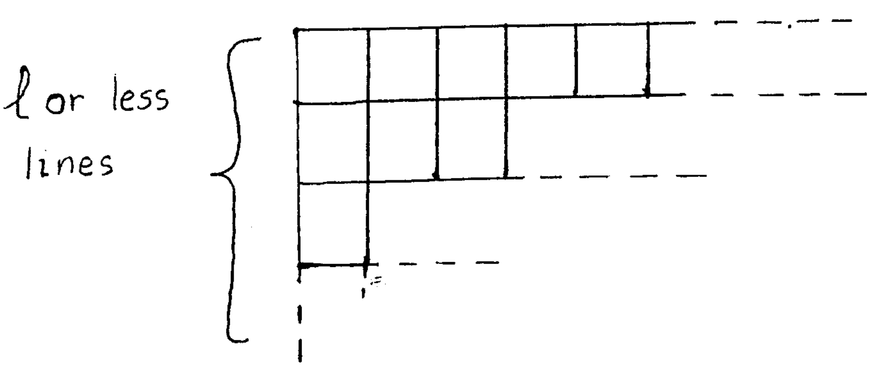
OLIVIER MATHIEU*

Abstract: *We compute the dimension of some irreducible representations of the symmetric groups in characteristic p (Theorem 2). The representations considered here are associated with Young diagrams* $\mathbf{m} : m_1 \geq m_2 \geq ... \geq m_l$ *such that* $m_1 - m_l \leq (p-l)$. *The formula is based on a variant of Verlinde's formula which computes some tensor product multiplicities of indecomposable modules for* $GL_l(\overline{\mathbf{F}}_p)$, *as it is proved in [7] [8].*

Mathematics Subject Classification (1991): 20 C 30

Introduction: In this paper we will compute the dimension of some modular irreducible representations of the symmetric group Σ_N, (see Theorem 2 below for a precise statement). By a classical formula of Frobenius, the dimension of a characteristic zero irreducible Σ_N-representation is given as the number of standard tableaux of a given shape. However in the modular case, it is not very convenient to use the standard tableaux to describe these dimensions. Instead, we will use a combinatorial description based on paths in the set of Young diagrams. For this reason, we will first "translate" the classical Frobenius formula in terms of paths.

Recall that a Young diagram of height $\leq l$ is a sequence of non-negative integers $\mathbf{m} : m_1 \geq m_2 \geq ... \geq m_l$. Pictorially one represents a Young diagram as follows,

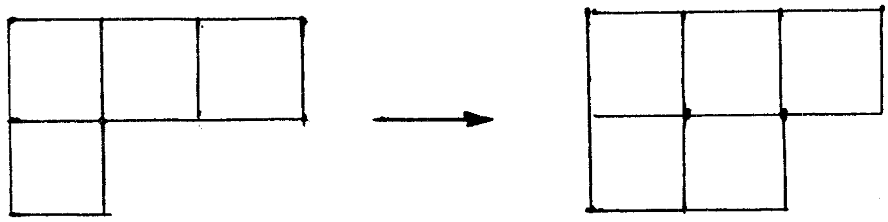


namely a set of boxes with m_1 boxes on the first line, m_2 boxes on the second line and so on.... The total number $m_1 + m_2 + ...$ of boxes will be called the size of the Young diagram $\mathbf{m}$. In order to give a completely rigorous definition, we also require that two Young diagrams which can be obtained one from the other one by adding or removing empty lines are considered as identical. For example the Young diagrams $3 \geq 1$ and $3 \geq 1 \geq 0$ are viewed as the same.

Let Y_l be the set of all Young diagrams of height $\leq l$. We consider Y_l as an oriented

* Institut de Recherches Mathèmatiques Avancées, Université Louis Pasteur, 7 rue René Descartes, F-67084 Strasbourg Cedex

graph. Actually there is an oriented edge going from $\mathbf{m}$ to $\mathbf{m}'$ if and only if we have $m'_i = m_i$ for all indices i except for one, say j, for which we have $m'_j = m_j + 1$. Pictorially, this means that we can get $\mathbf{m}'$ from $\mathbf{m}$ by adding exactly one box to $\mathbf{m}$, e.g.



Denote by $\emptyset$ the Young diagram with no boxes. To each Young diagram $\mathbf{m}$ of size N, Frobenius associated an irreducible $\mathbf{C}$-representation $E_{\mathbf{C}}(\mathbf{m})$ of Σ_N and he proved the following result.

THEOREM 1 (Frobenius formula in terms of paths). *The dimension of the complex representation $E_{\mathbf{C}}(\mathbf{m})$ is the number of oriented paths from $\emptyset$ to $\mathbf{m}$.*

Actually Frobenius Theorem was stated in terms of tableaux of shape $\mathbf{m}$. Recall that a standard tableau of shape $\mathbf{m}$ is a one-to-one labeling of the N boxes of $\mathbf{m}$ by the integers $1, 2, \ldots, N$ which is increasing along the lines and the columns. Actually it is easy to define a bijection between standard tableaux of shape $\mathbf{m}$ and paths from $\emptyset$ to $\mathbf{m}$. Given a standard tableau of shape $\mathbf{m}$, one can associate a path $\emptyset = \tau_0, \tau_1, \ldots, \tau_N = \mathbf{m}$ going from $\emptyset$ to $\mathbf{m}$ with the requirement that τ_k is the Young tableau of all boxes with label $\leq k$. Conversely one obtains a standard tableau from a path $\emptyset = \tau_0, \tau_1, \ldots, \tau_N = \mathbf{m}$ by labeling with k the unique box of $\tau_k \setminus \tau_{k-1}$.

Now fix a prime number p and two positive integers l and N. Set $k = \overline{\mathbf{F}}_p$. By using the Schur Weyl duality one can associate to any Young diagram $\mathbf{m}$ of size N a k-representation $E_k(\mathbf{m})$ of Σ_N. These representations $E_k(\mathbf{m})$ are irreducible or $\{0\}$, and the non-zero representations $E_k(\mathbf{m})$ form a complete set of irreducible representations of Σ_N (see Section 3 for more details).

Let $Y_l(p)$ the set of all Young diagrams $\mathbf{m} = m_1, \ldots, m_l$ of height $\leq l$ such that $m_1 - m_l \leq p - l$. We will prove:

THEOREM 2. (Assume $l < p$) *Let $\mathbf{m} \in Y_l(p)$ be a Young diagram. Then the dimension of the k-representation $E_k(\mathbf{m})$ is the number of oriented paths from $\emptyset$ to $\mathbf{m}$ entirely contained in $Y_l(p)$. In particular $E_k(\mathbf{m}) \neq 0$.*

For general irreducible representations of the symmetric group, it is still possible to describe the dimension in terms of paths. In section 5, we introduce a natural structure of oriented graph on the set of all Young diagrams. As the graph structure depends on p we will denote by $Z(p)$ this graph. By contrast with the characteristic zero case, or the case of the graph $Y_l(p)$, the graph $Z(p)$ contains multiple edges.

THEOREM 3. *Let $\mathbf{m}$ be a Young diagram of size N. Then the dimension of the Σ_N-module $E_k(\mathbf{m})$ is the number of oriented paths going from $\emptyset$ to $\mathbf{m}$ in $Z(p)$.*

However we do not known how to compute the multiplicities of edges in $Z(p)$. Thus Theorem 3 does not give an explicit formula, (as in Theorem 2) for the dimension of general simple representations of the symmetric group. However it explains why we believe that the combinatoric in terms of path is more adapted than the classical combinatoric of standard tableaux.

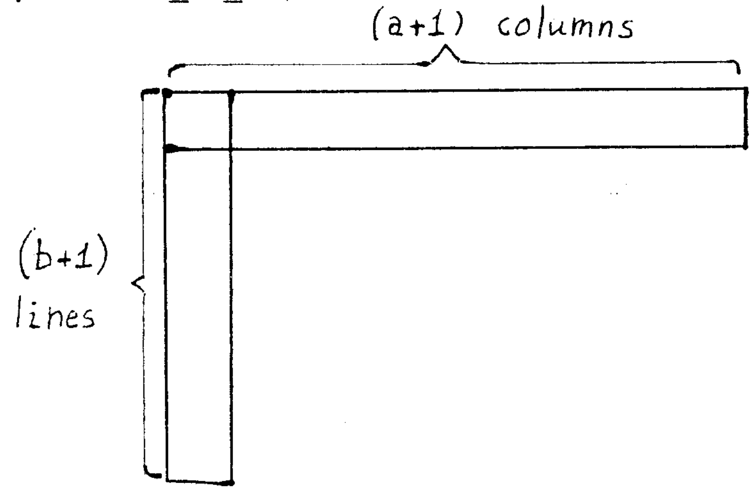
Remarks. 1. The formula and its proof are based on the Schur Weyl duality, Ringel's notion of tilting modules [12] and relies heavily on the work [8] (announced in [7]). In the work [8], it is proved that some tensor product multiplicities of tilting modules are given by Verlinde's formula [13]. However this formula and this work will not appear explicitly (although Lemma 12 is equivalent to the main statement of [8] for groups of type A).

2. K. Erdmann already used the tilting modules for the study of modular representations of Σ_N [4]. She recovered the classical dimension formula for all representations attached to a two-lines Young diagram (in [4], the author refers to Donkin's paper [6] for the basic idea).

3. A. S. Kleshchev proved independentely a lower bound for the dimension of representations in Theorem 1. His proof is based on a very different idea: he used his result about the Σ_{n-1}-socle of Σ_n-irreducible modules.

4. In his study [15] of representations of Hecke algebras at p-root of unity, H. Wenzl considered Hecke modules parametrized by Young diagrams $\mathbf{m} = m_1, ..., m_l$ satisfying exactly the same condition $m_1 - m_l \leq p - l$. Some authors, including R. Rouquier, told us that our formula and proof can be extended to Hecke algebras as well.

EXAMPLE 4. Denote by $Y_{a,b}$ be the Young diagram $\mathbf{m}$ such that $m_1 = a + 1$ and $m_i = 1$ for $2 \leq i \leq b + 1$.



In characteristic 0 (or characteristic $> a + b + 1$) the corresponding representation of Σ_{a+b+1} has dimension $(a + b)!/a!b!$. Now assume that $p = a + b + 1$ is a prime number. Let $\emptyset = \tau_0, \tau_1, ..., \tau_p = \mathbf{m}$ be a path in the set of all Young diagrams. We obtain τ_{p-1} by removing from $\mathbf{m}$ either the last box of the first line or the last box of the first column. In the second case we have $\tau_{p-1} \notin Y_l(p)$. Otherwise the full path $\emptyset = \tau_0, \tau_1, ..., \tau_p = \mathbf{m}$ belongs to $Y_l(p)$ and we have $\tau_{p-1} = Y_{a,b-1}$. Thus the dimension of $E_k(Y_{a,b})$ is the number of path from $\emptyset$ to $Y_{a,b-1}$. Thus we get dim $E_k(Y_{a,b}) = (a + b - 1)!/a!(b - 1)!$.

Acknowledgment I thank S. Donkin, M. Duflo, G. Georgiev and R. Rouquier for helpful conversations. This work has been supported by an NSF Grant and a Sloan Grant (at Rutgers University) and by IRMA (at CNRS at Strasbourg).

1. Root system of $GL_l(k)$. *From now on, set $k = \overline{\mathbf{F}}_p$.* In this section we will recall a few definitions and facts about the representation theory of reductive groups, for the particular case of the full linear group $GL_l(k)$.

Let H be the subgroup of all diagonal matrices of $GL_l(k)$. Let P be the group of all characters of H. We have $P = \mathbf{Z}\epsilon_1 \oplus \mathbf{Z}\epsilon_2 \oplus ... \oplus \mathbf{Z}\epsilon_l$ where ϵ_i is the character defined by $\epsilon_i(diag(\lambda_1, ..., \lambda_l)) = \lambda_i$, where $(diag(\lambda_1, ..., \lambda_l))$ is the diagonal matrix whith diagonal entries $\lambda_1, ..., \lambda_l$. Set $P^* = Hom(P, \mathbf{Z})$. For any i with $1 \leq i < l$, set $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $h_i = \epsilon_i^* - \epsilon_{i+1}^*$, where $(\epsilon_i^*)_{1 \leq i \leq l}$ is the dual basis of P^*. Also set $\alpha_0 = \epsilon_1 - \epsilon_l$, $h_0 = \epsilon_1^* - \epsilon_l^*$. Let W be the subgroup of $GL(P)$ generated by the reflection $s_i = 1 - h_i \otimes \alpha_i$, $1 \leq i \leq l$. Recall that W is naturally isomorphic to the symmetric group Σ_l acting by the permutation representation on $\mathbf{Z}^l$.

Define the affine reflection s_0 of P by $s_0(\lambda) = \lambda - (\lambda(h_0) - p)\alpha_0$. The affine Weyl group W_{aff} is by definition the group of affine transforms of P generated by W and s_0. Set

$P^+ = \{\lambda \in P | \lambda(h_i) \geq 0 \text{ for any } 1 \leq i \leq l\}$

$C = \{\lambda \in P^+ | \lambda(h_0) \leq p - l + 1\}$

$C^0 = \{\lambda \in P^+ | \lambda(h_0) \leq p - l\}$.

The following equivalent definitions of C and C^0 are more usual in the theory of reductive groups (see e.g. [9]). Choose any $\rho \in P$ such that $\rho(h_i) = 1$ for any i, $1 \leq i \leq l$. We have $\rho(h_0) = l - 1$. Thus an element $\lambda \in P^+$ belongs to C (respectively to C^0) iff we have $\lambda + \rho(h_0) \leq p$ (respectively $\lambda + \rho(h_0) < p$).

For any $\lambda \in P^+$ we will denote by $W(\lambda)$ the Weyl module with highest weight λ (see e.g. [5] or [9] for a definition). By definition a filtration of a rational $GL_l(k)$-module is called a Weyl filtration if its subquotients are Weyl modules $W(\lambda)$ for various $\lambda \in P^+$. For any rational module M we define its character as $ch(M) = \sum_{\mu \in P} (dim M_\mu) e^\mu \in \mathbf{Z}[P]$, where M_μ denotes the weight space corresponding to the weight μ.

The following result, which holds for any Chevalley group is usually called the Strong Linkage Principle. As it is stated below (namely for type A groups), it is due to Carter and Lusztig [3]. The general case is due to Andersen [1] (a convenient reference is [9]).

THEOREM 5 (Strong Linkage Principle). *If $W(\lambda)$ and $W(\mu)$ are in the same block, then we have $\lambda + \rho = w(\mu + \rho)$, for some $w \in W_{aff}$.*

The following two facts are well known consequences of the Strong Linkage Principle:

(i) for any $\lambda \in C$, the Weyl module $W(\lambda)$ is simple and its dual is again a Weyl module.

(ii) for any λ, $\mu \in C$ with $\lambda \neq \mu$, the Weyl modules $W(\lambda)$ and $W(\mu)$ are not in the same block.

By definition the fundamental weights of $GL_l(k)$ are the weights of the form $\omega_j = \epsilon_1 + ... + \epsilon_j$. If V denotes the natural l-dimensional representation of $GL_l(k)$, then $W(\omega_j) \simeq \wedge^j V$. Any weights ν of $W(\omega_j)$ is W-conjugated to ω_j and we have $|\nu(h_i)| \leq 1$ for any i, $0 \leq i \leq l$.

2. Tilting modules for $GL_l(k)$.

Set $G = GL_l(k)$. Recall that a finite dimensional rational G-module M is tilting if M and M^* have a Weyl filtration.

THEOREM 6 (Ringel [12], Donkin [6]).

(1) For any $\lambda \in P^+$ there exists a unique indecomposable tilting module $P(\lambda)$ which has λ as a unique highest weight. Moreover $P(\lambda)_\lambda$ has dimension 1.

(2) Any tilting modules is a direct sum of $P(\lambda)$ and for $\lambda \neq \mu$, $P(\lambda)$ and $P(\mu)$ are not isomorphic.

The following lemma follows immediately from the fact that any tensor product of modules having a Weyl filtration has a Weyl filtration (see ([6]). For a reductive group of type A, this result is proved in [14]. For general reductive groups see [5], [11].

LEMMA 7. *The tensor product of two tilting modules is tilting.*

It is easy to prove that the dual $W(\lambda)^*$ of a Weyl module $W(\lambda)$ has a Weyl filtration if and only if $W(\lambda)$ is simple. Thus any simple Weyl module is tilting. So by Lemma 7 we get.

COROLLARY 8. *For any N, the G-module $V^{\otimes N}$ is tilting, where V is the natural l-dimensional representation of G.*

The following lemma is well-known. Actually it is valid for any group G, and it is a very particular case of results in [2]. A quick proof can be found in [8].

LEMMA 9. *Let A and B be two rational G-modules. If A is indecomposable and $dim A$ is divisible by p, then any direct summand in $A \otimes B$ has dimension divisible by p.*

PROPOSITION 10. *Assume $l < p$. Let $\lambda \in P^+$.*

(1) If $\lambda \in C$ then $P(\lambda) \simeq W(\lambda)$ and $P(\lambda)$ is simple.

(2) If $\lambda \notin C^0$ then the dimension of $P(\lambda)$ is divisible by p.

Proof. Proof of (1): There is a filtration of $P(\lambda)$ whose subquotients are some $W(\mu)$. If $W(\mu)$ occurs as a subquotient then $\lambda - \mu$ is a linear combination of α_i with non-negative coefficients. Furthermore by the Strong Linkage Principle (Theorem 5), the weights $\mu + \rho$ and $\lambda + \rho$ are W_{aff}-conjugated. As $\lambda \in C$ this implies $\lambda = \mu$. Moreover $W(\lambda)$ occurs only once and is simple.

Proof of (2): We will prove (2) by induction on $(\lambda + \rho)(h_0)$, starting with the case $\lambda + \rho(h_0) = p$. First if $\lambda + \rho(h_0) = p$, then $\lambda \in C$ and by the first point of the proposition, we have $P(\lambda) = W(\lambda)$. Its dimension is given by Weyl's formula, namely $dim(W(\lambda)) = \prod_{\alpha \in \Delta^+} (\lambda + \rho)(h_\alpha)/\rho(h_\alpha)$ (where $\Delta^+ = \{\epsilon_i - \epsilon_j | i < j\}$ and for any $\alpha = \epsilon_i - \epsilon_j \in \Delta^+$ we set $h_\alpha = \epsilon_i^* - \epsilon_j^*$). Note that for any $\alpha \in \Delta^+$ we have $\rho(h_\alpha) < p$. Thus the denominator is prime to p. However the denominator is divisible by $p = (\lambda + \rho)(h_0)$. Thus $dim\, P(\lambda)$ is divisible by p.

Next let $\lambda \in P^+$ with $\lambda + \rho(h_0) > p$. There is a fundamental weight ω such that $\lambda - \omega \in P^+$. By Ringel's theorem, $P(\lambda) \otimes P(\omega)$ contains $P(\lambda)$ as a direct summand. Note that $(\lambda - \omega)(h_0) = \lambda(h_0) - 1$. Thus by induction hypothesis $P(\lambda - \omega)$ has dimension divisible by p and so is $P(\lambda)$ (Lemma 9). Q.E.D.

Let ω be a fundamental weight and set $\Omega(\omega) = \{W.\omega\}$. Recall that $\Omega(\omega)$ is the set of weights of $W(\omega)$, and all of them have multiplicity one.

LEMMA 11. *For any $\lambda \in P^+$, we have $ch(W(\lambda) \otimes W(\omega)) = \sum ch(W(\lambda + \nu))$ where the sum runs over all $\nu \in \Omega(\omega)$ such that $\lambda + \nu \in P^+$.*

Proof. Denote by $D : \mathbf{Z}[P] \to \mathbf{Z}[P]$ the linear operator defined by $D\, e^{\mu} = \chi_{\mu+\rho}/\chi_{\rho}$ where $\chi_{\mu} = \sum_{w \in W} \epsilon(w) e^{w \cdot \mu}$ and where $\epsilon(w)$ is the signature. Recall that we have

(i) $D\, e^{\lambda} = ch(W(\lambda)$ for any $\lambda \in P^{+}$,

(ii) $D(A.B) = (DA).B$ if B is W-invariant.

(iii) $D\, e^{\lambda} = 0$ if $\lambda(h_i) = -1$ for some $i \in \{1, ..., l\}$.

As ω is fundamental, we have $\nu(h_i) \geq -1$ for any i and any $\nu \in \Omega(\omega)$. Also either $\lambda + \nu$ is dominant or $(\lambda + \nu)(h_i) = -1$ for some $i \in \{1, ..., l\}$. Thus we get

$ch(W(\lambda) \otimes W(\omega))$
$= D(e^{\lambda}).ch(W(\omega)$
$= D(e^{\lambda}.ch(W(\omega))$
$= \sum_{\nu \in \Omega(\omega)} D(e^{\lambda+\nu})$
$= \sum_{\nu \in \Omega(\omega), \lambda+\nu \in P^{+}} D(e^{\lambda+\nu})$
$= \sum_{\nu \in \Omega(\omega), \lambda+\nu \in P^{+}} ch(W(\lambda+\nu))$.

LEMMA 12. *Assume $l < p$. Let ω be a fundamental weight and let $\lambda \in P^{+}$.*

(1) If $\lambda \in C^0$ then we have $W(\omega) \otimes W(\lambda) \simeq \oplus W(\lambda+\nu)$ where the sum runs over all $\nu \in \Omega(\omega)$ such that $\lambda + \nu \in P^{+}$.

(2) If $\lambda \notin C^0$ then $W(\omega) \otimes P(\lambda)$ is a sum of tilting modules $P(\nu)$ where all ν are outside C^0.

Proof. Proof of (1): By Lemma 11, we have $ch(W(\lambda) \otimes W(\omega)) = \sum ch(W(\lambda+\nu))$ where the sum runs over all $\nu \in \Omega(\omega)$ such that $\lambda + \nu \in P^{+}$. For any such ν, we have $\nu(h_0) \leq 1$ and $\lambda + \nu \in C$. Note that the tilting modules $P(\lambda+\nu) = W(\lambda+\nu)$ are simple and belongs to disjoint blocks. Thus the character identity corresponds to an isomorphism of G-modules.

Proof of (2): If $\lambda \in C^0$, then by Lemmas 7 and 9 and Proposition 10 all indecomposable summands of $W(\omega) \otimes P(\lambda)$ are tilting modules $P(\nu)$ with $\nu \notin C^0$. Q.E.D.

3. Modular representations of Σ_N.

Let A be an associative algebra, let M be a A-module of finite dimension and let B be the commutant of A in M. Let decompose the A-module M into indecomposable modules

$$(3.1)\ M = \sum_{\Lambda} m_{\Lambda} P(\Lambda),$$

where Λ runs over the set J of all isomorphism classes of indecomposable direct summands of M. Then we have $B/rad(B) \simeq \oplus_{\Lambda} Mat(m_{\Lambda})$. This allows to gives a natural bijection between the set J and the set of irreducible B-modules. Denote by $E \mapsto E(\Lambda)$ this bijection. Note that $dim E(\Lambda) = m_{\Lambda}$.

Let l, N be integers. Set $V = k^l$ and $M = V^{\otimes N}$. Let A be the subalgebra of $End(M)$ generated by the action of $GL_L(k)$ on M. Recall that the commutant B of A is generated by the action of Σ_N on M (see [3]).

Denote by Pol_N the set of all weights $\lambda = \sum_{1 \leq i \leq l} m_i \epsilon_i$ such that $m_i \geq 0$ for all i and $\sum_{1 \leq i \leq l} m_i = N$. Set $Pol_N^{+} = P^{+} \cap Pol_N$. Any weight of M belongs to Pol_N. So by Theorem 6 and Corollary 8, any indecomposable summand of the $GL_l(k)$-module M is of type $P(\lambda)$ with $\lambda \in Pol_N^{+}$ and there is an isomorphism

$$(3.2)\ V^{\otimes N} \simeq \oplus_{\lambda \in Pol_N^{+}} m_{\lambda} P(\lambda).$$

For any Young diagram $\mathbf{m} : m_1 \geq m_2 \geq ... \geq m_l$ of size N, set $\lambda(\mathbf{m}) = \sum m_i \epsilon_i$. Let $Y_{l,N}$ be the set of all Young diagrams $\mathbf{m}$ of height $\leq l$ of size N. The map $Y_{l,N} \to Pol_N^+$, $\mathbf{m} \mapsto \lambda(\mathbf{m})$ is a bijection. Thus the previous decomposition can be written as

$$(3.3)\ M = \oplus_{\mathbf{m} \in Y_{l,N}} m_{\lambda(\mathbf{m})} P(\lambda(\mathbf{m})).$$

By using the previous bijection between A-indecomposable summands of M and B-irreducible modules we can associate to any $\mathbf{m} \in Y_{l,N}$, such that $P(\lambda(\mathbf{m}))$ occurs effectively in M, a simple representation $E_k(\mathbf{m})$ of Σ_N. Moreover for $\mathbf{m} \in Y_{l,N}$ such that $m_{\lambda(\mathbf{m})} = 0$, we set $E_k(\mathbf{m}) = 0$. We have $dim\, E_k(\mathbf{m}) = m_{\lambda(\mathbf{m})}$.

It is easy to prove that $E_k(\mathbf{m})$ does not depend on l. More precisely by adding or removing empty lines, one can consider $\mathbf{m}$ as a Young diagram of height $\leq l$ for various values of l. However the Σ_N-modules $E_k(\mathbf{m})$ that one obtains as previously, by using the $GL_l - \Sigma_N$ duality for various l, are all isomorphic.

4. Proof of Theorem 2.

Let l be an integer. Set $Pol^+ = \cup_{N \geq 0} Pol_N^+$. The decomposition (3.2) allows us to define a multiplicity m_λ for any $\lambda \in Pol^+$.

LEMMA 13. *Assume $l < p$. Let $N \geq 1$ and $\lambda \in C^0 \cap Pol_N^+$. Then we have $m_\lambda = \sum m_{\lambda - \epsilon_i}$, where the sum runs over all i such that $\lambda - \epsilon_i \in C^0 \cap Pol_{N-1}^+$.*

Proof. The lemma follows by from Lemma 12.

Proof of the Theorem 2 stated in the introduction.

Let $\mathbf{m} \in Y_l(p)$. The assertions

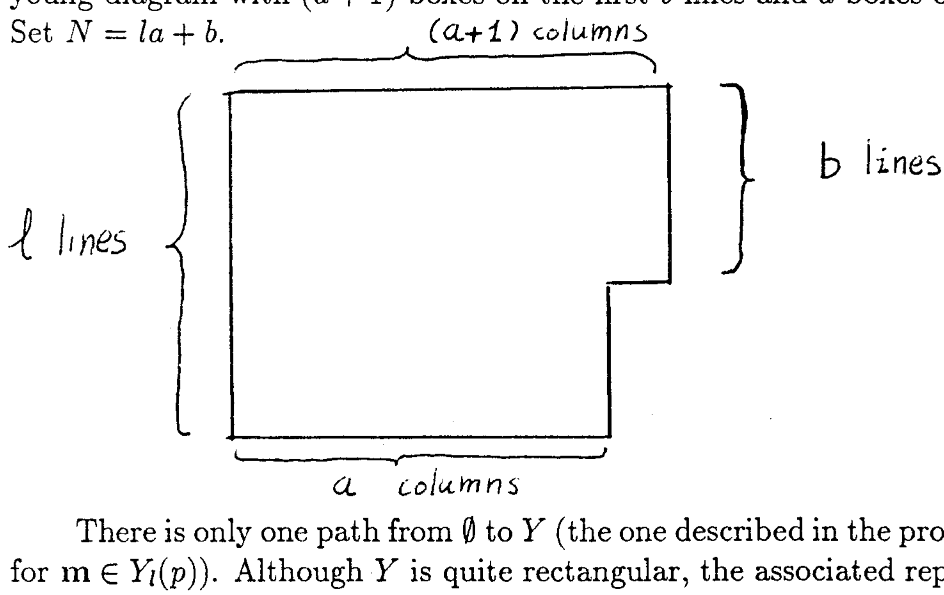
(i) $\lambda(\mathbf{m}) \in C^0$

(ii) $m_1 - m_l + l - 1 < p$

are equivalent. Thus the dimension formula follows easily by induction from Lemma 11. To show that this dimension is $\neq 0$ it suffices to exhibit a path going from $\emptyset$ to $\mathbf{m}$ inside $Y_l(p)$. This is done by filling the first column, then the second one and so on.

EXAMPLE 14.

Assume now that $p = l + 1$ and let a, b be integers with $1 \leq b < p$. Let Y be the young diagram with $(a + 1)$ boxes on the first b lines and a boxes on the last $(l - b)$ lines. Set $N = la + b$.



There is only one path from $\emptyset$ to Y (the one described in the proof that $dim\, E_k(\mathbf{m}) \neq 0$ for $\mathbf{m} \in Y_l(p)$). Although Y is quite rectangular, the associated representation $E_k(Y)$ has dimension 1. It is quite easy to prove that this representation is the signature representation of Σ_N.

5. Conclusion: the oriented graph structure on Y_l.

Let l be an integer. Set $G = GL_l(k)$ and $V = k^l$. For any $\lambda, \nu \in Pol^+$ define the mutiplicity $M_{\lambda,\nu}$ by the requirement $V \otimes P(\nu) = \oplus_\lambda M_{\lambda,\nu} P(\lambda)$. Now we define an oriented graph structure on Y_l by requiring that the number of edges going from $\mathbf{m}$ to $\mathbf{m}'$ is precisely $M_{\lambda(\mathbf{m}),\lambda(\mathbf{m}')}$.

We should notice that the multiplicities of the edges in Y_l depends on p. However it is easy to prove that these multiplicities do not depend on l. That is, for $\mathbf{m}$, $\mathbf{m}' \in Y_l$ the number of edges going from $\mathbf{m}$ to $\mathbf{m}'$ in Y_l and Y_{l+1} are the same.

Thus the set of all Young diagrams with the previous structure of oriented graph will be denoted by $Z(p)$ (note that the analogous graph in caracteristic zero is without multiplicities and it is described in the introduction).

Proof of Theorem 3. The result follows by induction on the size of $\mathbf{m}$ and from the following identities:

(i) $dim E_k(\mathbf{m}) = m_{\lambda(\mathbf{m})}$ (see section 3),

(ii) $m_\lambda = \sum_\nu M_{\lambda,\nu}\, m_\nu$.

Very unfortunately, the question of computing all the tensor product multiplicities of tilting modules is still open (see [8]). Theorem 3 means that explicit formulas for the dimensions of general irreducible representations of the symmetric groups follow from a precise knowledge of mutiplicities $M_{\lambda,\nu}$.

Bibliography

1. H.H.Andersen: *The strong linkage principle.* J.Reine Angew Math. 315 (1980) 53-55.

2. D. Benson, *Modular representations theory: new trends and methods*, L.N. Math. 1081, Springer Verlag.

3. R. Carter and G. Lusztig, *On the general linear and symmetric groups*, Math. Z., 136 (1974) 193-242.

4. K. Erdmann, *Symmetric groups and quasi-hereditary algebras*, Representations of algebras and related topics, ed. V. Dlab and L.L. Scott, Kluwer, Dordrecht 1994.

5. S.Donkin: *Rational representations of algebraic groups.* Springer Verlag, Lect.Notes in Math.1140 (1985).

6. S. Donkin, *On tilting modules for algebraic groups*, Math. Z. 212 (1993) 39-60.

7. G. Georgiev and O. Mathieu: *Categorie de fusion pour les groupes de Chevalley*, Comptes Rendus Acad. Sc. Paris, 315 (1992) 659-662.

8. G. Georgiev and O. Mathieu: *Fusion rings for modular representations of Chevalley groups*, Proceedings of "Math. Aspect of CTFT and Quantum Groups", Mount Holliock, June 1992 (ed. Flato, Lepowsky, Reshetikhin, Sally, Zuckerman). Compt. Math. 175 (1994) 89-100.

9. J.Jantzen: *Representations of algebraic groups*, Academic Press, Orlando(1987).

10. A.S. Kleshchev: *Branching rules for modular representations of symmetric groups III: some corollaries and a problem of Mullineux.*

11. O. Mathieu: *Filtrations of G-modules*, Ann. Ecole Norm. Sup. 23 (1990) 625-644.

12. C.M. Ringel: *The category of good modules over a quasi- hereditary algebra has an almost split sequence*, Preprint.

13. E. Verlinde: *Fusion rules and modular transformations in 2D conformal field theory*, Nucl. Phys. B 300 (1988) 360-375.

14. Wang Jian-Pian: *Sheaf cohomology on G/B and tensor products of Weyl modules.* J.of Algebra 77 (1982) 162-185.

15. H. Wenzl, *Hecke algebras of type A_n and subfactors*, Inv. Math. 92 (1988) 349-383.