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P. Di Francesco
C. Itzykson

Quantum Intersection Rings

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P. Di Francesco,

and

C. Itzykson

Service de Physique Théorique de Saclay *,
F-91191 Gif sur Yvette Cedex, France

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1. Introduction

Within the broadly defined subject of topological field theory E. Witten suggested in [1] to study generalized "intersection numbers" on a compactified moduli space $\bar{\mathcal{M}}_{g,n}$ of Riemann surfaces. These are computed by integrating pullbacks of appropriate forms on a target Kähler manifold obtained through holomorphic maps of marked surfaces. The corresponding axioms, discussed by R. Dijkgraaf and E. and H. Verlinde [2], were investigated by M. Kontsevich and Y. Manin [3] and lead for a subclass of targets to surprising results on the enumeration of rational curves. Our purpose here is to study a few illustrative examples and to check some of them using the most primitive tools of geometry.

While modern algebraic and differential geometry has reached such a high degree of abstraction and sophistication it is refreshing to return with methods inspired from physics to problems of elementary enumerative geometry such as those dealing with curves in projective spaces, initiated by Chasles, Schubert and Zeuthen. It is not clear to the

authors when these lines are written that complete proofs are in print to support the conjectures that the numbers computed below are indeed what they are meant to be. But it is likely that such proofs, if not available yet, should appear pretty soon in the literature.

Dealing with concrete examples of projective algebraic varieties covered by rational curves, we shall describe – to the best of our knowledge – the classical cohomology ring $H^*(M)$. The latter corresponds to some of the quantum observables of the topological field theory. Their correlation functions are however affected by quantum corrections due to non trivial maps $\mathbb{P}_1 \to M$ with the necessary markings to rigidify them. This leads to a deformation of the ring structure on $H^*(M)$ explaining the name "quantum cohomology ring". This deformation, parametrized by $H^*(M)$, should satisfy conditions expressing commutativity, associativity and the existence of a unit (the "puncture operator"). The structure constants of this deformed ring (the analog of a fusion ring in conformal theory) are derived from a generating function: the (perturbed) free energy. Once these elements are put together, one derives differential equations for the free energy. These generate recursion relations for the enumerative numbers, therefore entirely specified by boundary conditions.

The original example of the projective plane $\mathbb{P}_2$ due to Kontsevich will be discussed first as it provides the simplest case and can be studied in some detail. We will comment on various aspects of the sequence $\{N_d\}$ of numbers of rational plane curves of degree d passing through $(3d - 1)$ points in general position. In particular we investigate the asymptotic behaviour of N_d for large d. Following Dubrovin [4], we also consider an associated flat connection on a trivial bundle over $H^*(M) \times \mathbb{C}$, and record for completeness in appendix A his geometric interpretation of the differential equation leading to an equivalence with a Painlevé VI equation. The generalization of a geometric argument, in support of the basic differential equation, produces a formula constraining the "characteristic numbers" of rational plane curves which seems to be new.

To illustrate the generality of the method, the subsequent sections are devoted to higher projective spaces, quadrics and cubics in $\mathbb{P}_3$, the Plücker quadric in $\mathbb{P}_5$ describing line geometry in $\mathbb{P}_3$, finally the flag variety of $\mathbb{P}_2$. In a final section we turn to open questions and relations.

As physicists we are more interested in results that in the (painful and obviously necessary) process of justification of the interpretation. This might serve as an excuse for not quoting adequately the extensive mathematical literature, especially on intersection theory. Finally a word of caution. As one of the authors was warned by the referee of a

previous version [5], it would be often more appropriate to use the word "conjecture" than "proposition" in some statements. The reader will correct for himself.

C.I. is indebted to M. Kontsevich, S. Kleiman, R. Piene and I. Vainsencher who offered generously their help – and time – to introduce a novice to the subject. Most of what follows is to be considered as an application of their ideas, but the authors share the sole responsability for the presentation. P.D.F. thanks C. Procesi for illuminating discussions.

2. Rational curves in $\mathbb{P}_2$

Let N_d be the number of rational irreducible plane curves of degree d through $3d - 1$ points in general position.

Proposition 1. (Kontsevich)

$$(i) \ N_1 = 1$$

$$(ii) \ N_d = \sum_{\substack{d_1 + d_2 = d \\ d_1, d_2 \geq 1}} N_{d_1} N_{d_2} \left[d_1^2 d_2^2 \binom{3d - 4}{3d_1 - 2} - d_1^3 d_2 \binom{3d - 4}{3d_1 - 1} \right] , \ d > 1.$$

Supplied by the initial condition (i) (i.e. there exists a single line through two distinct points in the plane, or dually two distinct lines intersect in a single point), the recursion relation (ii) yields the set of N_d's, $d > 1$, as positive integers. It gives $N_2 = 1$ (a single conic through 5 points), $N_3 = 12$ (uninodal cubics through 8 points), $N_4 = 620$ (trinodal quartics through 11 points), a result due to Zeuthen and recorded on page 186 of Schubert's famous treatise [6]. The next number $N_5 = 87304$ (6–nodal quintics through 14 points) and has been recently confirmed by Vainsencher [7].

2.1. Preliminaries

An irreducible algebraic plane curve of degree d is described by the vanishing of an irreducible homogeneous polynomial of degree d in 3 variables $f(x_0, x_1, x_2) = 0$. The set of such curves is embedded in a projective space $\mathbb{P}_D$ describing homogeneous polynomials, of dimension

$$D = \frac{d(d + 3)}{2}. \tag{2.1}$$

Smooth curves form an open dense set in $\mathbb{P}_D$, the complement of a "discriminant locus" of singular curves for which either the polynomial is not irreducible and/or the tangent (line)

is not defined at certain points, i.e. the three partial derivatives $\frac{\partial f}{\partial x_i}(x_0, x_1, x_2) = 0$ admit simultaneous non trivial solutions. Assuming that the only singularities of an irreducible curve are δ ordinary double points (i.e. with distinct tangents) its class c, the degree of the dual curve, is

$$c = d(d-1) - 2\delta \tag{2.2}$$

from which the Riemann–Hurwitz theorem yields the genus g of the curve

$$g = \frac{(d-1)(d-2)}{2} - \delta \tag{2.3}$$

Since each double point reduces the freedom of curves by one unit we expect the set of irreducible curves with δ double points to depend on

$$\frac{d(d+3)}{2} - \delta = 3d - 1 + g \tag{2.4}$$

parameters, and therefore by imposing $3d - 1 + g$ independent conditions on such curves to find a finite number of those. We define $N_d^{(g)}$, and for short $N_d^{(0)} \equiv N_d$, to be the number of irreducible curves of degree d, genus g, with only simple nodes, through $3d - 1 + g$ points "in general position", since requiring that the curve go through a point implies a single linear condition. The statement "in general position" should be made precise in each concrete situation: it implies that the conditions as applied to irreducible curves are independent. Note that curves with simple nodes form the smallest codimension set among curves of a fixed degree having a given genus $g \leq (d-1)(d-2)/2$. In this sense they are the generic ones.

When $\delta = (d-1)(d-2)/2$, $g = 0$, we require that curves go through $3d - 1$ points. Alternatively such a parametrized rational curve is prescribed by 3 homogeneous polynomials of degree d in 2 variables, hence depends on $3(d+1)$ parameters. Quotienting by linear transformations on the homogeneous coordinates of $\mathbb{P}_1$ leaves again $3d - 1$ parameters. The values $N_1 = N_2 = 1$ are classical. To go slightly beyond, let $n_{d,\delta} \equiv N_d^{(g)}$, $g + \delta = (d-1)(d-2)/2$, denote the number of irreducible degree d plane curves with δ nodes through $d(d+3)/2 - \delta$ points in general position. For δ up to 6 the following general results hold.

Proposition 2.(Kleiman and Piene [8], Vainsencher [7])

(i) $n_{d,1} = 3(d-1)^2,\ d \geq 3$

(ii) $n_{d,2} = \dfrac{3}{2}(d-1)(d-2)(3d^2 - 3d - 11),\ d \geq 4$

(iii) $n_{d,3} = \dfrac{9}{2}d^6 - 27d^5 + \dfrac{9}{2}d^4 + \dfrac{423}{2}d^3 - 229d^2 - \dfrac{829}{2}d + 525 - \delta_{d,4}\dbinom{11}{2},\ d \geq 4$

(iv) $n_{d,4} = \dfrac{27}{8}d^8 - 27d^7 + \dfrac{1809}{4}d^5 - 642d^4 - 2529d^3 +$

$\qquad + \dfrac{37881}{8}d^2 + \dfrac{18057}{4}d - 8865 - \delta_{d,5}\dbinom{16}{2}, d \geq 5$

(v) $n_{d,5} = \dfrac{81}{40}d^{10} - \dfrac{81}{4}d^9 - \dfrac{27}{8}d^8 + \dfrac{2349}{4}d^7 - 1044d^6 - \dfrac{127071}{20}d^5 + \dfrac{128859}{8}d^4 +$

$\qquad + \dfrac{59097}{2}d^3 - \dfrac{3528381}{40}d^2 - \dfrac{946929}{20}d + 153513 - \delta_{d,5}27\dbinom{15}{2}, d \geq 5$

(vi) $n_{d,6} = \dfrac{81}{80}d^{12} - \dfrac{243}{20}d^{11} - \dfrac{81}{20}d^{10} + \dfrac{8667}{16}d^9 - \dfrac{9297}{8}d^8 - \dfrac{47727}{5}d^7 +$

$\qquad + \dfrac{2458629}{80}d^6 + \dfrac{3243249}{40}d^5 - \dfrac{6577679}{20}d^4 - \dfrac{25387481}{80}d^3 +$

$\qquad + \dfrac{6352577}{4}d^2 + \dfrac{8290623}{20}d - 2699706 - \delta_{d,5}\left[\dbinom{14}{5} + 225\dbinom{14}{2}\right], d \geq 5$

For $d = 3$, (i) yields $N_3 = 12$, from (iii) for $d = 4$ one recovers $N_4 = 675 - 55 = 620$, while for $d = 6$, (vi) yields $N_5 = 87304$.

Remarks.

a) The polynomial parts of these formulas include reducible curves. For instance, for $d = 3$, formula (ii) yields the $\binom{7}{2} = 21$ reducible cubics formed by a conic through 5 points and a line through the remaining two points. Similarly the subtractive term of (iii) for $d = 4$ represents the $\binom{11}{2} = 55$ reducible quartics formed by a cubic through 9 points and a line through the remaining two, with analogous interpretations for the remaining relations.

b) Including some reducible cases for small d these results would suggest that $n_{d,\delta}$, $1 \leq \delta \leq (d-1)(d-2)/2$, is a polynomial in $\mathbb{Q}[d]$ of degree 2δ with leading term of the form $(3d^2)^\delta/\delta! \simeq n_{d,1}^\delta/\delta!$, corresponding to the fact that one has essentially to pick δ points on a Jacobian variety (see below) of degree $3(d-1)^2$. We conjecture the following general

structure for the first few coefficients of the polynomial part of $n_{d,\delta}$, denoted $z_{d,\delta}$

$$z_{d,\delta} = \frac{3^\delta}{\delta!}\left[d^{2\delta} - 2\delta d^{2\delta-1} + \frac{\delta(4-\delta)}{3}d^{2\delta-2} + \frac{\delta(\delta-1)(20\delta-13)}{6}d^{2\delta-3} + \right.$$

$$- \frac{\delta(\delta-1)(69\delta^2 - 85\delta + 92)}{54}d^{2\delta-4} - \frac{\delta(\delta-1)(\delta-2)(702\delta^2 - 629\delta - 286)}{270}d^{2\delta-5} +$$

$$\left. + \frac{\delta(\delta-1)(\delta-2)(6028\delta^3 - 15476\delta^2 + 11701\delta + 4425)}{3240}d^{2\delta-6} + \cdots \right],$$

(2.5)

in agreement with the data of proposition 2.

An elementary derivation of (i) and (ii) (see for instance the book by Semple and Roth [9]) will illustrate how rapidly these problems become intricate.

Degree d curves through $d(d+3)/2 - 1$ points form a linear pencil

$$\lambda_0 f_0 + \lambda_1 f_1 = 0 \tag{2.6}$$

For nodal curves we look for simultaneous solutions of the system

$$\lambda_0 \frac{\partial f_0}{\partial x_i} + \lambda_1 \frac{\partial f_1}{\partial x_i} = 0, \ 0 \le i \le 2, \tag{2.7}$$

which entails (2.6) by virtue of Euler's relation. Each solution yields a value of the ratio $\lambda_0{:}\lambda_1$ provided

$$\begin{vmatrix} \frac{\partial f_0}{\partial x_0} & \frac{\partial f_1}{\partial x_0} \\ \frac{\partial f_0}{\partial x_2} & \frac{\partial f_1}{\partial x_2} \end{vmatrix} = \begin{vmatrix} \frac{\partial f_0}{\partial x_1} & \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_0}{\partial x_2} & \frac{\partial f_1}{\partial x_2} \end{vmatrix} = 0 , \tag{2.8}$$

omitting the $(d-1)^2$ points where $\frac{\partial f_0}{\partial x_0} = \frac{\partial f_1}{\partial x_2} = 0$. From Bezout's theorem, this gives $[2(d-1)]^2 - (d-1)^2 = 3(d-1)^2$ points, and proves (i).

To derive (ii) for binodal curves of degree $d \ge 4$, consider the two dimensional system of degree d curves C_λ through

$$\nu_d = \frac{d(d+3)}{2} - 2 = \frac{(d-1)(d+4)}{2} \tag{2.9}$$

points, represented as

$$C_\lambda : \qquad \sum_{0 \le i \le 2} \lambda_i f_i(x) = 0. \tag{2.10}$$

with $f_i(x)$ three homogeneous polynomials of degree d, generically such that $f_i(x) = 0$ is irreducible and smooth. Consider a map

$$T : \mathbb{P}_2(x) \to \mathbb{P}_2(f)$$
$$x_0, x_1, x_2 \to f_0(x), f_1(x), f_2(x) \tag{2.11}$$

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sending curves C_λ to lines in $\mathbb{P}_2(f)$, or points of the dual $^*\mathbb{P}_2(f) = \mathbb{P}_2(\lambda)$, the space of curves C_λ. Different choices of bases f_0, f_1, f_2 for C_λ amount to a $PGL(3)$ automorphism of $\mathbb{P}_2(f)$. The map T is locally one to one except on the Jacobian curve $J \subset \mathbb{P}_2(x)$

$$J \ : \ \frac{D(f)}{D(x)} \equiv \det(\frac{\partial f_i}{\partial x_j}) = 0 \quad \deg(J) = 3(d-1). \tag{2.12}$$

Let $\Gamma \equiv T(J) \subset \mathbb{P}_2(f)$ and $\tilde{\Gamma}$ its dual in $\mathbb{P}_2(\lambda)$. The assigned points (2.9) are generically simple nodes of J. Indeed if $x^{(0)}$ is an assigned point, from Euler's identity

$$\frac{1}{d} \sum_{0 \leq j \leq 2} x_j^{(0)} \frac{\partial f_i}{\partial x_j^{(0)}} = f_i(x^{(0)}) = 0 \ , \tag{2.13}$$

it belongs to J. On the other hand J is the locus of nodes of curves of the family C_λ. Hence there exists a curve in C_λ with a node at $x^{(0)}$ and we can choose it to be of the form $f_0 = 0$ (this induces the harmless presence of a singular curve in the basis). Pick a coordinate system in $\mathbb{P}_2(x)$ such that $x^{(0)} = (0,0,1)$ and such that $x_0 = 0$ is tangent to $f_1 = 0$ (assumed smooth at $x^{(0)}$). Thus

$$f_0(x) = x_2^{d-2}(ax_0^2 + 2bx_0 x_1 + cx_1^2) + \cdots$$
$$f_1(x) = x_2^{d-1}(\alpha x_0) + \cdots \tag{2.14}$$
$$f_2(x) = x_2^{d-1}(\beta x_0 + \gamma x_1) + \cdots$$

Calculation yields $D(f)/D(x) = x_2^{3d-5} p_2(x_0, x_1) + \cdots$ with $p_2(x_0, x_1)$ an homogeneous second degree polynomial generically irreducible, thus confirming that $x^{(0)}$ is a simple node of J. Consequently, the genus of J reads

$$g(J) = \frac{(3d-4)(3d-5)}{2} - \frac{(d-1)(d+4)}{2} = 4d^2 - 15d + 12 \tag{2.15}$$
$$2g(J) - 2 = 2(d-1)(4d-11) \ .$$

The degree of $\Gamma = T(J)$ is the number of variable points in the intersection of a curve C_λ with J

$$m = \deg(\Gamma) = 3(d-1) \times d - 2 \times \frac{(d-1)(d+4)}{2} = 2(d-1)(d-2) \tag{2.16}$$

where in the subtraction the assigned points, as nodes of J, are counted twice. On the other hand the degree of $\tilde{\Gamma}$ is the number of singular (hence generically uninodal) curves in a linear pencil of curves C_λ computed in (i)

$$n = \deg(\tilde{\Gamma}) = 3(d-1)^2 \tag{2.17}$$

while

$$g(\tilde{\Gamma}) \equiv g(\Gamma) = g(J).$$

(2.18)

Clearly the number $n_{d,2}$ of binodal curves is the number of nodes of $\tilde{\Gamma}$ which are distinct from the ν_d ones, images of those of J. With the following notations

$\delta = n_{d,2} + \nu_d = $ number of nodes of $\tilde{\Gamma}$

$\kappa = $ number of cusps of $\tilde{\Gamma} = $ number of cuspidal curves in $\mathcal{C}_\lambda$

$\tau = $ number of bitangents of $\tilde{\Gamma} = $ number of nodes of Γ

$i = $ number of flexes of $\tilde{\Gamma} = $ number of cusps of Γ

(2.19)

$$g = g(\tilde{\Gamma}) = \frac{(n-1)(n-2)}{2} - \delta - \kappa$$

$$= g(\Gamma) = \frac{(m-1)(m-2)}{2} - \tau - i$$

the Plücker formulas [9] read

$$m = 2n + 2g - 2 - \kappa$$
$$n = 2m + 2g - 2 - i .$$

(2.20)

Since we know m, n and g, this gives the number of cuspidal curves of degree d through $(d-1)(d+4)/2$ points

$$\kappa = 2n - m + 2g - 2 = 6(d-1)^2 - 2(d-1)(d-2) + 2(d-1)(4d-11)$$
$$= 12(d-1)(d-2) ,$$

(2.21)

from which the formula for the genus g yields

$$\delta = n_{d,2} + \nu_d = \frac{(n-1)(n-2)}{2} - g - \kappa$$

(2.22)

Finally the number of binodal curves through $(d-1)(d+4)/2$ points reads

$$n_{d,2} = \frac{(3d^2 - 6d + 2)(3d^2 - 6d + 1)}{2} - (4d^2 - 15d + 12) - 12(d-1)(d-2) - \frac{(d-1)(d+4)}{2}$$

$$= \frac{3}{2}(d-1)(d-2)[3d(d-1) - 11] .$$

(2.23)

Although more rigor must be provided to justify a number of implicit assumptions, this gives at least a heuristic proof of part (ii) of Proposition 2 and gives a flavour of how intricate the proof of parts $(iii)-(vi)$ can be.

2.2. Quantum ring

A "topological σ-model coupled to two dimensional gravity" is a field theory defined on some covering of the moduli space of marked Riemann surfaces equipped with metrics, with target space a Kähler manifold– here $\mathbb{P}_2$. This means that the basic fields are maps to the target $\mathbb{P}_2$ together with the metric. A suitable compactification of the space of maps is necessary. Critical points of the action correspond to conformal metrics on the surface and maps to stable irreducible curves, satisfying certain conditions to be specified below. Finally a set of observables is selected with the property that their correlations are pure numbers, generally rational. This is where the topological nature of the model manifests itself. The concept of short distance expansion for products of fields, familiar from the field theory point of view, translates into a consistent "fusion ring". A subset of observables are in correspondence with cohomology classes of the target, and for genus 0 source, the corresponding fusion ring is a deformation – due to quantum corrections – of the classical cohomology (or intersection) ring of the target. This "jargon" can eventually be translated into well defined axioms, the role of "physical intuition" being reduced to the interpretation, which remains to be put on a more respectable mathematical footing. Following Kontsevich and Manin, the deformation is parametrized by the (dual of) the cohomology space as a complex vector space (as we will only deal here with even cohomology, this requires no $\mathbb{Z}_2$ grading of the space).

For $\mathbb{P}_2$ the cohomology ring is $\mathbb{C}[u]/u^3$ with basis $t_i = u^i$, $i = 0, 1, 2$. The multiplication table is therefore $t_i t_j = t_{i+j}$ for $i + j \leq 2$, and 0 otherwise. The only non–trivial relation $t_1^2 = t_2$ is dually equivalent to the fact that two lines intersect in a point in $\mathbb{P}_2$. A general element in H^* is therefore of the form $\sum_{0 \leq i \leq 2} y_i t_i.^1$

The free energy of the would–be topological field theory is split into contributions of genus 0, 1, ... and we call F the genus 0 contribution. The formal power series $F(y_0, y_1, y_2)$ defined up to a second degree polynomial, is a sum of a "classical" and a "quantum" part

$$F = f_{cl} + f \qquad (2.24)$$

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[1] For typographical reasons we use subscripts to index the coordinates y_i instead of the more appropriate y^i, which might induce a confusion when raised to some powers. We return to upper indices in the appendix.

where f_{cl} is a cubic polynomial encoding the multiplication rules of the classical intersection ring, namely

$$\frac{\partial f_{cl}}{\partial y_i \partial y_j \partial y_k} = \text{coeff. of the top class } t_2 \text{ in the product } t_i t_j t_k , \qquad (2.25)$$

in accordance with the fact that it requires at least 3 points to stabilize $\mathbb{P}_1$. In particular the intersection form is

$$\eta_{ij} \equiv \frac{\partial f_{cl}}{\partial y_0 \partial y_i \partial y_j} \qquad (2.26)$$

For $\mathbb{P}_2$ as a target the only non–vanishing values are $\eta_{02} = \eta_{11} = 1$, hence the inverse $\eta^{ij} = (\eta^{-1})_{ij} = \eta_{ij}$, and

$$f_{cl}(y_0, y_1, y_2) = \frac{1}{2}(y_0^2 y_2 + y_0 y_1^2) . \qquad (2.27)$$

The fact that all relations derived below only involve third derivatives of F explains why it is defined only up to a second degree polynomial. The splitting of the free energy into f_{cl} and f is according to maps for which the image of $\mathbb{P}_1$ is respectively a point or an irreducible curve. An obvious invariant of the latter is its degree, proportional to the integral over the pre–image of the Kähler class of $\mathbb{P}_2$ represented by t_1. In a path integral each "instanton configuration" (non–trivial critical point of the action or alternatively rational curve) would appear weighted by its "area" (Kähler class). To rigidify the curve one requires that it intersects $(3d - 1)$ indistinguishable points, dual to the class t_2. This serves as motivation to postulate that

$$f = \sum_{d=1}^{\infty} N_d \frac{y_2^{3d-1}}{(3d-1)!} e^{dy_1} . \qquad (2.28)$$

The factorial in the denominator accounts for the indiscernability of the points and a physicist would say that in (2.28) N_d counts the number of degree d rational curves through $(3d - 1)$ points. In particular $N_1 = 1$. Fusion implies the following:

Axiom. There exists a commutative, associative ring with a unit (rather an algebra over $\mathbb{Q}[[y_0, y_1, y_2]]$) with basis T_0 (the unit), T_1, T_2, such that

$$T_i T_j = F_{ijk}(y_0, y_1, y_2)\eta^{kl}T_l . \qquad (2.29)$$

For short, subindices on F stand for derivatives w.r.t. the corresponding y variables. Requiring T_0 to be the unit is equivalent to $F_{0ij} = \eta_{ij} = $ constant, thus agrees with

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$\partial f/\partial y_0 = 0$. Commutativity is explicit. The most important constraint is the associativity of the deformed ring. For $\mathbb{P}_2$, this reduces to a single equation

$$\boxed{f_{222} = f_{112}^2 - f_{111}f_{122}}$$

(2.30)

which from the expansion (2.27) is equivalent to the statement of proposition 1.

This is obviously *not* a proof but, from the suggestion of an underlying path integral, a strong hint – supported as we saw by checks for N_d, $d \le 5$. We give below a sketch of a direct enumerative proof, along lines suggested by Kontsevich.

There is an important homogeneity relation crucial in later generalizations. It is possible to assign weights denoted as [.] to y_0, e^{y_1}, y_2 so as to make F homogeneous up to a second degree polynomial, which does not affect third derivatives. These are

$$[y_0] = 1 \quad [e^{y_1}] = 3 \quad [y_2] = -1 \quad [F] = 1 .$$

(2.31)

This means that y_1 is a parabolic element of weight 0, i.e. under $y_0 \to \lambda y_0$, $y_1 \to y_1 + 3\ln\lambda$, $y_2 \to \lambda^{-1}y_2$, then $F \to \lambda F+$ second degree polynomial. This type of property will appear again and again in further examples.
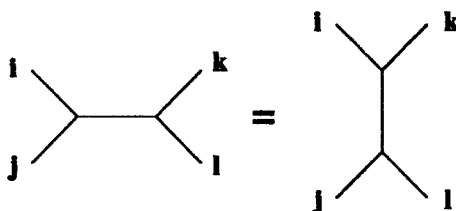
Fig. 1: Duality relations in a topological field theory. Each trivalent vertex stands for a 3 point correlation of observables in the theory. The intermediate link symbolizes the sum over intermediate states of the form $(ab) = a\eta^{ab}b$, where η^{ab} plays the role of propagator between the two states.

In topological field theories, the associativity conditions above are the "duality" relations familiar to physicists. They express the equality between the two different ways of writing a 4 point correlation by decomposing it onto a complete set of "intermediate states", leading to products of 3 point correlations F_{ijk} interpreted as correlations of the observables dual to y_i, y_j, y_k, namely the deformed classes $\langle T_iT_jT_k \rangle$ (Fig.1). Here the only non–trivial relation is obtained for $i = j = 1$, $k = l = 2$, and the intermediate states to be summed over are (02), (11) and (20) respectively, so that

$$\sum_{0 \le j \le 2} F_{11j}F_{2-j22} = \sum_{0 \le j \le 2} F_{12j}F_{2-j12} ,$$

(2.32)

equivalent to (2.30) by writing $F = f_{cl} + f$, with f_{cl} as in (2.25).

2.3. Sketch of an enumerative proof (according to Kontsevich)

An enumerative proof of proposition 1 is suggested by the very form of the recursion relation for N_d. It is obtained by studying degenerate cases i.e. boundary cycles on the moduli space. Consider an algebraic family of rational irreducible curves through $(3d-2)$ points split into a subset of $(3d-4)$ points $\{q_*\}$ and two distinguished points p_1 and p_2. Let two fixed lines l_3 and l_4 intersect in p_0. For each curve C of the family pick a point $p_i \in C \cap l_i$. Since $\{p_1, p_2, p_3, p_4\}$ all lie on $C \simeq \mathbb{P}_1$, it makes sense to consider their cross ratio

$$x = \frac{p_1 - p_3}{p_1 - p_4} \frac{p_2 - p_4}{p_2 - p_3} \tag{2.33}$$

which defines a map from the family to $\mathbb{P}_1 - \{0, 1, \infty\}$. One computes its degree in two different ways by letting x go to 0 or 1. For $x = 0$, C degenerates in all possible ways into a union of two necessarily rational curves C_1, C_2 of degrees d_1, d_2, with $d_1 + d_2 = d$. C_1 contains p_1, p_3 and $(3d_1 - 2)$ points among the q_*, which together with p_1 make up the required number $(3d_1 - 1)$ to fix N_{d_1} rational curves. Similarly there are N_{d_2} possibilities for C_2, containing p_2, p_4 and the other $(3d_2 - 2)$ q_* points. These two curves C_1 and C_2 correspond to pinching a generic curve C at one of their $d_1 d_2$ intersection points, while p_3 (resp. p_4) is among the d_1 points in $C_1 \cap l_3$ (resp. $C_2 \cap l_4$). This results in the following degree of the map

$$\sum_{\substack{d_1 + d_2 = d \\ d_1, d_2 \geq 1}} N_{d_1} N_{d_2} \binom{3d-4}{3d_1 - 2} \times d_1 \times d_2 \times d_1 d_2 \tag{2.34}$$

As $x \to 1$, there are two possibilities. Either $p_3 = p_4 = p_0 = l_3 \cap l_4$, giving a contribution N_d to the degree, or the curve degenerates with p_3, p_4 on C_1 of degree d_1, which should therefore also contain $(3d_1 - 1)$ of the q_*, and p_1, p_2 on C_2 of degree d_2, which contains the $(3d_2 - 3)$ other q_*. The pinching point is again one of the $d_1 d_2$ points of $C_1 \cap C_2$ while this time p_3 and p_4 are to be chosen among the intersections of C_1 with l_3 and l_4, d_1^2 in number. Altogether the degree is

$$N_d + \sum_{\substack{d_1 + d_2 = d \\ d_1, d_2 \geq 1}} N_{d_1} N_{d_2} \binom{3d-4}{3d_1 - 1} d_1 \times d_2 \times d_1^2 \tag{2.35}$$

Equating (2.34) and (2.35), we find the recursion relation of proposition 1.

The very spirit of this reasoning is not so different from the one which led to the consideration of the quantum cohomology ring in the first place, and the degenerations of C leading to the expressions (2.34) and (2.35) are reminiscent of the two diagrams of Fig.1, which correspond to two ways of degenerating a 4 point correlation function.

2.4. Asymptotics

The solutions of the recursion formula for N_d exhibits a factorial growth shown up to degree 12 in table I.

d	N_d	$(3d - 1)$
1	1	2
2	1	5
3	12	8
4	620	11
5	87304	14
6	26312976	17
7	14616808192	20
8	13525751027392	23
9	19385778269260800	26
10	40739017561997799680	29
11	120278021410937387514880	32
12	4821136806180292923686860080	35

Table I: The numbers N_d of irreducible, degree d, rational, plane curves through $(3d - 1)$ points, up to $d = 12$.

Proposition 3. There exists two real positive numbers a and b such that

$$\frac{N_d}{(3d - 1)!} = a^d d^{-7/2} b \left(1 + O(d^{-1})\right) \tag{2.36}$$

Numerically the asymptotic behavior is in agreement with the exponent $7/2$ and yields

$$a = 0.138 \qquad b = 6.1 \tag{2.37}$$

The proposition implies that, as a function of the variable $y_2^3 e^{y_1}$, $y_2 f(y_1, y_2)$ admits a convergent power series in a disk of radius a^{-1}, with a singularity on the real axis (a square

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root branch point). One might think of $\ln a$ as an analog of entropy of rational curves of large degree.

The proof is divided into two parts. The first uses the "explicit" solution of the recursion relation for N_d to prove the convergence of the power series for f. The second is based on an analysis of the differential equation to extract the exponent $7/2$.

Let us rewrite the recursion relation in proposition 1 by symmetrizing the sum on the r.h.s. as

$$\frac{N_d}{(3d-1)!} = \sum_{\substack{d_1+d_2=d \\ d_1,d_2 \geq 1}} \frac{N_{d_1}}{(3d_1-1)!} \frac{N_{d_2}}{(3d_2-1)!} q(d_1,d_2)$$

$$q(d_1,d_2) = \frac{d_1 d_2 (3 d_1 d_2 (d+2) - 2d^2)}{2(3d-1)(3d-2)(3d-3)} \quad (2.38)$$

$$= \frac{d_1 d_2 [(3d_1-2)(3d_2-2)(d+2) + 8(d-1)]}{6(3d-1)(3d-2)(3d-3)}$$

where $d = d_1 + d_2$. The second form exhibits the positivity of $q(d_1,d_2)$. In general, consider a recursion relation of the type

$$u_1 > 0$$

$$u_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2 \geq 1}} u_{d_1} u_{d_2} X(d_1,d_2) \quad d \geq 2 \quad (2.39)$$

where $X(d_1,d_2)$ need not be symmetric.

Lemma 1.

(i) For $d > 1$, u_d is $(u_1)^d$ times a sum of contributions, each one assigned to a rooted trivalent tree graph with oriented and labeled edges. At each vertex there is one incoming edge labeled d_a, and there are two outgoing edges labeled d_b, d_c, with $d_a = d_b + d_c$. Vertices, $(d-1)$ in number, are assigned a weight $X(d_b, d_c)$. The contribution of a tree is the product of its vertex weights.

(ii) The number of trees contributing to u_d is the Catalan number $(2d-2)!/[d!(d-1)!]$.

(iii) Let $u_d^{(1)}$ and $u_d^{(2)}$ be two solutions of a recursion relation of the type (2.39) with kernels $X^{(1)}$ and $X^{(2)}$, such that $u_1^{(1)} \geq u_1^{(2)}$, and $X^{(1)}(d_1,d_2) \geq X^{(2)}(d_1,d_2)$, then for all $d \geq 1$, $u_d^{(1)} \geq u_d^{(2)}$.

Part (i) is obtained by iterating the recursion relation until one gets rid of all u_d, $d > 1$. As for (ii), it is sufficient to replace $X(d_1,d_2)$ by a constant X and to consider the generating function

$$\Phi(t) = \sum_{d=1}^{\infty} u_d t^d \quad (2.40)$$

satisfying the quadratic relation

$$\Phi(t) - tu_1 = X\Phi(t)^2 \tag{2.41}$$

hence

$$\Phi(t) = \frac{1 - \sqrt{1 - 4Xtu_1}}{2X} = \sum_{d=1}^{\infty} \frac{(2d-2)!}{d!(d-1)!} u_1^d X^{d-1} t^d \tag{2.42}$$

proving (ii). Property (iii) follows immediately from (i).
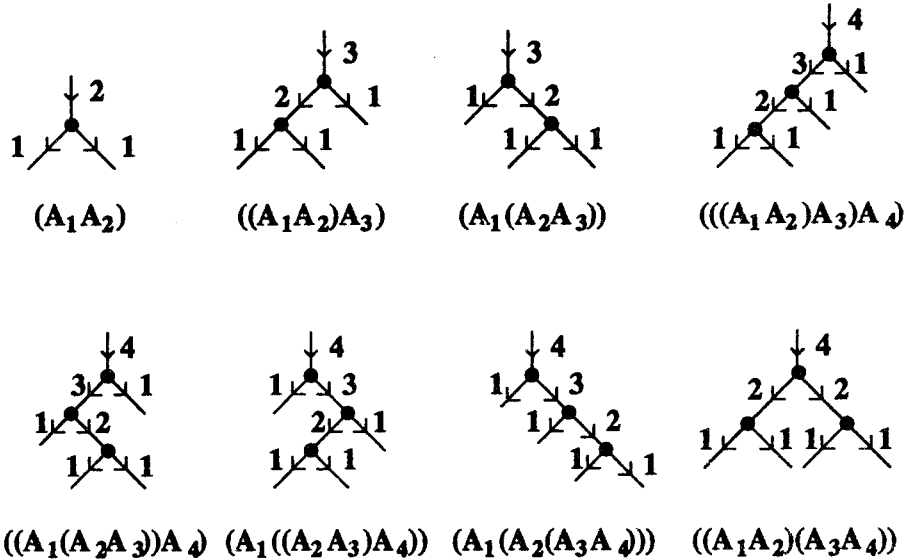
Fig. 2: Rooted trees, the flow of degrees and the corresponding "parenthesing" for $d = 2$, 3 and 4.

The above trees are in one to one correspondence with "parenthesings". Given a sequence of symbols A_1, A_2, ...,A_d, we insert pairs of open/closed parentheses in all possible ways and compute the corresponding weight accordingly, as exemplified in Fig.2, for $d = 2$, 3, and 4. Thus, with $Q(d_1, d_2) = Q(d_2, d_1) \equiv q(d_1, d_2)(3d - 1)!/[(3d_1 - 1)!(3d_2 - 1)!]$, and $Q(1,1) = 1$, $Q(2,1) = 6$, $Q(2,1) = 224$, $Q(3,1) = 33/2$, $N_1 = 1$, we find

$$N_2 = Q(1,1) = 1$$

$$N_3 = 2Q(1,1)Q(2,1) = 12 \tag{2.43}$$

$$N_4 = 4Q(1,1)Q(2,1)Q(3,1) + Q(1,1)^2 Q(2,2) = 4.6.33/2 + 224 = 620$$

in agreement with table I.

There is a natural equivalence relation on the kernels $X(d_1, d_2)$, namely

$$X(d_1, d_2) \sim X(d_1, d_2)\frac{X(d_1)X(d_2)}{X(d_1 + d_2)} \tag{2.44}$$

for arbitrary non vahishing $X(d)$. It amounts to replace $u_d \to X(d)v_d$. The equivalence class of $X(d_1, d_2) = X =$ constant yields the Catalan numbers.

Since $N_d/(3d-1)!$ is positive, to show that the series for $y_2 f(y_1, y_2)$ converges as claimed in proposition 3, it is sufficient to prove, for large d, inequalities of the type

$$(a_<)^d \leq \frac{N_d}{(3d-1)!} \leq (a_>)^d . \tag{2.45}$$

To obtain the lower bound we start from

$$q(d_1, d_2) = \frac{d_1 d_2}{6} \frac{[(3d_1-2)(3d_2-2)(d+2) + 8(d-1)]}{(3d-1)(3d-2)(3d-3)} . \tag{2.46}$$

In the numerator we omit the positive term $8(d-1)$ and replace $(d+2)$ by $(d-1)$ while in the denominator we replace $(3d-1)$ by $3d$ to obtain

$$q(d_1, d_2) > \frac{d_1 d_2}{54d} \frac{(3d_1-2)(3d_2-2)}{(3d-2)} \equiv q_<(d_1, d_2) \tag{2.47}$$

with $q_<(d_1, d_2)$ in the equivalence class of a constant. According to lemma 1, taking into account $N_1/(3.1-1)! = 1/2$, we get

$$\frac{N_d}{(3d-1)!} > \frac{27(2d)!}{(108)^d d(2d-1)(3d-2)[d!]^2} \tag{2.48}$$

and using Stirling's formula $(2d)!/[d!]^2 \sim 2^{2d}/\sqrt{\pi d}$, for d large enough

$$\frac{N_d}{(3d-1)!} > \frac{9}{2\sqrt{\pi}(27)^d d^{7/2}}(1 + O(1/d)) \tag{2.49}$$

proving that the series for F has a non–vanishing radius of convergence. Similarly to obtain a rough upper bound starting from $q(d_1, d_2)$, we increase the numerator by replacing $8(d-1)$ by $8(d-1)(3d_1-2)(3d_2-2)$, $(d+2)$ by $4(d-1)$ and $d_1 d_2$ by $(3d_1-1)(3d_2-1)/4$. Hence

$$q(d_1, d_2) < \frac{(3d_1-1)(3d_1-2)(3d_2-1)(3d_2-2)}{6(3d-1)(3d-2)} \equiv q_>(d_1, d_2) \tag{2.50}$$

so that

$$\frac{N_d}{(3d-1)!} < \frac{3(2d)!}{6^d(2d-1)(3d-1)(3d-2)[d!]^2} \tag{2.51}$$

and asymptotically

$$\frac{N_d}{(3d-1)!} < \frac{1}{6\sqrt{\pi}d^{7/2}}\left(\frac{2}{3}\right)^d (1 + O(1/d)) \tag{2.52}$$

From this we conclude that the series for $y_2 f$ has a finite radius of convergence $a = \lim_{d \to \infty} [N_d/(3d-1)!]^{1/d}$, satisfying

$$\frac{1}{108} < a < \frac{2}{3} \ . \tag{2.53}$$

Our numerical value .138 satisfies these bounds.

To find the power law prefactor $d^{-7/2}$, we set

$$G(x) = \frac{1}{3} \sum_{d=1}^{\infty} \frac{N_d}{(3d-1)!} e^{dx} \tag{2.54}$$

with a finite abscissa of convergence $Re(x) < x_0 = \ln 1/a$ and a behavior $e^x/6$ for $x \to -\infty$. It satisfies the translation invariant differential equation

$$(9 + 2G' - 3G'')G''' = 2G + 11G' + 18G'' + (G'')^2 \ . \tag{2.55}$$

The functions G, G', G'', G''' are all positive with $G < G' < G'' < G'''$ for real $x < x_0$. Therefore G, G' and G'' remain finite at x_0 with G'''' blowing up. With $0 < \alpha < 1$, we have in the vicinity of x_0

$$G(x) = g_0 + g_1(x_0 - x) + g_2 \frac{(x_0 - x)^2}{2} + \lambda(x_0 - x)^{2+\alpha} + \cdots \tag{2.56}$$

hence

$$-[(9 - 2g_1 - 3g_2) - 3\lambda(1+\alpha)(2+\alpha)(x_0 - x)^\alpha + \cdots]\lambda\alpha(1+\alpha)(2+\alpha)(x_0 - x)^{\alpha-1}$$
$$= 2g_0 - 11g_1 + 18g_2 + g_2^2 + O(x) \tag{2.57}$$

leading to

$$9 - 2g_1 - 3g_2 = 0 \qquad 2\alpha - 1 = 0$$
$$3\alpha(1+\alpha)^2(2+\alpha)^2\lambda^2 = 2g_0 - 11g_1 + 18g_2 + g_2^2 \tag{2.58}$$

thus $\alpha = 1/2$ which corresponds to a behavior

$$\lim_{d \to \infty} \frac{d^{7/2} N_d}{a^d(3d-1)!} = \text{const.} \tag{2.59}$$

This concludes the proof of proposition 3. It is remarkable that our very poor upper and lower bounds both exhibit the same power law exponent 7/2. If, as physicists, we write it as $3 - \gamma_{\text{string}}$ (the expected area dependence of the free energy of a two dimensional quantum gravity theory), we find $\gamma_{\text{string}} = -1/2$, which corresponds to the "pure gravity"

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case[2]. As shown in appendix A, following Dubrovin [4], the differential equation (2.55) is related to a particular case of a Painlevé VI equation.

2.5. Flat connection

For generic values of y_i within the domain of convergence of f, the commutative algebra generated by the T_i's is semi–simple. Its regular representation as linear operators acting on the basis $(T_2, T_1, T_0)^T$ reads

$$T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_1 = \begin{pmatrix} 0 & f_{112} & f_{122} \\ 1 & f_{111} & f_{112} \\ 0 & 1 & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & f_{122} & f_{222} \\ 0 & f_{112} & f_{122} \\ 1 & 0 & 0 \end{pmatrix} \tag{2.60}$$

Associativity is granted for multiplication of linear operators, the actual constraint is the commutativity $[T_i, T_j] = 0$, and reduces to $[T_1, T_2] = 0$, equivalent to (2.30). Moreover from (2.60) we get

$$\frac{\partial T_i}{\partial y_j} = \frac{\partial T_j}{\partial y_i} . \tag{2.61}$$

Conversely, let

$$T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_1 = \begin{pmatrix} 0 & a_{11} & a_{12} \\ 1 & a_{21} & a_{22} \\ 0 & 1 & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \\ 1 & 0 & 0 \end{pmatrix} \tag{2.62}$$

where the matrix elements a_{ij}, b_{ij} are a priori functions of y_0, y_1, y_2. For $a_{ij} = b_{ij} = 0$, we find the classical cohomology ring of $\mathbb{P}_2$. Let us require that

$$[T_i, T_j] = 0 \quad \text{and} \quad \frac{\partial T_i}{\partial y_j} = \frac{\partial T_j}{\partial y_i} \quad i \neq j . \tag{2.63}$$

Hence $0 = \partial T_0 / \partial y_j - \partial T_j / \partial y_0 = -\partial T_j / \partial y_0$ so a_{ij} and b_{ij} are functions of y_1, y_2 only while

$$\frac{\partial a_{ij}}{\partial y_2} = \frac{\partial b_{ij}}{\partial y_1} . \tag{2.64}$$

---

[2] Using this analogy with quantum gravity, we are led to the following conjecture for the asymptotics of higher genus $g > 0$ numbers of curves through $(3d - 1 + g)$ points

$$\frac{N_d^{(g)}}{(3d - 1 + g)!} \propto \frac{a^d}{d^{1 + \frac{5}{2}(1 - g)}}$$

Commutativity requires

$$a_{12} = b_{11} = b_{22} \qquad a_{11} = a_{22} = b_{21}$$

$$b_{12} = (a_{11})^2 - a_{12}a_{21} \ , \tag{2.65}$$

which combines with (2.64) to yield

$$\frac{\partial a_{11}}{\partial y_2} = \frac{\partial a_{22}}{\partial y_2} = \frac{\partial a_{12}}{\partial y_1}$$

$$\frac{\partial a_{11}}{\partial y_1} = \frac{\partial a_{21}}{\partial y_2} \tag{2.66}$$

$$\frac{\partial a_{12}}{\partial y_2} = \frac{\partial b_{12}}{\partial y_1} \ .$$

This implies the existence of a function $\Psi(y_1, y_2)$ such that

$$a_{11} = a_{22} = b_{21} = \frac{\partial^2 \Psi}{\partial y_1 \partial y_2} \ ; \ a_{21} = \frac{\partial^2 \Psi}{\partial y_1^2} \ ; \ a_{12} = b_{11} = b_{22} = \frac{\partial \Psi}{\partial y_2^2} \tag{2.67}$$

while $\partial b_{12}/\partial y_2 = \partial^3 \Psi/\partial y_2^3$, which is solved if we write $\Psi = \partial f/\partial y_1$. Finally

$$a_{11} = a_{22} = b_{21} = f_{112} \ ; \ a_{21} = f_{111} \ ; \ a_{12} = b_{11} = b_{22} = f_{122} \ ; \ b_{12} = f_{222} \ . \tag{2.68}$$

The last condition from commutativity amounts to the familiar relation (2.30).

Proposition 4. (Dubrovin)

(i) The necessary and sufficient condition that the covariant derivatives

$$D_i \equiv \frac{\partial}{\partial y_i} - zT_i \ , \tag{2.69}$$

with T_i of the form (2.62), commute for arbitrary z is that there exists a function $f(y_1, y_2)$ solution of (2.30) such that the T's have the form (2.60).

(ii) Set

$$T_{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{2.70}$$

With $f = \sum_{d \geq 1} N_d y_2^{3d-1} e^{d y_1}/(3d-1)!$ (in its region of convergence), the D_i commute with

$$D_z \equiv z\frac{\partial}{\partial z} - \left[T_{-1} + z(y_0 T_0 + 3T_1 - y_2 T_2) \right] \ . \tag{2.71}$$

The commutation relations $[D_i, D_j] = 0$ for arbitrary z amount to $\partial_i T_j = \partial_j T_i$ and $[T_i, T_j] = 0$ proving (i). As for (ii), it follows from the homogeneity of f, namely $f(y_1 + 3\ln\lambda, \lambda^{-1}y_2) = \lambda f(y_1, y_2)$. We have $[D_0, D_z] = 0$, since $\partial_0 T_1 = \partial_0 T_2 = 0$. Moreover from $[T_1, T_2] = 0$ and $\partial_2 T_1 = \partial_1 T_2$ we get

$$
\begin{aligned}
[D_1, D_z] &= -z\partial_1(3T_1 - y_2 T_2) + zT_1 + z[T_1, T_{-1}] \\
&= z\big[-3\partial_1 T_1 + y_2\partial_2 T_2 + T_1 + [T_1, T_{-1}]\big] \\
&= z\begin{pmatrix} 0 & (2 - 3\partial_1 + y_2\partial_2)f_{112} & (3 - 3\partial_1 + y_2\partial_2)f_{122} \\ 0 & (1 - 3\partial_1 + y_2\partial_2)f_{111} & (2 - 3\partial_1 + y_2\partial_2)f_{112} \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{2.72}
$$

Since the degrees of homogeneity are $[f_{111}] = 1$, $[f_{112}] = 2$ and $[f_{122}] = 3$, all the terms vanish, for instance

$$
(1 - 3\partial_1 + y_2\partial_2)f_{111} = \lambda\frac{\partial}{\partial\lambda}\big[\lambda f_{111}(y_1, y_2) - f_{111}(y_1 + 3\ln\lambda, \lambda^{-1}y_2)\big]_{\lambda=1} = 0. \tag{2.73}
$$

Similarly one checks that $[D_2, D_z] = 0$.

The system

$$
D_i L = D_z L = 0 , \tag{2.74}
$$

where L is a 3×3 matrix, admits therefore a consistent solution and we have a trivial bundle over a covering of that part of the dual $H^*(\mathbb{P}_2) \times (\mathbb{C} - \{0\})$ (for the variable z), where f is defined. Thus, at least locally, T_i is a pure gauge $T_i = z^{-1}L^{-1}\partial_i L$.

On the plane $y_2 = 0$ the ring generated by $\{T_i\}$ reduces to a simpler one since the only non–vanishing f_{ijk} is $f_{122} = e^{y_1} \equiv q^3$. Its multiplication laws read

$$
T_1^2 = T_2 ; \quad T_1 T_2 = q^3 T_0 ; \quad T_2^2 = q^3 T_1 \tag{2.75}
$$

i.e. it is the ring $\mathbb{C}[x]/(x^3 - q^3)$ with the identification $T_i \to x^i$. In the vicinity of this plane the covariant derivatives D_i generate "monodromy preserving" flows of the solutions of $D_z L = 0$. To identify this monodromy, it is therefore sufficient to look at this equation at a specific point, say $y_0 = y_2 = 0$, where it takes the form $(z \neq 0)$

$$
\left[\frac{\partial}{\partial z} - \begin{pmatrix} -1/z & 0 & 3q^3 \\ 3 & 0 & 0 \\ 0 & 3 & 1/z \end{pmatrix}\right] L = 0 . \tag{2.76}
$$

This equation is easily solved as follows. For $z \neq 0$, set

$$A(z) = z \sum_{n=0}^{\infty} \frac{(zq)^{3n}}{[n!]^3}$$

$$B(z) = -2z \sum_{n=1}^{\infty} \frac{(zq)^{3n}}{[n!]^3} \sum_{p=1}^{n} \frac{1}{p} \qquad (2.77)$$

$$C(z) = z \sum_{n=1}^{\infty} \frac{(zq)^{3n}}{[n!]^3} \left[\left(\sum_{p=1}^{n} \frac{1}{p} \right)^2 + \frac{1}{3} \sum_{p=1}^{n} \frac{1}{p^2} \right]$$

and define the matrix L_0 as

$$\begin{pmatrix} \frac{1}{9}\partial(\partial - 1/z)A & \frac{4}{9z}(\partial - 1/z)A + \frac{1}{9}\partial(\partial - 1/z)B & \frac{2A}{9z^2} + \frac{2}{9z}(\partial - 1/z)B + \frac{1}{9}\partial(\partial - 1/z)C \\ \frac{1}{3}(\partial - 1/z)A & \frac{2A}{3z} + \frac{1}{3}(\partial - 1/z)B & \frac{B}{3z} + \frac{1}{3}(\partial - 1/z)C \\ A & B & C \end{pmatrix}$$

$$(2.78)$$

where we use the shorthand notation ∂ for $\partial/\partial z$. Finally let us introduce the matrix

$$X \equiv \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad X^3 = 0 . \qquad (2.79)$$

Lemma 2.

(i) Up to multiplication to the right by a z–independent matrix, the solution L of (2.76) is of the form

$$L = L_0 z^X . \qquad (2.80)$$

(ii) Therefore, up to conjugacy the generator of the monodromy of the solution L around the origin is

$$L \to L\, e^{2i\pi X} . \qquad (2.81)$$

The matrix $\exp(2i\pi X)$ is unipotent

$$e^{2i\pi X} = 1 + 2i\pi X - 2\pi^2 X^2 = \begin{pmatrix} 1 & 4i\pi & -4\pi^2 \\ 0 & 1 & 2i\pi \\ 0 & 0 & 1 \end{pmatrix} \qquad (2.82)$$

It is perhaps not unexpected that X is a generator of the nilpotent ring $\mathbb{C}[x]/x^3$, i.e. of the classical cohomology ring.

The monodromy around the origin is independent of $q^3 = e^{y_1}$. This is part of a more general property (Dubrovin). Namely, returning to the case of general (y_0, y_1, y_2) we can

solve locally *simultaneously* $D_zL = D_iL = 0$, since all the differential operators commute when they are defined. Thus to lowest order in $\epsilon \equiv (\epsilon_0, \epsilon_1, \epsilon_2)$

$$L(y + \epsilon, z) = L(y, z) + \sum_{0 \leq i \leq 2} z\epsilon_i T_i(y) L(y, z) + O(\epsilon^2) \tag{2.83}$$

implying that $L(y + \epsilon, z)$ has the same monodromy around the origin as $L(y, z)$. Up to conjugation this is the same as on the line $y_0 = y_2 = 0$, computed above.

In the completed plane the only other singularity of the general equation $D_zL = 0$ is an essential singularity at infinity. Kontsevich and Manin suggest to perform a formal Fourier transform

$$L(y, z) = \int dp\, e^{pz} \tilde{L}(y, p) \ . \tag{2.84}$$

We should then solve for

$$\left[\frac{\partial}{\partial p} - (p - (y_0 T_0 + 3T_1 - y_2 T_2))^{-1}(1 + T_{-1}) \right] \tilde{L}(y, p) = 0 \ . \tag{2.85}$$

This differential equation has now 4 singular points, namely the roots of

$$\det\left[p - (y_0 T_0 + 3T_1 - y_2 T_2) \right] = 0 \tag{2.86}$$

and the point at infinity. Again we see that as y varies the monodromy is preserved. In particular when $y_0 = y_2 = 0$, (2.85) reduces to

$$\left[\frac{\partial}{\partial p} - \frac{1}{p^3 - 27q^3} \begin{pmatrix} 0 & 9q^3 & 6q^3 p \\ 0 & p^2 & 18q^3 \\ 0 & 3p & 2p^2 \end{pmatrix} \right] \tilde{L}(y, p) = 0 \ , \tag{2.87}$$

exhibiting the three simple poles at $p = 3q$ and $p = 3qe^{\pm 2i\pi/3}$.

Remark. For generic values of y, the commutative algebra generated by the T's over $\mathbb{C}$ is semi–simple and assumes the form $\mathbb{C}[x]/P(x)$, where P is a monic cubic polynomial obtained as follows. Set

$$T_0 = 1 \ ,$$
$$T_1 = x \ , \tag{2.88}$$
$$T_2 = x^2 - f_{111}x - f_{112} \ ,$$

From $T_1 T_2 = f_{112}T_1 + f_{122}T_0$, we deduce that

$$P(x) \equiv x^3 - f_{111}x^2 - 2f_{112}x - f_{122} = 0. \tag{2.89}$$

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When $y_0 = y_2 = 0$, P reduces to $x^3 - e^{y_1} = x^3 - q^3$. Such a property will extend to projective spaces in arbitrary dimension. Let $P(x)$ be a monic polynomial with distinct roots of arbitrary degree n, then the commutative algebra $\mathbb{C}[x]/P(x)$ admits a set of idempotent generators $T_1, T_2, ... T_n$, satisfying

$$T_\alpha \, T_\beta = T_\beta \, T_\alpha = \delta_{\alpha,\beta} \, T_\alpha \tag{2.90}$$

with the unit being $1 = \sum_{1 \leq \alpha \leq n} T_\alpha$. Indeed let a_α be the distinct roots of $P(x) = \prod_{1 \leq \alpha \leq n}(x - a_\alpha)$, one has

$$T_\alpha \equiv \prod_{\beta \neq \alpha} \frac{x - a_\beta}{a_\alpha - a_\beta} \qquad \mod P(x) \tag{2.91}$$

hence for any polynomial $R(x) = \sum_{1 \leq \alpha \leq n} R(a_\alpha) T_\alpha \mod P(x)$. In particular, $x^k = \sum_{1 \leq \alpha \leq n} T_\alpha a_\alpha^k$. Presented in the form (2.90), the algebra conveys no information except n (the degree of the polynomial $P(x)$), and its semi–simplicity. The crucial information is coded in the T's as linear combinations of the T's or equivalently in the metric η. We refer to appendix A for a discussion of the matter.

2.6. Higher genera

A generalization of the preceding enumerative theory in higher genus is lacking at present. However we can collect some data. Set

$$f^{(g)} = \sum_{d:(d-1)(d-2) \geq 2g} N_d^{(g)} \frac{y_2^{3d-1+g}}{(3d-1+g)!} e^{dy_1} \tag{2.92}$$

where $f^{(0)} \equiv f$, $N_d^{(0)} = N_d$, and $N_d^{(g)}$ counts the number of degree d, genus g, stable irreducible curves through $(3d - 1 + g)$ points in $\mathbb{P}_2$, i.e. with $\delta = (d-1)(d-2)/2 - g$ ordinary double points. The sum in (2.92) therefore starts from $d = 1$ for $g = 0$, and from integer $d \geq [3 + \sqrt{1 + 8g}]/2$ for $g \geq 1$. From propositions 1 and 2, the available data are,

up to degree 5

$$\frac{y_2^2}{2}e^{y_1} + \frac{y_2^5}{5!}e^{2y_1} + 12\frac{y_2^8}{8!}e^{3y_1} + 620\frac{y_2^{11}}{11!}e^{4y_1} + 87304\frac{y_2^{14}}{14!}e^{5y_1} + ... = f^{(0)}$$

$$\frac{y_2^9}{9!}e^{3y_1} + 225\frac{y_2^{12}}{12!}e^{4y_1} + 87192\frac{y_2^{15}}{15!}e^{5y_1} + ... = f^{(1)}$$

$$27\frac{y_2^{13}}{13!}e^{4y_1} + 36855\frac{y_2^{16}}{16!}e^{5y_1} + ... = f^{(2)}$$

$$\frac{y_2^{14}}{14!}e^{4y_1} + 7915\frac{y_2^{17}}{17!}e^{5y_1} + ... = f^{(3)} \qquad (2.93)$$

$$882\frac{y_2^{18}}{18!}e^{5y_1} + ... = f^{(4)}$$

$$48\frac{y_2^{19}}{19!}e^{5y_1} + ... = f^{(5)}$$

$$\frac{y_2^{20}}{20!}e^{5y_1} + ... = f^{(6)} .$$

The challenge is to find the topological recursion relation, or the non–linear extension of the Dubrovin flows, or else a manageable path integral which would generate these higher genus contributions (on the so–called "little phase space"). Note that all the enumerative coefficients $N_d^{(g)}$ are presumed to be non–negative integers. This is to be contrasted with general intersection numbers on the orbifold compactification of the moduli space of punctured Riemann surfaces of fixed genus, which tend to be rational.

2.7. Characteristic numbers

Let us return to rational curves of degree d. When discussing the deformed ring we only dealt with the number N_d of such curves through $3d - 1$ points. Classically one also considered mixed conditions involving points and lines. Let $N_{\alpha,\beta}$, $\alpha + \beta = 3d - 1$, denote the number of rational plane curves through α fixed points and tangent to β fixed lines, so that $N_{3d-1,0} = N_d$. The generating function

$$\phi(y_1, y_2, z) = \sum_{\substack{d \geq 1 \ \alpha,\beta \geq 0 \\ \alpha+\beta=3d-1}} N_{\alpha,\beta} \frac{y_2^\alpha}{\alpha!} \frac{z^\beta}{\beta!} e^{dy_1} \qquad (2.94)$$

reduces to $f(y_1, y_2)$ defined in (2.28) for $z = 0$. The integers $N_{\alpha,\beta}$ are refered to as *characteristic numbers* (an analogous definition holds for higher genera).

It is possible to obtain for the derivatives of ϕ a quadratic relation which generalizes equation (2.30) by extending the argument of section 2.3 to mixed conditions. We have

only to modify the "spectator" set of conditions (which included $3d - 4$ fixed points q_*), to involve $\alpha - 3$ points and β lines of tangency. The reasoning is completely analogous provided we take into account degenerating curves intersecting on a line of tangency or on one of their intersections and their multiplicities. If this is done carefully, the result reads

$$\phi_{222} = (\phi_{112}^2 - \phi_{111}\phi_{122}) + 2z(\phi_{112}\phi_{122} - \phi_{111}\phi_{222}) + 2z^2(\phi_{112}^2 - \phi_{112}\phi_{222})$$

$$(2.95)$$

Readily available initial data are

$$
\begin{aligned}
d = 1 &: \quad N_{2,0} = 1 \quad N_{1,1} = N_{0,2} = 0 \\
d = 2 &: \quad N_{\alpha,\beta} = N_{\beta,\alpha}
\end{aligned}
$$

$$(2.96)$$

where the second line expresses duality for the conics.

Proposition 5. Equation (2.95) determines all characteristic numbers $N_{\alpha,\beta}$ for $\beta \leq 3d_0 - 1$, in terms of the $N_{\alpha,\beta}$ in degree $d \leq d_0$. Beyond, i.e. for $\beta \geq 3d_0$, one only gets quadratic relations. In particular the initial conditions (2.96) determine all the $N_{\alpha,\beta}$ for $\beta \leq 5$.

For low degree we present below the characteristic numbers for $\beta \leq 5$ using only eq.(2.96). In degree $d \leq 4$ they agree with known results [6] [10]. Complementing these data with the missing values in degrees 3 and 4 from the above references, we obtain $N_{14-\beta,\beta}$ in degree 5 up to $\beta = 11$. In the process we also check the relations arising from

powers of z^6 to z^8 in ϕ in degree 4.

$$\mathbf{d{=}1:} \quad N_{2,0} = 1 \quad N_{1,1} = 0 \quad N_{0,2} = 0$$

$$\mathbf{d{=}2:} \quad N_{5,0} = 1 \quad N_{4,1} = 2 \quad N_{3,2} = 4$$

$$N_{2,3} = 4 \quad N_{1,4} = 2 \quad N_{0,5} = 1$$

$$\mathbf{d{=}3:} \quad N_{8,0} = 12 \quad N_{7,1} = 36 \quad N_{6,2} = 100$$

$$N_{5,3} = 240 \quad N_{4,4} = 480 \quad N_{3,5} = 712$$

$$N_{2,6} = 756 \quad N_{1,7} = 600 \quad N_{0,8} = 400$$

$$\mathbf{d{=}4:} \quad N_{11,0} = 620 \quad N_{10,1} = 2184 \quad N_{9,2} = 7200$$

$$N_{8,3} = 21776 \quad N_{7,4} = 59424 \quad N_{6,5} = 143040$$

$$N_{5,6} = 295544 \quad N_{4,7} = 505320 \quad N_{3,8} = 699216$$

$$N_{2,9} = 783584 \quad N_{1,10} = 728160 \quad N_{0,11} = 581904$$

$$\mathbf{d{=}5:} \quad N_{14,0} = 87304 \quad N_{13,1} = 335792 \quad N_{12,2} = 1222192$$

$$N_{11,3} = 4173280 \quad N_{10,4} = 13258208 \quad N_{9,5} = 38816224$$

$$N_{8,6} = 103544272 \quad N_{7,7} = 248204432 \quad N_{6,8} = 526105120$$

$$N_{5,9} = 969325888 \quad N_{4,10} = 1532471744 \quad N_{3,11} = 2069215552$$

(2.97)

Equation (2.95) can also be interpreted as the associativity condition for a deformed ring. With z as parameter, the genus 0 free energy is now

$$\Phi(y_0, y_1, y_2; z) = \phi_{cl}(y_0, y_1, y_2; z) + \phi(y_0, y_1, y_2; z)$$

$$\phi_{cl}(y_0, y_1, y_2; z) = \frac{y_0^2 y_2 + y_0 y_1^2}{2} - 2z \frac{y_0^2 y_1}{2} + 4z^2 \frac{y_0^3}{6}$$

(2.98)

It corresponds to a modified "metric" η_{ij} written in matrix form

$$\begin{pmatrix} 2z^2 & -2z & 1 \\ -2z & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \left(|2\rangle - z|1\rangle + \frac{z^2}{2}|0\rangle \right)\langle 0|$$

$$+ \left(|1\rangle - z|0\rangle \right)\left(\langle 1| - z\langle 0| \right) + |0\rangle\left(\langle 2| - z\langle 1| + \frac{z^2}{2}\langle 0| \right)$$

and its inverse η^{ij} which appears as a propagator (cf. Fig.1 and eq.(2.32))

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2z \\ 1 & 2z & 2z^2 \end{pmatrix} = \left(|0\rangle + z|1\rangle + \frac{z^2}{2}|2\rangle \right)\langle 2|$$

$$+ \left(|1\rangle + z|2\rangle \right)\left(\langle 1| + z\langle 2| \right) + |2\rangle\left(\langle 0| + z\langle 1| + \frac{z^2}{2}\langle 2| \right)$$

On the rhs of these expressions we used Dirac's bra–ket notation to emphasize the structure, which suggests a natural generalization in higher projective spaces.

For large β characteristic numbers can be difficult to obtain. At the very least, equation (2.95) gives useful constraints, in the form of recursion relations.

3. Rational manifolds

We consider in this section some examples of rational target manifolds and the corresponding enumeration of rational curves. Details will be omitted except for a few elementary comments.

3.1. Projective spaces

We first generalize from the plane $\mathbb{P}_2$ to $\mathbb{P}_n$, $n \geq 2$, in which case the classical cohomology ring may be identified with $\mathbb{C}[x]/x^{n+1}$. Correspondingly, we introduce deformation parameters y_i, $0 \leq i \leq n$, with weights $[y_i] = 1 - i$, $i \neq 1$, and $[e^{y_1}] = n + 1$, also $[F^{(g)}] = (1 - g)(3 - n)$, which gives for $g = 0$ ($F^{(0)} \equiv F$) $[F] = 3 - n$. These assignments are consistent with the following facts

(i) The intersection form

$$\eta_{ij} = \delta_{i+j,n} = \frac{\partial^3 F}{\partial y_0 \partial y_i \partial y_j} \tag{3.1}$$

is dimensionless, hence $[F] = [y_0] + [y_i] + [y_{n-i}]$.

(ii) A generic rational curve of degree d in $\mathbb{P}_n$ depends on $(d+1)(n+1)-4$ parameters and intersection with a linear subspace of codimension k imposes $k-1$ linear conditions on these parameters. Since y_k is dual to cycles of codimension k one expects with arguments similar to those of section 2 that the free energy F decomposes into a classical and a quantum part $F(y) = f_{cl}(y) + f(y)$, where as before f_{cl} is a cubic polynomial encoding the multiplication rules of the classical cohomology ring

$$f_{cl}(y) = \frac{1}{3!} \sum_{i+j+k=n} y_i y_j y_k \tag{3.2}$$

and the quantum corrections take the form

$$f(y) = \sum N(a_2, a_3, ..., a_n | d) \frac{y_2^{a_2}}{a_2!} \frac{y_3^{a_3}}{a_3!} ... \frac{y_n^{a_n}}{a_n!} e^{dy_1} , \tag{3.3}$$

where the sum is running over non negative a_i's such that

$$(n+1)d + \sum_{i=2}^{n}(1-i)a_i = (3-n) , \qquad (3.4)$$

expressing the global homogeneity of f. Finally $N(a_2, a_3, ..., a_n|d)$ is interpreted as the number of rational curves of degree d intersecting a_n points, a_{n-1} lines, ... a_{n-k} linear spaces of dimension k, ... in "general position". In particular when $d = 1$, we have $\sum_{2 \le i \le n}(i-1)a_i = 2(n-1)$ and $N(0,0,...,0,2|1) = 1$ is the number of lines through 2 points.

The study of the quantum cohomology ring for general n deserves a specific combinatorial treatment since the equations expressing associativity become quite cumbersome. We limit ourselves here to $\mathbb{P}_3$ in which case

$$f_{cl} = \frac{1}{2}y_0^2 y_3 + y_0 y_1 y_2 + \frac{1}{6}y_1^3$$

$$f = \sum_{\substack{a+2b=4d \\ a,b \ge 0 \, d \ge 1}} N_{a,b} \frac{y_2^a}{a!} \frac{y_3^b}{b!} e^{dy_1} \qquad (3.5)$$

where for short $N_{a,b} \equiv N(a,b|d)$. As before, we introduce the deformed ring with basis T_0, T_1, T_2, T_3

$$T_i T_j = \sum_{0 \le k,l \le n} F_{ijk}\eta^{kl}T_l \qquad (3.6)$$
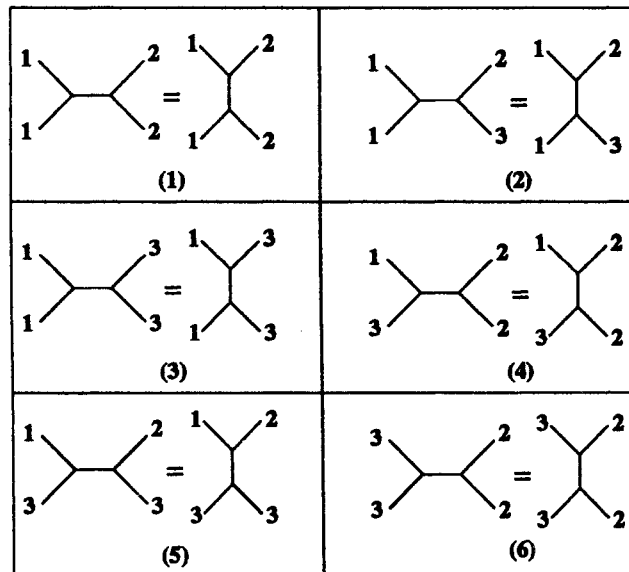
and T_0 is the identity (f does not depend on y_0).

Fig. 3: The six duality relations for $\mathbb{P}_3$.

The six associativity conditions corresponding to the "duality" diagrams of Fig.3 read

$$(1) \quad 2f_{123} - f_{222} = f_{111}f_{222} - f_{112}f_{122}$$

$$(2) \quad f_{133} - f_{223} = f_{111}f_{223} - f_{113}f_{122}$$

$$(3) \quad -f_{233} = f_{111}f_{233} + f_{112}f_{133} - 2f_{113}f_{123}$$

$$(4) \quad f_{233} = f_{113}f_{222} - f_{112}f_{223}$$

$$(5) \quad f_{333} = f_{123}^2 + f_{113}f_{223} - f_{112}f_{233} - f_{122}f_{133}$$

$$(6) \quad 0 = f_{122}f_{233} + f_{133}f_{222} - 2f_{123}f_{223} \ .$$

$$(3.7)$$

With f of weight 0, the equations are respectively of weight 3, 4, 5, 6, 7. They yield (with the convention that $a + 2b = 4d$, $a_i + 2b_i = 4d_i$ for $i = 1$ and 2, $d_1, d_2 \geq 0$, $d_1 + d_2 = d \geq 1$)

Proposition 6. The integers $N_{a,b}$ are determined by $N_{0,2} = 1$ and

$$(1) \quad 2dN_{a-2,b+1} - N_{a,b} = \sum N_{a_1,b_1}N_{a_2,b_2}\binom{b}{b_1}\left[d_1^3\binom{a-3}{a_1} - d_1^2d_2\binom{a-3}{a_1-1}\right]$$

$$(2) \quad dN_{a-2,b+1} - N_{a,b} = \sum N_{a_1,b_1}N_{a_2,b_2}\binom{a-2}{a_1}\left[d_1^3\binom{b-1}{b_1} - d_1^2d_2\binom{b-1}{b_1-1}\right]$$

$$(3) \quad -N_{a,b} = \sum N_{a_1,b_1}N_{a_2,b_2} \times$$

$$\times \left[d_1^3\binom{a-1}{a_1}\binom{b-2}{b_1} + d_1^2d_2\binom{a-1}{a_1-1}\binom{b-2}{b_1} - 2d_1^2d_2\binom{a-1}{a_1}\binom{b-2}{b_1-1}\right]$$

$$(4) \quad N_{a-2,b+1} = \sum N_{a_1,b_1}N_{a_2,b_2}d_1^2\left[\binom{a-3}{a_1}\binom{b-1}{b_1-1} - \binom{a-3}{a_1-1}\binom{b-1}{b_1}\right]$$

$$(5) \quad N_{a-2,b+1} = \sum N_{a_1,b_1}N_{a_2,b_2}\left[d_1d_2\binom{a-2}{a_1-1}\binom{b-2}{b_1-1} + \right.$$

$$\left. +d_1^2\binom{a-2}{a_1}\binom{b-2}{b_1-1} - d_1^2\binom{a-2}{a_1-1}\binom{b-2}{b_1} - d_1d_2\binom{a-2}{a_1-2}\binom{b-2}{b_1}\right]$$

$$(6) \quad 0 = \sum N_{a_1,b_1}N_{a_2,b_2} \times$$

$$\times d_1\left[\binom{a-3}{a_1-1}\binom{b-2}{b_1} + \binom{a-3}{a_1}\binom{b-2}{b_1-2} - 2\binom{a-3}{a_1-1}\binom{b-2}{b_1-1}\right]$$

$$(3.8)$$

By convention, the combinatorial factors $\binom{n}{n_1}$ are non–vanishing only for n, n_1 and $n-n_1 \geq 0$. The single input $N_{0,2} = 1$ for the number of lines through 2 points suffices to determine all the $N_{a,b}$. The six conditions appear consistent although the system looks strongly

overdetermined. One finds

$$d=1 : \quad N_{4,0} = 2 \quad N_{2,1} = 1 \quad N_{0,2} = 1$$

$$d=2 : \quad N_{8,0} = 92 \quad N_{6,1} = 18 \quad N_{4,2} = 4$$

$$N_{2,3} = 1 \quad N_{0,4} = 0$$

$$d=3 : \quad N_{12,0} = 80160 \quad N_{10,1} = 9864 \quad N_{8,2} = 1312$$

$$N_{6,3} = 190 \quad N_{4,4} = 30 \quad N_{2,5} = 5$$

$$N_{0,6} = 1$$

$$d=4 : \quad N_{16,0} = 383306880 \quad N_{14,1} = 34382544 \quad N_{12,2} = 3259680$$

$$N_{10,3} = 327888 \quad N_{8,4} = 35104 \quad N_{6,5} = 4000$$

$$N_{4,6} = 480 \quad N_{2,7} = 58 \quad N_{0,8} = 4$$

$$d=5 : \quad N_{20,0} = 6089786376960 \quad N_{18,1} = 429750191232 \quad N_{16,2} = 31658432256$$

$$N_{14,3} = 2440235712 \quad N_{12,4} = 197240400 \quad N_{10,5} = 16744080$$

$$N_{8,6} = 1492616 \quad N_{6,7} = 139098 \quad N_{4,8} = 13354$$

$$N_{2,9} = 1265 \quad N_{0,10} = 105$$

<div style="text-align: right">(3.9)</div>

In degree 1, there is of course $N_{2,1} = 1$ line through a point which meets two lines, and $N_{4,0} = 2$ lines meeting 4 lines in general position. This classical result may be obtained as follows. The set of lines intersecting three given lines l_1, l_2, l_3 is a linear pencil (for instance for each point p of l_1 there corresponds the unique line of the pencil defined by the intersection of the planes containing p and respectively l_2 and l_3) hence span a surface isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$, i.e. a quadric, cut by a fourth line l_4 in two points, corresponding to two lines of the pencil meeting the four lines l_i, $i = 1, ..., 4$. For conics there is a single conic ($N_{2,3} = 1$) through 3 points which meets two lines. The three points define a plane, which is intersected by the two lines in two further points, yielding altogether the necessary 5 points in this plane. But to find for instance that there are $N_{8,0} = 92$ conics meeting 8 lines in general position is already non trivial. With (twisted) cubics, we have rational normal curves in $\mathbb{P}_3$ (i.e. not lying in a $\mathbb{P}_k$, $k \leq 2$). In general the standard rational curve in $\mathbb{P}_n$ of degree n parametrized by $(1, t, t^2, .., t^n)$ in an appropriate coordinate system is uniquely stabilized by $(n + 3)$ points (see [11] for a general proof), so for $\mathbb{P}_3$, $d = 3$ we have $N_{0,6} = 1$ and for $\mathbb{P}_n$, $N(0, 0, ..., 0, n + 3 | n) = 1$. These twisted cubics can be interpreted as

the residual intersection of a linear pencil of quadrics having a line in common. A quadric is fixed by 9 points, but if 3 of them are aligned we only get a linear pencil of quadrics intersecting in a line and a twisted cubic through the 6 remaining points. Thus, given 6 points in general position, if we add 3 other aligned points, we get the linear pencil determining uniquely the pencil of quadrics intersecting on the line and residually on a twisted cubic through the 6 points. The remaining enumerative numbers of twisted cubics are not so easily described.

3.2. Quadric surfaces

For the cases of quadric and cubic surfaces we shall be sketchy as the matter has been already reported in [5]. A smooth quadric in $\mathbb{P}_3$ is isomorphic with $\mathbb{P}_1 \times \mathbb{P}_1$, H^* is therefore 4–dimensional, with Betti numbers $b^0 = b^4 = 1$, $b^2 = 2$. We denote the generators as t_0, t_A, t_B and $t_2 = t_A t_B$. The non vanishing intersections are therefore $\eta_{02} = \eta_{AB} = 1$. Let y_0, y_A, y_B, y_2 be the deformation parameters and F the genus zero free energy (modulo a second degree polynomial), split into $F = f_{cl} + f$ with

$$f_{cl} = \frac{1}{2}y_0^2 y_2 + y_0 y_A y_B$$

$$f = \sum_{\substack{a,b \geq 0 \\ a+b \geq 1}} N(2(a+b) - 1|a,b)\frac{y_2^{2(a+b)-1}}{(2(a+b) - 1)!}e^{ay_A + by_B} . \tag{3.10}$$

This corresponds to the weights

$$[y_0] = 1 \quad ; \quad [e^{y_A}] = [e^{y_B}] = 2 \quad ; \quad [y_2] = -1 \quad ; \quad [F] = 1 \tag{3.11}$$

while we will use the short hand notation $N_{a,b} = N_{b,a} \equiv N(2(a+b) - 1|a,b)$ for the number of rational curves on the quadric, with bidegree (a, b), through $2(a + b) - 1$ points.

A rational curve of bidegree (a, b) on $\mathbb{P}_1 \times \mathbb{P}_1$ is described by two homogeneous polynomials of degree a in a pair of parametrizing variables for the first $\mathbb{P}_1$ and similarly with degree b for the second $\mathbb{P}_1$, modulo PSL_2 on the parameters times $\mathbb{C}^* \times \mathbb{C}^*$ for homogeneity in each target $\mathbb{P}_1$. We are left with $2(a+1) + 2(b+1) - 3 - 1 - 1 = 2(a+b) - 1$ parameters explaining that a rational curve of bidegree (a, b) is fixed by $2(a + b) - 1$ points. Its degree as a curve in $\mathbb{P}_3$ is $d = a + b$. Moreover a general curve on $\mathbb{P}_1 \times \mathbb{P}_1$ with δ simple nodes has genus

$$g = (a - 1)(b - 1) - \delta , \tag{3.12}$$

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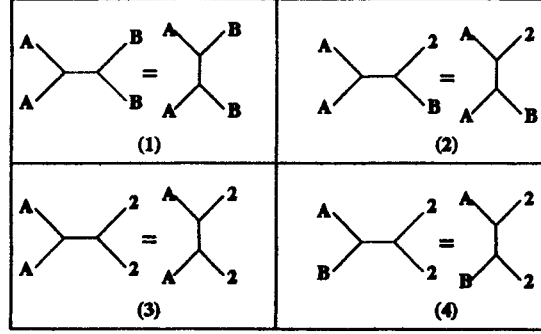
Fig. 4: Duality relations for the quadric $\mathbb{P}_1 \times \mathbb{P}_1$.

hence the rational curves we are counting have $\delta = (a-1)(b-1)$ simple nodes.

Let T_0, T_A, T_B and T_2 be a basis of the deformed ring with $T_i T_j = \sum_{j,k} F_{ijk} \eta^{kl} T_l$. Associativity yields the 4 conditions depicted on Fig.4 and their symmetric $A \leftrightarrow B$ counterparts (when the relation is not symmetric itself), leading to the 4 recursion relations of

Proposition 7. The integers $N_{a,b} = N_{b,a}$ are determined by the initial condition $N_{1,0} = 1$ and the system

(1) $\quad 2ab N_{a,b} = \sum N_{a_1,b_1} N_{a_2,b_2} a_1^2 b_2^2 (a_1 b_2 - a_2 b_1) \begin{pmatrix} 2(a+b)-2 \\ 2(a_1+b_1)-1 \end{pmatrix}$

(2) $\quad a N_{a,b} = \sum N_{a_1,b_1} N_{a_2,b_2} a_1 (a_1^2 b_2^2 - a_2^2 b_1^2) \begin{pmatrix} 2(a+b)-3 \\ 2(a_1+b_1)-1 \end{pmatrix}$

(3) $\quad 0 = \sum N_{a_1,b_1} N_{a_2,b_2} a_1^2 [(a_2 + b_2 - 1)(b_1 a_2 + b_2 a_1) - a_2 b_2 (2(a_1+b_1) - 1)] \begin{pmatrix} 2(a+b)-3 \\ 2(a_1+b_1)-1 \end{pmatrix}$

(4) $\quad N_{a,b} = \sum N_{a_1,b_1} N_{a_2,b_2} (a_1 b_2 + a_2 b_1) b_2 \left[a_1 \begin{pmatrix} 2(a+b)-4 \\ 2(a_1+b_1)-2 \end{pmatrix} - a_2 \begin{pmatrix} 2(a+b)-4 \\ 2(a_1+b_1)-3 \end{pmatrix} \right]$

$$(3.13)$$

where $a_1, a_2 \geq 0$, $b_1, b_2 \geq 0$, $a_1 + a_2 = a$, $b_1 + b_2 = b$, and the convention for combinatorial numbers is as before.

Up to degree $d = a + b = 10$, we find

d=1 : $N_{1,0} = 1$

d=2 : $N_{1,1} = 1$

d=3 : $N_{2,1} = 1$

d=4 : $N_{3,1} = 1$ $\quad N_{2,2} = 12$

d=5 : $N_{4,1} = 1$ $\quad N_{3,2} = 96$

d=6 : $N_{5,1} = 1$ $\quad N_{4,2} = 640$ $\quad N_{3,3} = 3510$ $\hfill (3.14)$

d=7 : $N_{6,1} = 1$ $\quad N_{5,2} = 3840$ $\quad N_{4,3} = 87544$

d=8 : $N_{7,1} = 1$ $\quad N_{6,2} = 21504$ $\quad N_{5,3} = 1763415$ $\quad N_{4,4} = 6508640$

d=9 : $N_{8,1} = 1$ $\quad N_{7,2} = 114688$ $\quad N_{6,3} = 30940512$ $\quad N_{5,4} = 348005120$

d=10 : $N_{9,1} = 1$ $\quad N_{8,2} = 589824$ $\quad N_{7,3} = 492675292$ $\quad N_{6,4} = 15090252800$

$\qquad N_{5,5} = 43628131782$

It is readily seen that $N_{a,1} = N_{1,b} = 1$ in particular $N_{2,1} = N_{1,2} = 1$ twisted cubic through 5 points on a quadric. The first non trivial result is $N_{2,2} = 12$ for the number of rational quartics through 7 points on a quadric. This number follows from the Zeuthen–Segre formula, giving the number n of singular elements in a linear pencil of curves of genus g on a surface of Euler characteristic χ through s base points: $n = \chi - 4(1 - g) + s$. Here we take the linear pencil of elliptic curves (smooth quartics, intersection of two quadrics) through 7 points on the quadric of characteristic 4, having thus an eighth base point, so $s = 8$. Since the singular elements are rational quartics, we have $N_{2,2} = 4 + 0 + 8 = 12$.

Remark The recursion relations (1)–(4) yield closed expressions for the first few $N_{a,b}$, with $b = 0, 1, 2, 3, 4$, as functions of a

$$N_{a,0} = \delta_{a,1}$$

$$N_{a,1} = 1$$

$$N_{a,2} = 2^{2a} \frac{a(a+1)}{8}$$

$$N_{a,3} = 3^{2a} \frac{512a^4 + 1280a^3 + 448a^2 - 368a - 75}{2^{14}} + \frac{32a^2 + 104a + 75}{2^{14}} \qquad (3.15)$$

$$N_{a,4} = 4^{2a} \frac{a(a+1)(8a^4 + 28a^3 + 6a^2 - 31a + 1)}{768} + 2^{2a} \frac{a(a+1)(a+2)(a+3)}{192} \ .$$

In general, $N_{a,b}$ has the following structure as a function of a for fixed b

$$N_{a,b} = \sum_{0 \le 2j < b} (b - 2j)^{2a} P^{(b)}_{2b-2j-2}(a) \ , \qquad (3.16)$$

where $P_n^{(b)}$ are degree n polynomials with rational coefficients depending on b only. It is a non trivial check that this leads to a symmetric $N_{a,b} = N_{b,a}$. For instance $N_{4,3} = N_{3,4} = 87544$ are both obtained from (3.15). The apparent simplicity of the structure of $N_{a,b}$ for $\mathbb{P}_1 \times \mathbb{P}_1$ may be related to the fact that unlike the $\mathbb{P}_n$, $n > 1$ cases, $\mathbb{P}_1$ has a very simple free energy

$$F = \frac{y_0^2}{2}y_1 + e^{y_1} , \qquad (3.17)$$

with the weights $[y_0] = 1$ and $[e^{y_1}] = 2$. It would be desirable to understand better the product $\mathbb{P}_1 \times \mathbb{P}_1$ in terms of free energies.

3.3. Cubic surfaces.

A classical description of cubic surfaces is by blowing up to lines six points of $\mathbb{P}_2$ not lying on a conic, with any triplet not lying on a line, using the $\mathbb{P}_3$ of plane cubics through the 6 points. These surfaces posses a system of 27 lines with a configuration (and intersection form) invariant under a finite group G isomorphic to the Weyl group of E_6 of order $72 \times 6!$ (G is transitive on the 72 sextets of non intersecting lines). For an in–depth study we refer to the book by Manin [12]. In the plane, any two cubics through 6 points intersect in $3 \times 3 = 6 + 3$ points. Hence a line in $\mathbb{P}_3$ corresponding to a linear pencil of cubics in the plane intersects the surface in 3 variable points, proving that the surface is indeed a cubic.

In the sequel we identify the second cohomology group of the surface $H^2 = H^{1,1}$ and the divisor class group, isomorphic with $\mathbb{Z}^7 \otimes \mathbb{C}$. The latter is generated by the six blown lines $(l_i, i = 1, 2, ..., 6)$ and the image of a line in the plane l_0. A plane section of the surface (generically a genus 1 cubic) is then equivalent to the divisor $3l_0 - \sum_{1 \leq i \leq 6} l_i$. Generically we write a divisor (class)

$$a = a^0 l_0 - a^i l_i \qquad (3.18)$$

with a degree

$$d_a = 3a^0 - \sum_{1 \leq i \leq 6} a^i \qquad (3.19)$$

and self–intersection

$$a.a \equiv (a^0)^2 - \sum_{1 \leq i \leq 6} (a^i)^2 . \qquad (3.20)$$

We use physicists' notation with $\{a^\mu\} = \{a^0, a^i\}$ and a Lorentzian scalar product denoted with a dot

$$a.b = a^0 b^0 - \sum_{1 \leq i \leq 6} a^i b^i . \tag{3.21}$$

Greek (resp. latin) indices run from 0 to 6 (resp. 1 to 6) and ω is the anticanonical divisor

$$\omega \equiv \{3, 1, 1, 1, 1, 1, 1\} \quad , \quad \omega.\omega = 3 . \tag{3.22}$$

With the interpretation in terms of curves of $\mathbb{P}_2$, when $a^0 > a^i \geq 0$, a^0 is the degree of the corresponding plane curve, and a^i the multiplicity at the i-th assigned point (we assume generically multiple points with distinct tangents at the six assigned points). The apparent (or arithmetic) genus of a curve in the divisor class a, denoted p_a, is then

$$\begin{aligned} p_a &= \frac{1}{2}[(a^0 - 1)(a^0 - 2) - \sum_{1 \leq i \leq 6} a^i(a^i - 1)] \\ &= 1 + \frac{1}{2}(a.a - \omega.a) . \end{aligned} \tag{3.23}$$

The topological genus g of a representative in the class can be smaller than p_a since the above formula does not take into account other singularities outside the assigned points. As before we assume generically that these are simple nodes. In this description, the only other effective divisor classes are given by the six lines l_i on the cubic surface.

For the quantum cohomology ring we change slightly our notations to avoid confusion. We use T_x for the identity (instead of T_0) and T_z instead of T_2, and keep the seven generators T_μ. The intersection form has non vanishing entries $\eta_{xz} = 1$ and $\eta_{\mu\nu} = g_{\mu\nu}$ the 7–dimensional diagonal Lorentzian metric with $g_{00} = -g_{ii} = 1$, numerically equal to its inverse.

We are interested in counting rational irreducible curves belonging to an effective class a of degree $d = \omega.a$, satisfying extra requirements. In the above presentation this implies that either $a = \{0, -1, 0, 0, 0, 0, 0\}$ up to "space" (indices $i = 1, ..., 6$) permutations or else

$$\begin{aligned} &(i) \quad 0 \leq a^i < a^0 \quad , \quad 1 \leq i \leq 6 \\ &(ii) \quad a^i + a^j \leq a^0 \quad , \quad 1 \leq i < j \leq 6 , \, d > 1 \\ &(iii) \quad \sum_{1 \leq i \leq 6} a^i \leq 2a^0 + a^k \quad , \quad 1 \leq k \leq 6 , \, d > 1 . \end{aligned} \tag{3.24}$$

These conditions, necessary but not sufficient for irreducibility, mean respectively that (i) in the plane an irreducible curve of degree a^0 cannot have a point of multiplicity as large

as its degree, (ii) that no line cuts it in more that a^0 points and (iii) that for $a^0 > 2$, a conic through 5 points cannot cut the curve in more that a^0 points. Higher conditions are subsummed under the inequality $p_a \geq 0$. Let $\Delta_p \equiv \sum \Delta_{p,d}$ be the set of effective divisors corresponding to the rational curves of arithmetic genus p, split according to the degree and $\Delta \equiv \sum_{p \geq 0} \Delta_p$. Rational curves in $\Delta_{p,d}$ have p extra double points and their divisors form invariant sets under the action of G. In particular Δ_0 corresponds to smooth rational curves on the cubic. Plane curves of degree $a^0 \geq 1$ with assigned multiplicities a^i at the base points form a projective space of dimension $a^0(a^0+3)/2 - \sum_{1 \leq i \leq 6} a^i(a^i+1)/2 = p_a + d_a - 1$. Imposing p_a extra conditions to reduce their topological genus to 0 leaves a space of rational curves of dimension $d_a - 1 = \omega . a - 1$. Only a finite number N_a of such curves are therefore expected to intersect $d_a - 1$ generic points.

The action of the group G which leaves ω invariant is generated by

(i) space permutations

(ii) reflections of the type $a \to a + (a.e)e$

where e is one of the $\binom{6}{3} = 20$ vectors of square -2 having only the time and 3 space components non vanishing and equal to 1. If $a \in \Delta$ and $\gamma \in G$, the $\gamma(a) \in \Delta$ and $N_a = N_{\gamma(a)}$, a useful check on the forthcoming computation.

As usual, we write the genus 0 free energy $F = f_{\rm cl} + f$, with

$$f_{\rm cl} = \frac{1}{2}(x^2 z + xy.y)$$

$$f = \sum_{a \in \Delta} N_a \frac{z^{d_a - 1}}{(d_a - 1)!} e^{a.y}$$

(3.25)

corresponding to the weights

$$[x] = 1 \quad ; \quad [z] = -1 \quad ; \quad [e^{y^0}] = 3 \quad ; \quad [e^{y^i}] = 1 \quad ; \quad [F] = 1 .$$
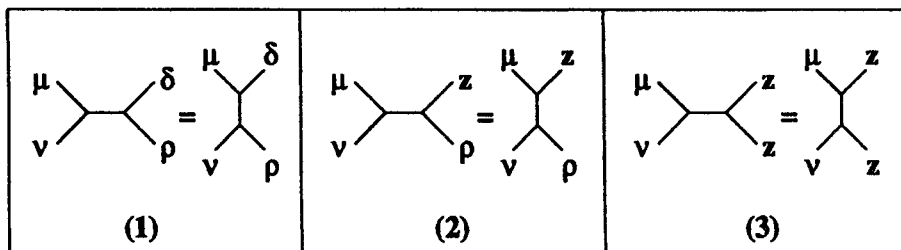
(3.26)

Fig. 5: Duality relations for the lines of $\mathbb{P}_3$.

Associativity of the deformed ring leads to the (G–covariant) conditions

$$\text{(1)}\quad g_{\mu\rho}f_{\nu\delta z} - g_{\nu\rho}f_{\mu\delta z} + g_{\nu\delta}f_{\mu\rho z} - g_{\mu\delta}f_{\rho\nu z} = \sum_{\sigma,\sigma'} g^{\sigma\sigma'}(f_{\mu\delta\sigma}f_{\nu\rho\sigma'} - f_{\nu\delta\sigma}f_{\mu\rho\sigma'})$$

$$\text{(2)}\quad g_{\mu\rho}f_{\nu zz} - g_{\nu\rho}f_{\mu zz} = \sum_{\sigma,\sigma'} g^{\sigma\sigma'}(f_{\mu\sigma z}f_{\nu\rho\sigma'} - f_{\nu\sigma z}f_{\mu\rho\sigma'}) \tag{3.27}$$

$$\text{(3)}\quad g_{\mu\nu}f_{zzz} = \sum_{\sigma,\sigma'} g^{\sigma\sigma'}(f_{\mu\sigma z}f_{\nu\sigma' z} - f_{\mu\nu\sigma}f_{\sigma' zz}) \ .$$

Substituting the expansion of f, we get

Proposition 8. The integers N_a interpreted as numbers of rational curves on a cubic surface belonging to the divisor class a, through $d_a - 1$ points are uniquely determined by the initial condition $N_a = 1$ for $d_a = 1$ and, with $a, b, c \in \Delta$, $a = b + c$

$$\text{(1)}\quad 2(g_{\mu\rho}a_\nu a_\delta - g_{\nu\rho}a_\mu a_\delta + g_{\nu\delta}a_\mu a_\rho - g_{\mu\delta}a_\rho a_\nu)N_a =$$

$$= \sum N_b N_c \binom{d_a - 2}{d_b - 1} b.c(b_\mu c_\nu - b_\nu c_\mu)(b_\delta c_\rho - b_\rho c_\delta)$$

$$\text{(2)}\quad (g_{\mu\rho}a_\nu - g_{\nu\rho}a_\mu)N_a = \sum N_b N_c \binom{d_a - 3}{d_b - 1} b.c(b_\mu c_\nu - b_\nu c_\mu)c_\rho \tag{3.28}$$

$$\text{(3)}\quad g_{\mu\nu}N_a = \sum N_b N_c\, b.c \left[b_\mu c_\nu \binom{d_a - 4}{d_b - 2} - b_\mu b_\nu \binom{d_a - 4}{d_b - 1} \right] \ .$$

For smooth rational curves, $p_a = 0$, i.e. $a \in \Delta_0$, we expect and find that $N_a = 1$. Once more the system of equations for the N_a's is overdetermined but consistent. For explicit calculations it is better to get scalar equations. For instance contracting relation (1) with $\omega^\mu \omega^\rho g^{\nu\delta}$, we find

$$2(5d_a^2 + 3a^2)N_a = \sum N_b N_c \binom{d_a - 2}{d_b - 1} b.c(2d_b d_c b.c - d_b^2 c.c - d_c^2 b.b) \ . \tag{3.29}$$

Starting with the 27 lines of the cubic (a complete orbit of G), the formula (3.29) yields a single conic through a generic point in each of the 27 planes containing this point and one of the 27 lines (again a simple orbit of G). We then tabulate with the same formula all N_a's for $d_a \leq 3$ with divisors all belonging to Δ_0 (for which we indeed find $N_a = 1$), except for $a = \omega$, an orbit of G reduced to a single element, with $N_\omega = 12$, corresponding to rational cubic curves through 2 generic points of the cubic surface. To proceed further it is convenient to take the trace of relation (3)

$$7N_a = \sum N_b N_c b.c \left[b.c \binom{d_a - 4}{d_b - 2} - b.b \binom{d_a - 4}{d_b - 1} \right] \ . \tag{3.30}$$

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	$a.a$	p_a	N_a		Orb.
0	-1	0	0	0	0	0	1	-1	0	1	6	
1	1	1	0	0	0	0	1	-1	0	1	15	27
2	1	1	1	1	1	0	1	-1	0	1	6	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	$a.a$	p_a	N_a		Orb.
1	1	0	0	0	0	0	2	0	0	1	6	
2	1	1	1	1	0	0	2	0	0	1	15	27
3	2	1	1	1	1	1	2	0	0	1	6	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	$a.a$	p_a	N_a		Orb.
1	0	0	0	0	0	0	3	1	0	1	1	
2	1	1	1	0	0	0	3	1	0	1	20	
3	2	1	1	1	1	0	3	1	0	1	30	72
4	2	2	2	1	1	1	3	1	0	1	20	
5	2	2	2	2	2	2	3	1	0	1	1	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	$a.a$	p_a	N_a		Orb.
3	1	1	1	1	1	1	3	3	1	12	1	1

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	$a.a$	p_a	N_a		Orb.
2	1	1	0	0	0	0	4	2	0	1	15	
3	2	1	1	1	0	0	4	2	0	1	60	
4	3	1	1	1	1	1	4	2	0	1	6	216
4	2	2	2	1	1	0	4	2	0	1	60	
5	3	2	2	2	1	1	4	2	0	1	60	
6	3	3	2	2	2	2	4	2	0	1	15	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	$a.a$	p_a	N_a		Orb.
3	1	1	1	1	1	0	4	4	1	12	6	
4	2	2	1	1	1	1	4	4	1	12	15	27
5	2	2	2	2	2	1	4	4	1	12	6	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	$a.a$	p_a	N_a		Orb.
2	1	0	0	0	0	0	5	3	0	1	6	
3	2	1	1	0	0	0	5	3	0	1	60	
4	3	1	1	1	1	0	5	3	0	1	30	
4	2	2	2	1	0	0	5	3	0	1	60	
5	3	2	2	2	1	0	5	3	0	1	120	432
6	4	2	2	2	2	1	5	3	0	1	30	
6	3	3	3	2	1	1	5	3	0	1	60	
7	4	3	3	2	2	2	5	3	0	1	60	
8	4	3	3	3	3	3	5	3	0	1	6	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	$a.a$	p_a	N_a		Orb.
3	1	1	1	1	0	0	5	5	1	12	15	
4	2	2	1	1	1	0	5	5	1	12	60	
5	3	2	2	1	1	1	5	5	1	12	60	216
5	2	2	2	2	2	0	5	5	1	12	6	
6	3	3	2	2	2	1	5	5	1	12	60	
7	3	3	3	3	2	2	5	5	1	12	15	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	$a.a$	p_a	N_a		Orb.
4	2	1	1	1	1	1	5	7	2	96	6	
5	2	2	2	2	1	1	5	7	2	96	15	27
6	3	2	2	2	2	2	5	7	2	96	6	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	$a.a$	p_a	N_a		Orb.
2	0	0	0	0	0	0	6	4	0	1	1	
4	2	2	2	0	0	0	6	4	0	1	20	72
6	4	2	2	2	0	0	6	4	0	1	30	
8	4	4	4	2	2	2	6	4	0	1	20	
10	4	4	4	4	4	4	6	4	0	1	1	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	$a.a$	p_a	N_a		Orb.
3	2	1	0	0	0	0	6	4	0	1	30	
4	3	1	1	1	0	0	6	4	0	1	60	
5	4	1	1	1	1	1	6	4	0	1	6	
5	3	2	2	2	0	0	6	4	0	1	60	432
6	3	3	3	2	1	0	6	4	0	1	120	
7	5	2	2	2	2	2	6	4	0	1	6	
7	4	3	3	3	1	1	6	4	0	1	60	
8	5	3	3	3	2	2	6	4	0	1	60	
9	5	4	3	3	3	3	6	4	0	1	30	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	$a.a$	p_a	N_a		Orb.
3	1	1	1	0	0	0	6	6	1	12	20	
4	2	2	1	1	0	0	6	6	1	12	90	
5	3	2	2	1	1	0	6	6	1	12	180	
6	3	3	3	1	1	1	6	6	1	12	20	
6	3	3	2	2	2	0	6	6	1	12	60	720
6	4	2	2	2	1	1	6	6	1	12	60	
7	4	3	3	2	2	1	6	6	1	12	180	
8	4	4	3	3	2	2	6	6	1	12	90	
9	4	4	4	3	3	3	6	6	1	12	20	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	$a.a$	p_a	N_a		Orb.
4	2	1	1	1	1	0	6	8	2	96	30	
5	2	2	2	2	1	0	6	8	2	96	30	
5	3	2	1	1	1	1	6	8	2	96	30	
6	3	3	2	2	1	1	6	8	2	96	90	270
7	4	3	2	2	2	2	6	8	2	96	30	
7	3	3	3	3	2	1	6	8	2	96	30	
8	4	3	3	3	3	2	6	8	2	96	30	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	$a.a$	p_a	N_a		Orb.
4	1	1	1	1	1	1	6	10	3	620	1	
5	2	2	2	1	1	1	6	10	3	620	20	
6	3	2	2	2	2	1	6	10	3	620	30	72
7	3	3	3	2	2	2	6	10	3	620	20	
8	3	3	3	3	3	3	6	10	3	620	1	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	$a.a$	p_a	N_a		Orb.
6	2	2	2	2	2	2	6	12	4	2376	1	1

Table II: Rational curves on a cubic surface. We display respectively the divisor classes, degree, self–intersection, arithmetic genus, N_a, number of conjugates under space permutation, order of the G–orbit.

Table II exhibits N_a for divisors in the form $a^0 > a^1 \geq a^2 \geq ... \geq a^6 \geq 0$ up to degree 6, with the number of distinct equivalent ones under space permutations recorded in the column before the last and grouped in complete orbits under G. The last column yields the order of the orbit. We observe that the 72 divisors of degree 3 and $p_a = 0$, $N_a = 1$ form a single orbit of G. Indeed [5] they are in one to one correspondence with the system $\mathcal{R}$ of 72 non zero roots of E_6 ($\dim(E_6) = 72 + 6 = 78$) $\mathcal{R} = \{a \in \mathbb{Z}^7 | \omega.a = 0, a.a = -2\}$, and $\Delta_{3,0} = \mathcal{R} + \omega$. Some numbers are recognizable. For instance for $a = \omega$, $d_a = 3$ interpreted in the plane as uninodal cubics through $6 + (3 - 1) = 8$ points, we find $N_a = 12$ as in section 2. Similarly for $a = \{4,1,1,1,1,1,1\}$, $d_a = 6$ interpreted in the plane as 3–nodal quartics through $6 + (6 - 1) = 11$ points, we recover $N_a = 620$ again as in section 2.

One could similarly discuss other Del Pezzo surfaces in particular those with intersection form on divisor classes invariant under the Weyl groups of E_7 and E_8.

3.4. Lines in $\mathbb{P}_3$.

As an example of Grassmannian we shall content ourselves to consider the 4 dimensional Plücker quadric in $\mathbb{P}_5$, corresponding to the variety of lines in $\mathbb{P}_3$. In physicist's notation, this is also the (complex) electromagnetic fields up to scale $F \equiv (\vec{E}, \vec{B})$, such that $\vec{E}.\vec{B} = 0$. If a line in $\mathbb{P}_3$ is given by 2 points with homogeneous coordinates (x_0, x_1, x_2, x_3) and (y_0, y_1, y_2, y_3), the coordinates of the line are $F_{\mu\nu} = x_\mu y_\nu - x_\nu y_\mu$, $0 \leq \mu < \nu \leq 3$, i.e.

$E_i = F_{0i}$ and $B_i = F_{jk}$, where i, j, k is a cyclic permutation of $1, 2, 3$. One checks that $\vec{E}.\vec{B} = 0$ is the necessary and sufficient condition that F lies on the Plücker quadric Q in $\mathbb{P}_5$ [11]. The cohomology ring is spanned by t_0, t_1, t_{2a}, t_{2b}, t_3 and t_4 dual to divisors given in the table below, where we first give an interpretation of the cycles as lines in $\mathbb{P}_3$, then as points on the quadric Q.

	dim.	lines in $\mathbb{P}_3$	points on Q
t_0	4	all lines	Q
t_1	3	lines meeting a line	hyperplane sect. of Q
t_{2a}	2	lines through a point	α–plane $\subset Q$
t_{2b}	2	lines in a plane	β–plane $\subset Q$
t_3	1	lines in a plane through a point	line $\subset Q$
t_4	0	fixed line	point $\in Q$

Table III: Homology cycles for the lines of $\mathbb{P}_3$. We display the cohomology basis, dimension of the divisor, and the dual cycles respectively as lines in $\mathbb{P}_3$, and on the Plücker quadric in $\mathbb{P}_5$.

The classical ring with unit t_0 is given by the relations

$$t_1^2 = t_{2a} + t_{2b}$$

$$t_1 t_{2a} = t_1 t_{2b} = t_3 \tag{3.31}$$

$$t_1 t_3 = t_{2a}^2 = t_{2b}^2 = t_4 \ ,$$

all other products being 0. Consequently the non vanishing elements of the intersection form are $\eta_{04} = \eta_{13} = \eta_{2a,2a} = \eta_{2b,2b} = 1$. We introduce deformation parameters y_0, y_1, y_{2a}, y_{2b}, y_3 and y_4 and the genus 0 free energy F, with weights

$$[y_0] = 1 \ ; \ [e^{y_1}] = 4 \ ; \ [y_{2a}] = [y_{2b}] = -1 \ ; \ [y_3] = -2 \ ; \ [y_4] = -3 \ ; \ [F] = -1 \ . \tag{3.32}$$

As usual the free energy is split into $F = f_{cl} + f$ with

$$f_{cl} = \frac{1}{2} y_0 (y_0 y_4 + y_{2a}^2 + y_{2b}^2) + y_0 y_1 y_3 + \frac{1}{2} y_1^2 (y_{2a} + y_{2b})$$

$$f = \sum_{\substack{\alpha,\beta,\gamma,\delta \geq 0 \\ \alpha+\beta+2\gamma+3\delta=4d+1}} N(\alpha,\beta,\gamma,\delta|d) \frac{y_{2a}^\alpha}{\alpha!} \frac{y_{2b}^\beta}{\beta!} \frac{y_3^\gamma}{\gamma!} \frac{y_4^\delta}{\delta!} e^{dy_1} \ . \tag{3.33}$$

To confirm this assignment of weights, we note that there is a single linear pencil of lines $(d = 1)$, i.e. a single set of lines in a plane through a point, containing a fixed line l_0 and intersecting another generic such pencil in one line. Indeed the former is the set of lines through a point on l_0 contained in a plane through l_0. To intersect a pencil given by the set of lines in a plane π_0 through a point $p_0 \in \pi_0$, in a single line l_1, we consider the point $p_1 = \pi_0.l_0$, the line $l_1 = [p_0, p_1]$ and the plane $\pi_1 = [l_0, l_1]$, then the pencil looked for is the set of lines through p_1 in π_1. Hence we find $N(0,0,1,1|1) = 1$, and obtain a first relation on the weights $[F] = [y_3] + [y_4] + [e^{y_1}]$.

Curves on Q correspond to ruled surfaces in $\mathbb{P}_3$. The integers $N(\alpha, \beta, \gamma, \delta|d)$ are interpreted as counting the number of rational ruled surfaces of degree d subject to auxiliary relations dictated by the exponents α, β, γ, δ in terms of their ruling. For instance we expect a single quadric ($\mathbb{P}_1 \times \mathbb{P}_1$) through 3 lines in general position, i.e. $N(0,0,0,3|2) = 1$ which implies that $[F] = 3[y_4] + 2[e^{y_1}]$. With the assignments $[y_0] = 1$, $[y_{2a,b}] = -1$, $[y_3] = -2$ and $[y_4] = -3$, these relations specify the weights of F and e^{y_1} to be -1 and 4 respectively. We also have the obvious symmetry $N(\alpha, \beta, \gamma, \delta|d) = N(\beta, \alpha, \gamma, \delta|d)$.

We will not write explicitly the 15 relations expressing the associativity of the deformed ring and their symmetric counterparts. It is interesting to note however than only 5 of them are sufficient to completely specify the numbers $N(\alpha, \beta, \gamma, \delta|d)$. We content ourselves with the following

Proposition 9. For the Grassmannian of lines in $\mathbb{P}_3$ the associativity conditions, together with the initial condition $N(0,0,1,1|1) = 1$ determine recursively all $N(\alpha, \beta, \gamma, \delta|d)$. The

first few N's for $d \leq 2$ read (we denote for short $N(\alpha, \beta, \gamma, \delta | d) \equiv N(\alpha, \beta, \gamma, \delta)$)

d=1 : $N(5,0,0,0) = 0$ $N(4,1,0,0) = 0$ $N(3,2,0,0) = 1$ $N(3,0,1,0) = 0$

        $N(2,1,1,0) = 1$ $N(1,0,2,0) = 1$ $N(2,0,0,1) = 0$ $N(1,1,0,1) = 1$

        $N(0,0,1,1) = 1$

d=2 : $N(9,0,0,0) = 2$ $N(8,1,0,0) = 6$ $N(7,2,0,0) = 18$ $N(6,3,0,0) = 34$

        $N(5,4,0,0) = 42$ $N(7,0,1,0) = 3$ $N(6,1,1,0) = 9$ $N(5,2,1,0) = 17$

        $N(4,3,1,0) = 21$ $N(5,0,2,0) = 5$ $N(4,1,2,0) = 9$ $N(3,2,2,0) = 11$

        $N(3,0,3,0) = 5$ $N(2,1,3,0) = 6$ $N(1,0,4,0) = 3$ $N(6,0,0,1) = 1$

        $N(5,1,0,1) = 3$ $N(4,2,0,1) = 5$ $N(3,3,0,1) = 5$ $N(4,0,1,1) = 2$

        $N(3,1,1,1) = 3$ $N(2,2,1,1) = 3$ $N(2,0,2,1) = 2$ $N(1,1,2,1) = 2$

        $N(0,0,3,1) = 1$ $N(3,0,0,2) = 1$ $N(2,1,0,2) = 1$ $N(1,0,1,2) = 1$

        $N(0,0,0,3) = 1$

$$(3.34)$$

We get a number of enumerative data on quadrics. We recover $N(0,0,0,3) = 1$. Also $N(9,0,0,0) = N(0,9,0,0) = 2$ correspond to the fact that a quadric is uniquely fixed by 9 points or dually by 9 tangent planes, and that there are two rulings on such a quadric. Similarly there are 3 quadrics through 8 points tangent to a plane. Indeed through 8 points a linear pencil of quadrics cuts a plane in a linear pencil of conics through 4 points, which degenerates in 3 ways in a pair of lines corresponding to a quadric tangent to the plane. Since again a quadric has two rulings, this explains $N(8,1,0,0) = N(1,8,0,0) = 2 \times 3 = 6$. Similarly $N(7,0,1,0) = N(0,7,1,0) = 3$ since now we have quadrics through 8 points tangent to a plane but one of the rulings is selected. One can similarly check part of our results for $d = 2$ against table VI on page 329 of the book by Semple and Roth [9], namely there are 9 quadrics through 7 points and tangent to 2 planes ($N(7,2,0,0) = 2N(6,1,1,0) = 18$), 17 quadrics through 6 points and tangent to 3 planes ($N(6,3,0,0) = 2N(5,2,1,0) = 34$), and 21 quadrics through 5 points and tangent to 4 planes ($N(5,4,0,0) = 2N(4,3,1,0) = 42$).

Remark. Consider the deformed ring when $y_0 = y_{2a} = y_{2b} = y_3 = y_4 = 0$. Setting

$e^{y_1} = q^4$, the multiplication table of the deformed ring, with identity T_0, reduces to

$$T_1^2 = T_{2a} + T_{2b} \quad , \quad T_1 T_{2a} = T_1 T_{2b} = T_3$$

$$T_1 T_3 = T_4 + q^4 T_0 \quad , \quad T_1 T_4 = q^4 T_1$$

$$T_{2a}^2 = T_{2b}^2 = T_4 \quad , \quad T_{2a} T_{2b} = q^4 T_0$$

$$T_{2a} T_3 = T_{2b} T_3 = 0 \quad , \quad T_{2a} T_4 = q^4 T_{2b} \tag{3.35}$$

$$T_{2b} T_4 = q^4 T_{2a} \quad , \quad T_3^2 = 0$$

$$T_3 T_4 = q^4 T_3 \quad , \quad T_4^2 = q^8 T_0 \; ,$$

since the only surviving third derivatives of f are $f_{134} = f_{2a,2b,4} = q^4$ and $f_{444} = q^8$. Indeed in the sum over non negative exponents the condition $\alpha + \beta + 2\gamma + 3\delta = 4d + 1$ entails that for $d > 2$, $\alpha + \beta + \gamma + \delta > 3$, and for $d = 2$ the only possibility for $\alpha + \beta + \gamma + \delta \leq 3$ is $\alpha = \beta = \gamma = 0$, and $\delta = 3$.

The intermediate ring (3.35) for the Grassmannian $G(2,4)$ of lines in $\mathbb{P}_3$ was considered by Witten [13], elaborating earlier work by Gepner, in relation to Verlinde's formula. It is identified as the ring of symmetric functions in two variables quotiented by the gradient of a symmetric polynomial as follows. Let the two variables be λ_1, λ_2, and $s = \lambda_1 + \lambda_2$, $p = \lambda_1 \lambda_2$. The polynomial ring of symmetric functions in λ_1, λ_2 is $\mathbb{C}[s,p]$. With $W \in \mathbb{C}[s,p]$ defined as

$$W(s,p) = \frac{\lambda_1^5 + \lambda_2^5}{5} + q^4(\lambda_1 + \lambda_2) \; . \tag{3.36}$$

the corresponding Gepner–Witten ring is $\mathbb{C}[s,p]/\mathrm{grad}W(s,p)$. The gradient conditions read

$$\frac{\partial W}{\partial s} = \frac{\lambda_1^5 - \lambda_2^5}{\lambda_1 - \lambda_2} + q^4 = s^4 - 3ps^2 + p^2 + q^4$$

$$\frac{\partial W}{\partial p} = \frac{\lambda_1^4 - \lambda_2^4}{\lambda_1 - \lambda_2} = s^3 - 2ps \tag{3.37}$$

So we can also write $\mathbb{C}[s,p]/\{s^4 - 3ps^2 + p^2 + q^4, s^3 - 2ps\}$. When $q = 0$, the Hilbert polynomial for this ring is

$$\frac{(1 - t^3)(1 - t^4)}{(1 - t)(1 - t^2)} = 1 + t + 2t^2 + t^3 + t^4 \tag{3.38}$$

indeed the same as for the classical Grassmannian. The correspondence

$$T_0 \leftrightarrow 1 \quad ; \quad T_1 \leftrightarrow s \quad ; \quad T_{2a} \leftrightarrow s^2 - p$$

$$T_{2b} \leftrightarrow p \quad ; \quad T_3 \leftrightarrow sp \quad ; \quad T_4 \leftrightarrow p^4 \tag{3.39}$$

identifies the two rings as the reader can easily check (the roles of T_{2a} and T_{2b} could be permuted).

3.5. Flags.

The flag manifold $\mathcal{F}_n$ in $\mathbb{P}_n$ is the set of sequences formed by a point on a line in a plane ... Viewing $\mathbb{P}_n$ as $\mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*$, it may be identified either as GL_{n+1}/B_{n+1}, quotient of the linear group by the Borel subgroup of upper triangular matrices, which exhibits $\mathcal{F}_n$ as a complex variety, or equivalently as U_{n+1}/U_1^{n+1} proving that it is compact, with complex dimension $\dim(\mathcal{F}_n) = n(n+1)/2$. Its intersection ring is generated by divisors u_k, $k = 1, ..., n$, corresponding to the n cycles of codimension 1, $C_k \equiv$ *the linear* $(k-1)$ *subvariety in the flag intersects a fixed linear subvariety of codimension k*. For the following presentation see the book of Macdonald [14], and references therein. Set $u_0 = u_{n+1} = 0$. It is convenient to introduce $x_r = u_{r+1} - u_r$ for $r = 0, 1, ..., n$ with the relation $\sum_{0 \le r \le n} x_r = 0$, or equivalently $u_k = \sum_{0 \le r \le k-1} x_r$ for $k = 1, 2, ..., n$. Let $\mathbb{C}[x_0, ..., x_n]$ be the (graded) polynomial ring on x_0, ..., x_n, with x_k of grade 1, and $\mathcal{S}[x_0, ..., x_n]$ the ideal generated by the elementary symmetric functions in x_0, ..., x_n,

$$H^*(\mathcal{F}_n) = \mathbb{C}[x_0, ..., x_n]/\mathcal{S}[x_0, ..., x_n] . \tag{3.40}$$

As a $\mathbb{Z}$-module a basis is given by the so-called "Schubert polynomials" indexed by permutations of $(n+1)$ objects (group S_{n+1}). The Hilbert polynomial counting the number of elements in the ring of given degree is

$$P(x) = \frac{1}{(1-x)^{n+1}}(1-x)(1-x^2)...(1-x^{n+1})$$
$$= (1+x)(1+x+x^2)...(1+x+x^2+....+x^n) , \tag{3.41}$$

expressing the fact that we have one relation in each degree from 1 to $(n+1)$ (given by the elementary symmetric functions in x_0, ..., x_n), and showing that the dimension of the ring is $(n+1)!$. Let $p(x_0, ..., x_n)$ and $q(x_0, ..., x_n)$ be two polynomials representative of classes in H^* of eq.(3.40). The group S_{n+1} acts by permuting the arguments, and this action is still well defined on the quotient (3.40). The intersection form on the ring is then obtained as

$$\langle p, q \rangle = \oint \frac{dx_0}{2i\pi x_0} ... \oint \frac{dx_n}{2i\pi x_n} \frac{A[pq](x_0, ..., x_n)}{\prod_{0 \le i < j \le n}(x_i - x_j)} , \tag{3.42}$$

where, for any polynomial $r(x_0, ..., x_n)$, $A[r]$ denotes the antisymmetrized polynomial $A[r](x_0, ..., x_n) \equiv \sum_{\sigma \in S_{n+1}} \epsilon(\sigma) r(x_{\sigma(0)}, ..., x_{\sigma(n)})$, $\epsilon(\sigma)$ the signature of the permutation σ. The integrals in (3.42) are over the unit circle. For instance, a representative of top degree $n(n+1)/2$ is $p_{max} = x_0^n x_1^{n-1} x_2^{n-2} ... x_n^0$, while a representative of lowest (0) degree
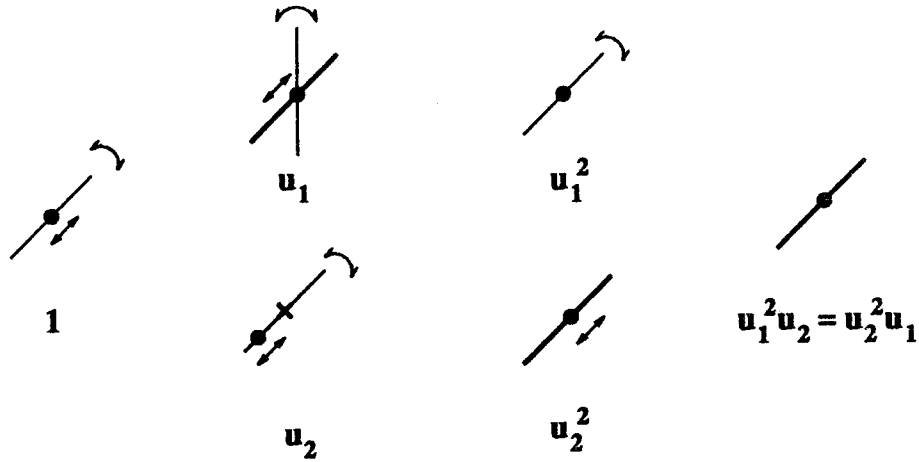
Fig. 6: Cycles in the flags of $\mathbb{P}_2$. Each drawing is indexed by the corresponding dual cohomology class. Arrows indicate the freedom of the flag cycles.

is 1, with $\langle 1, p_{max} \rangle = 1$. Rather than developing the general theory let us now concentrate on the first non–trivial case i.e. the flags of $\mathbb{P}_2$, in which case we have the generators

$u_1 = x_0$: *the point lies on a fixed line*,

$u_2 = x_0 + x_1$: *the line goes through a fixed point*,

and one adds x_2 such that $x_0 + x_1 + x_2 = 0$.

The number of elements in the ring $\mathbb{C}[x_0, x_1, x_2]/\mathcal{S}[x_0, x_1, x_2]$ is 6, with basis

$$1 \qquad \begin{matrix} x_0 \\ x_0 + x_1 \end{matrix} \qquad \begin{matrix} x_0^2 \\ x_0 x_1 \end{matrix} \qquad x_0^2 x_1 \tag{3.43}$$

i.e., as depicted on Fig.6,

$$1 \left\{ \begin{matrix} any \\ flag \end{matrix} \right\} \quad \begin{matrix} u_1 \\ \\ u_2 \end{matrix} \quad \begin{matrix} \left\{ \begin{matrix} the\ point\ lies \\ on\ a\ line \end{matrix} \right\} \\ \left\{ \begin{matrix} the\ line\ goes \\ through\ a\ point \end{matrix} \right\} \end{matrix} \quad \begin{matrix} u_1^2\ \{the\ point\ is\ fixed\ \} \\ \\ u_2^2\ \ \{the\ line\ is\ fixed\} \end{matrix} \quad u_1^2 u_2\ \{the\ flag\ is\ fixed\} \tag{3.44}$$

The relations read

$$x_0 + x_1 + x_2 = 0 \quad ; \quad x_0 x_1 + x_1 x_2 + x_2 x_0 = 0 \quad ; \quad x_0 x_1 x_2 = 0 \tag{3.45}$$

or eliminating x_2

$$x_0^2 + x_0 x_1 + x_1^2 = 0 \quad \text{and} \quad x_0^2 x_1 + x_1^2 x_0 = 0, \tag{3.46}$$

which entails $x_0^3 = x_1^3 = x_2^3 = 0$. Expressed in u_1, u_2, this yields

$$u_1 u_2 = u_1^2 + u_2^2 \qquad u_1^2 u_2 = u_2^2 u_1 \tag{3.47}$$

and therefore $u_1^3 = u_2^3 = 0$. The non–vanishing intersections are

$$\langle 1, x_0^2 x_1 \rangle = \langle x_0, x_0 x_1 \rangle = \langle x_0 + x_1, x_0^2 \rangle = 1$$
$$\langle 1, u_1^2 u_2 \rangle = \langle u_2, u_1^2 \rangle = \langle u_1, u_2^2 \rangle = 1 \ , \tag{3.48}$$

which are readily checked geometrically, using the representations of Fig.6.

The flag manifold of $\mathbb{P}_2$ is 3 dimensional and can also be described by the incidence relation of points and lines in $\mathbb{P}_2 \times {}^*\mathbb{P}_2$ (where ${}^*\mathbb{P}_2$ stands for the dual plane). If x, y, z (resp. X, Y, Z) stand for the coordinates in $\mathbb{P}_2$ (resp. ${}^*\mathbb{P}_2$), we have a "quadric" in $\mathbb{P}_2 \times {}^*\mathbb{P}_2$ of equation $xX + yY + zZ = 0$. Rational curves in this space have a bidegree (a, b) for their projections on the two planes. Such a curve is described by three homogeneous polynomials of degree a (resp. b) in two variables (u, v) parametrizing $\mathbb{P}_1$, namely $x(u,v)$, $y(u,v)$ and $z(u,v)$ (resp. $X(u,v)$, $Y(u,v)$, $Z(u,v)$), modulo the action of PSL_2 on each triple of polynomials, and subject to the incidence condition $xX + yY + zZ = 0$. The number of free parameters is therefore

$$[3(a+1) - 3] + [3(b+1) - 3] - [(a+b+1) - 1] = 2(a+b) \ . \tag{3.49}$$

To fix the curve we therefore need $2(a + b)$ conditions. Intersection with a codimension 2 cycle in the classes u_1^2 or u_2^2 counts as one condition, whereas intersection with a codimension 3 cycle (in the class $u_1^2 u_2$) i.e. a fixed flag counts for two conditions. Using these remarks, one can generate the deformed ring. With obvious notations, let T_0, T_{1a}, T_{1b}, T_{2a}, T_{2b} and T_3 correspond to the deformations of 1, u_1, u_2, u_1^2, u_2^2 and $u_1^2 u_2 = u_2^2 u_1$ respectively. The associated parameters y_0, y_{1a}, y_{1b}, y_{2a}, y_{2b} and y_3 and the genus 0 free energy F have weights

$$[y_0] = 1 \ ; \ [e^{y_{1a}}] = [e^{y_{1b}}] = 2 \ ; \ [y_{2a}] = [y_{2b}] = -1 \ ; \ [y_3] = -2 \ ; \ [F] = 0 \ . \tag{3.50}$$

From the preceding, we have $F = f_{cl} + f$, with

$$f_{cl} = \frac{y_0^2}{2} y_3 + y_0(y_{1a} y_{2b} + y_{1b} y_{2a}) + \frac{1}{2} y_{1a} y_{1b}(y_{1a} + y_{1b})$$

$$f = \sum_{\substack{\lambda+\mu+2\nu=2(a+b) \\ \lambda,\mu,\nu,a,b \geq 0; a+b \geq 1}} N(\lambda, \mu, \nu | a, b) \frac{y_{2a}^\lambda}{\lambda!} \frac{y_{2b}^\mu}{\mu!} \frac{y_3^\nu}{\nu!} e^{a y_{1a} + b y_{1b}} \tag{3.51}$$

Obviously $N(\lambda, \mu, \nu | a, b) = N(\mu, \lambda, \nu | b, a)$, reflecting the duality between point and line.

A curve in the flag space such that the line is fixed intersects a generic cycle in the class u_1 in a single flag, and does not intersect a generic cycle in u_2. Hence its bidegree is $(1,0)$. It is fixed if we require that it contains a fixed flag (which fixes the line), and obviously does not intersect generic cycles in the classes u_1^2 or u_2^2, hence

$$N(0,0,1|1,0) = N(0,0,1|0,1) = 1 .$$

(3.52)

Similarly a curve of bidegree $(1,0)$ required to intersect two cycles in the class u_1^2, which fixes 2 distinct points, hence the line, is uniquely determined, hence

$$N(2,0,0|1,0) = N(0,2,0|0,1) = 1 .$$

(3.53)

We refrain again from writing explicitly all associativity conditions for the deformed intersection ring. We checked up to $a + b \leq 5$ that they were all consistent and claim

Proposition 10. Either initial condition (3.52) or (3.53), together with the associativity conditions, determine uniquely the integers $N(\lambda, \mu, \nu|a, b)$, and one finds $(d = a + b)$

$\mathbf{d=1}$: $N(2,0,0|1,0) = 1$ $N(0,0,1|1,0) = 1$

$\mathbf{d=2}$: $N(0,0,2|1,1) = 1$ $N(1,1,1|1,1) = 1$ $N(2,2,0|1,1) = 1$

$\mathbf{d=3}$: $N(2,0,2|2,1) = 1$ $N(4,0,1|2,1) = 1$ $N(3,1,1|2,1) = 1$ $N(5,1,0|2,1) = 2$

$\qquad N(4,2,0|2,1) = 1$

$\mathbf{d=4}$: $N(4,0,2|3,1) = 1$ $N(6,0,1|3,1) = 4$ $N(5,1,1|3,1) = 1$ $N(8,0,0|3,1) = 12$

$\qquad N(7,1,0|3,1) = 6$ $N(6,2,0|3,1) = 1$ $N(0,0,4|2,2) = 1$ $N(2,0,3|2,2) = 1$

$\qquad N(1,1,3|2,2) = 2$ $N(3,1,2|2,2) = 2$ $N(2,2,2|2,2) = 3$ $N(4,2,1|2,2) = 4$

$\qquad N(3,3,1|2,2) = 5$ $N(5,3,0|2,2) = 8$ $N(4,4,0|2,2) = 10$

$\mathbf{d=5}$: $N(10,0,0|4,1) = 60$ $N(9,1,0|4,1) = 12$ $N(8,2,0|4,1) = 1$ $N(8,0,1|4,1) = 9$

$\qquad N(7,1,1|4,1) = 1$ $N(6,0,2|4,1) = 1$ $N(8,2,0|3,2) = 108$ $N(7,3,0|3,2) = 150$

$\qquad N(6,4,0|3,2) = 96$ $N(5,5,0|3,2) = 32$ $N(7,1,1|3,2) = 36$ $N(6,2,1|3,2) = 60$

$\qquad N(5,3,1|3,2) = 40$ $N(4,4,1|3,2) = 16$ $N(6,0,2|3,2) = 12$ $N(5,1,2|3,2) = 26$

$\qquad N(4,2,2|3,2) = 18$ $N(3,3,2|3,2) = 8$ $N(4,0,3|3,2) = 12$ $N(3,1,3|3,2) = 9$

$\qquad N(2,2,3|3,2) = 4$ $N(2,0,4|3,2) = 5$ $N(1,1,4|3,2) = 2$ $N(0,0,5|3,2) = 1$

(3.54)

Apart from the two simple sets of initial conditions (3.52)-(3.53), we recognize a few numbers. Take for instance $N(8,0,0|3,1) = 12$. We deal with a curve in flag space intersecting in 3 flags a cycle in the class u_1. Clearly the projection from our curve to the first $\mathbb{P}_2$ should be a uninodal cubic (since our curves are rational). Moreover, this cubic should pass through 8 points ($\lambda = 8$). Now the curve in flag space should be such that as the point varies on the cubic with accompanying line, it intersects in a single flag a cycle in the class u_2. This forces the line to pass through the node of the cubic. Indeed given a cycle in the class u_2, with lines through a fixed point of the plane, we join this point to the node to obtain the line of the desired flag, its point being the only other intersection of this line with the cubic. Finally it is readily seen that such a family does not intersect a generic cycle in either u_1^2 (the point is fixed) or u_2^2 (the line is fixed), hence $\lambda = \mu = 0$, a fortiori it does not pass through an arbitrarily fixed flag, $\gamma = 0$. Now we know from section 2 that there are $N_3 = 12$ uninodal cubics through 8 generic points of the plane, confirming $N(8,0,0|3,1) = 12$.

Finally, following the remark at the end of the previous subsection, we exhibit the deformed ring multiplication table, when restricted to the plane $y_0 = y_{2a} = y_{2b} = y_3 = 0$, setting $e^{y_{1a}} = q_a^2$ and $e^{y_{1b}} = q_b^2$, and with unit T_0

$$T_{1a}^2 = T_{2a} + q_a^2 T_0 \quad ; \quad T_{1a}T_{1b} = T_{2a} + T_{2b} \quad ; \quad T_{1a}T_{2a} = q_a^2 T_{1b}$$

$$T_{1a}T_{2b} = T_{1b}T_{2a} = T_3 \quad ; \quad T_{1a}T_3 = q_a^2 T_{2b} + q_a^2 q_b^2 T_0 \quad ; \quad T_{1b}^2 = T_{2b} + q_b^2 T_0$$

$$T_{1b}T_{2b} = q_b^2 T_{1a} \quad ; \quad T_{1b}T_3 = q_b^2 T_{2a} + q_a^2 q_b^2 T_0 \quad ; \quad T_{2a}^2 = q_a^2 T_{2b} \tag{3.55}$$

$$T_{2a}T_{2b} = q_a^2 q_b^2 T_0 \quad ; \quad T_{2a}T_3 = q_a^2 q_b^2 T_{1a} \quad ; \quad T_{2b}^2 = q_b^2 T_{2a}$$

$$T_{2b}T_3 = q_a^2 q_b^2 T_{1b} \quad ; \quad T_3^2 = q_a^2 q_b^2 (T_{2b} + T_{2a}) \, .$$

This reduced ring may be interpreted as $\mathbb{C}[u_1, u_2]/\mathcal{I}$, where the ideal $\mathcal{I}$ is generated by the polynomials

$$
\begin{aligned}
u_1^2 + u_2^2 - u_1 u_2 - q_a^2 - q_b^2 \\
u_1 u_2^2 - u_2 u_1^2 - q_b^2 u_1 + q_a^2 u_2
\end{aligned}
\tag{3.56}
$$

with the identifications

$$T_{1a} \leftrightarrow u_1$$

$$T_{1b} \leftrightarrow u_2$$

$$T_{2a} \leftrightarrow u_1^2 - q_a^2 \tag{3.57}$$

$$T_{2b} \leftrightarrow u_2^2 - q_b^2$$

$$T_3 \leftrightarrow u_1 u_2^2 - q_b^2 u_1 \equiv u_2 u_1^2 - q_a^2 u_2 \, .$$

Unfortunately T_{1a} and T_{1b} satisfy now sixth degree equations (instead of a third degree one in the classical case $u_1^3 = u_2^3 = 0$)

$$(u_1^2 - q_a^2)^3 = q_a^4 q_b^2 \qquad (u_2^2 - q_b^2)^3 = q_a^2 q_b^4 \tag{3.58}$$

and therefore it is impossible (even for $q_a = q_b$) to write it as a deformation of $\mathbb{C}[x_0, x_1, x_2]/\mathcal{S}[x_0, x_1, x_2]$, insisting that the elementary symmetric functions of x_0, x_1, x_2 take assigned values. Rather the ideal $\mathcal{I}$ of (3.56) is the one of the six points of intersection of a conic and a cubic in the affine plane (u_1, u_2), with coordinates

$$u_1 = q_a \left[1 + \left(q_b^2/q_a^2 \right)^{1/3} \right]^{1/2} \qquad u_2 = u_1 \left(\frac{u_1^2}{q_a^2} - 1 \right). \tag{3.59}$$

The six determinations arise from the cubic and square roots, while u_2 is rational in u_1. Thus the reduced ring is also of the form $\mathbb{C}[x]/P(x)$ with

$$P(x) = (x^2 - q_a^2)^3 - q_a^4 q_b^2 \tag{3.60}$$

and

$$
\begin{aligned}
u_1 &\to x \\
u_2 &\to x \left(\frac{x^2}{q_a^2} - 1 \right)
\end{aligned}
\tag{3.61}
$$

or equivalently

$$
\begin{aligned}
x_0 &\to x \\
x_1 &\to x \left(\frac{x^2}{q_a^2} - 2 \right) \\
x_2 &\to x \left(\frac{x^2}{q_a^2} - 1 \right),
\end{aligned}
\tag{3.62}
$$

i.e. a specific deformation of the A_5 singularity. The six points in (3.59) are not in general position since they lie on a conic, i.e. a rational curve, explaining why the ring can be expressed in terms of the roots of a sixth–degree polynomial in a single variable.

4. Comments and questions

In this (in)conclusive section, we add a few remarks, questions and results from the literature.

4.1. Manifolds with rational curves

In a systematic discussion we should have commented on the type of generic projective varieties for which one would expect to enumerate rational curves. For simplicity we restrict ourselves to irreducible hypersurfaces X of degree d in $\mathbb{P}_n$. It is known [11] that the variety $\mathcal{F}_k(X)$ of k-planes in X has dimension

$$dim\mathcal{F}_k(X) = (n - k)(k + 1) - \binom{d + k}{k}$$ (4.1)

More precisely for $d > 2$, when this number is non negative, it is the correct dimension. There are exceptions for $d = 2$. Applying this to lines, $k = 1$, we get

$$dim\mathcal{F}_1(X) = 2n - 3 - d$$ (4.2)

So one expects to find lines up to degree $d = 2n - 3$, i.e. up to cubics (27 lines) in $\mathbb{P}_3$, or quintics (2875 lines) for threefolds in $\mathbb{P}_4$, the famous example for mirror symmetry, etc... A general formula for the number L_d of lines on a generic hypersurface of degree $2n - 3$ in $\mathbb{P}_n$ is due to Harris [15]

$$L_d = d \times d! \sum_{k=0}^{n-2} \frac{(2k)!}{(k+1)!k!} \sum_{I_k} \prod_{i \in I_k} \frac{(d - 2i)^2}{i(d - i)}$$ (4.3)

where I_k runs over subsets of $\{1, 2, ..., n - 2\}$ with $n - k - 2$ elements and an empty product is equal to 1.

The above does not mean that smooth hypersurfaces in $\mathbb{P}_n$ of degree larger than $2n - 3$ do not possess lines, rather such hypersurfaces form a positive codimension submanifold in the space of degree $2n - 3$ hypersurfaces (i.e. they are not generic). On the other hand generic hypersurfaces of higher degree might still possess rational curves (but these will not be lines).

As an example take the generic smooth quartic surface in $\mathbb{P}_3$, a typical case of a K_3 surface. It is known that all curves on such a surface are complete intersections with another surface – hence, by Bezout, have degrees multiple of 4. The dimension of the space of curves of degree $4d$ is then expected to be $h_d - 1$, where h_d is the dimension of the space of homogeneous polynomials in 4 variables modulo those that vanish on the quartic X. A generating function is

$$\sum_{d=0}^{\infty} h_d \lambda^d = \frac{1 - \lambda^4}{(1 - \lambda)^4} = 1 + 2 \sum_{d=0}^{\infty} (1 + d^2) \lambda^d$$ (4.4)

For $d \geq 1$ we then have a space of curves of degree $4d$ on X of dimension $2d^2 + 1$. The arithmetic genus of such a curve C (i.e. disregarding possible singularities) is

$$p = 1 + \frac{C.C + C.K}{2} \qquad (4.5)$$

with the canonical divisor K vanishing for a K_3 surface, while $C.C = 4d^2$ (the value $C.C'$, where C and C' are intersections with two surfaces Y and Y' of degree d, reads $C.C' = 4 \times d \times d$ by Bezout). Consequently $p = 1 + 2d^2$. We conclude (with Kontsevich) that requiring C to be rational, i.e. imposing $1 + 2d^2$ conditions (generically to have $1 + 2d^2$ double points) in the space of the same dimension, leads to expect finitely many rational curves in each degree $4d$. In the first non trivial instance on a smooth quartic surface in $\mathbb{P}_3$ there are 3200 rational (plane, trinodal) quartics cut by 3-tangent planes according to the classical formula of Salmon for the number of tritangent planes to a smooth surface of degree r

$$tritg(r) = \frac{1}{6}(r^9 - 6r^8 + 15r^7 - 59r^6 + 204r^5 - 339r^4 + 770r^3 - 2056r^2 + 1920r) \qquad (4.6)$$

evaluated at $r = 4$. To extend this to curves of higher degree on a quartic is apparently an open problem. For instance in degree 8 one is required to count 9-fold tangent quadrics to X!

To emphasize the point, non generic quartic surfaces (indeed a codimension 1 in a 19 dimensional space) do possess lines, as examplified by the Fermat quartic, $\sum_{0 \leq i \leq 3} x_i^4 = 0$, on which we find at least 48 lines of the type $\lambda(1, \omega, 0, 0) + \mu(0, 0, 1, \omega')$ with $\omega^4 = \omega'^4 = -1$. In general the most naive reasoning goes as follows. Using homogeneous coordinates in $\mathbb{P}_1$ a parametrized curve of degree k in $\mathbb{P}_n$ depends on $(n+1)(k+1)$ parameters. The constraint to lie on a degree d hypersurface implies the vanishing of a homogeneous polynomial in the two coordinates, of degree kd, hence $kd + 1$ conditions. If the difference

$$(n + 1)(k + 1) - (kd + 1) = k(n + 1 - d) + n \qquad (4.7)$$

is larger or equal to 4 (the arbitrariness in the parametrization) one expects rational curves (this extends readily to complete intersections). A marginal case occurs when $d = n + 1$ (so that k does not matter). If moreover n is precisely equal to 4, i.e. for the famous quintic threefold in $\mathbb{P}_4$, the expectation (Clemens) is the existence of finitely many rational curves of any degree. Conjecturally mirror symmetry enables one to compute these numbers in any degree k (and more).

On the other hand for $n = 3$ and $d = 4$ the naive reasoning fails for k a multiple of 4 in which case one of the conditions must be redundant, while if $d = 2$ the counting of degrees of freedom (i.e. subtracting 4 from (4.7)) yields $2k - 1$ in agreement with equation (3.10) where $k = c + b$, and when $d = 3$ one gets $k - 1$, again in agreement with equation (3.25) where $k = d_a$.

4.2. Parallel with matrix models

Topological field theories assumed to underlie this paper are generalizations of those arising from matrix models of $2D$–quantum gravity as interpreted and elaborated in references [1] [2] [3] [16] [17]. Similar formulas as those presented above also hold in these cases for the genus zero "little phase space" free energies. Take for instance the generalized Airy matrix integrals which we denote, as models, by W_{n+2}, $n \geq 0$, the original case being W_2 with reference to the corresponding W–algebra constraints satisfied by the partition function[3]

Using for W_{n+2} models similar notations as for the $\mathbb{P}_n$ case of this paper, let us denote the little phase space variables y_i, $i = 0, 1, ..., n$, and the genus zero free energy F, with weights

$$[y_i] = 1 - \frac{i}{n+2} \qquad [F] = 3 - \frac{n}{n+2} \qquad (4.8)$$

to be compared with those in the $\mathbb{P}_n$ case

$$[y_i] = 1 - i \qquad [F] = 3 - n \qquad (4.9)$$

That all weights are positive implies that F is a polynomial in the W_{n+2} case. It splits as $F = f_{\mathrm{cl}} + f$ where f_{cl} is the same cubic polynomial as in the $\mathbb{P}_n$ case

$$f_{\mathrm{cl}} = \frac{1}{3!} \sum_{i_1+i_2+i_3=n} y_{i_1} y_{i_2} y_{i_3} \qquad (4.10)$$

---

[3] One also finds the notation A_{n+1} from singularity theory. After the parallel we will make between W_{n+2} models and enumerations in $\mathbb{P}_n$, this denomination raises the following question: what are the "natural" families of targets in the enumerative context, corresponding to the other D and E series of singularities and matrix models if any?

The quantum part f contains finitely many higher degree terms in the y's. From unpublished work by J.-B. Zuber (whom we take this opportunity to thank) we extract the following table up to W_6

$$W_2(n = 0) \; : \; f = 0$$

$$W_3(n = 1) \; : \; f = \frac{1}{3}\frac{y_1^4}{4!}$$

$$W_4(n = 2) \; : \; f = \frac{1}{4}\frac{y_1^2}{2!}\frac{y_2^2}{2!} + \frac{1}{8}\frac{y_2^5}{5!}$$

$$W_5(n = 3) \; : \; f = \frac{1}{5}[\frac{y_1^2}{2!}\frac{y_3^2}{2!} + y_1\frac{y_2^2}{2!}y_3 + 2\frac{y_2^4}{4!}] + \frac{2}{5^2}\frac{y_2^2}{2!}\frac{y_3^3}{3!} + \frac{6}{5^3}\frac{y_3^6}{6!}$$

$$W_6(n = 4) \; : \; f = \frac{1}{6}[\frac{y_1^2}{2!}\frac{y_4^2}{2!} + y_1 y_2 y_3 y_4 + y_1\frac{y_3^3}{3!} + 2\frac{y_2^2}{2!}\frac{y_3^2}{2!} + \frac{y_2^3}{3!}y_4]$$

$$+ \frac{2}{6^2}[\frac{y_2^2}{2!}\frac{y_4^3}{3!} + y_2\frac{y_3^2}{2!}\frac{y_4^2}{2!} + 2\frac{y_3^4}{4!}y_4] + \frac{3!}{6^3}\frac{y_3^2}{2!}\frac{y_4^4}{4!} + \frac{4!}{6^4}\frac{y_4^7}{7!}$$

(4.11)

One checks that the equations for the associativity of the quantum ring are indeed satisfied. Take for instance W_4 in parallel to target $\mathbb{P}_2$, with $[y_0] = 1$, $[y_1] = 3/4$, $[y_2] = 1/2$, $[F] = 5/2$. Writing

$$f = \sum_{3a+2b=10} \nu_{a,b}\frac{y_1^a}{a!}\frac{y_2^b}{b!} = \nu_{2,2}\frac{y_1^2}{2!}\frac{y_2^2}{2!} + \nu_{0,5}\frac{y_2^5}{5!}$$

(4.12)

the familiar equation $f_{222} = f_{112}^2 - f_{111}f_{122}$ yields

$$\nu_{0,5} = 2\nu_{2,2}^2$$

(4.13)

and the "initial condition" $\nu_{2,2} = 1/4$ gives agreement with the above table, for W_4.

There also exist polynomial solutions for the same associativity conditions but with different weights. For instance with three generators as in W_4 but with weights $[y_0] = 1$, $[y_1] = 2/3$, $[y_2] = 1/3$ and $[F] = 7/3$, with the same f_{cl}, we find

$$f = \alpha\frac{y_1^3}{3!}y_2 + 2\alpha^2\frac{y_1^2}{2!}\frac{y_2^3}{3!} + 24\alpha^4\frac{y_2^7}{7!}$$

(4.14)

where the remaining arbitrary coefficient α would be determined by an hypothetic geometric interpretation. That the cubic part f_{cl} is common to W_{n+2} and $\mathbb{P}_n$ remains slightly mysterious to the authors. Is there some geometric connection, or could one "twist" the matrix model to yield the same results as for $\mathbb{P}_n$?

4.3. Partition function

There are indications [8] that formulas generalizing those in proposition 2, as well as formula (2.5), hold for quadrics. This points to some system of equations satisfied by $\exp \sum F^{(g)}$, where the free energy is decomposed according to the genus g, analogous to the Virasoro constraints and their W-generalizations for matrix models, which remain to be found.

To elaborate the case for $\mathbb{P}_2$, assume that for $g > 0$, $F^{(g)}$ has no polynomial part and define a partition function

$$Z = e^{-\left[\frac{y_0^2 y_2 + y_0 y_1^2}{2}\right] + \sum_{s \geq 0} F^{(s)}}$$

$$= \sum_{d \geq 0} \sum_{0 \leq \delta \leq d(d-1)/2} z_{d,\delta} \frac{y_2^{d(d+3)/2 - \delta}}{(d(d+3)/2 - \delta)!} e^{dy_1} \qquad (4.15)$$

Here $z_{d,\delta}$ are for fixed δ the polynomials in d of degree 2δ introduced in (2.5) and given explicitly [7] [8] up to $\delta = 6$ by the polynomial parts of proposition 2. These numbers count reducible as well as irreducible degree d plane curves with δ simple nodes through $d(d + 3)/2 - \delta$ generic points. From homogeneity, if g_i stands for the genus of the i-th irreducible part (the same irreducible part may occur several times) then

$$\sum_i (g_i - 1) = \frac{d(d-3)}{2} - \delta \qquad (4.16)$$

The sum over δ extends up to $d(d-1)/2$ (and not $(d-1)(d-2)/2$ for irreducible curves) corresponding to systems of d lines through $2d$ points in which case

$$z_{d,d(d-1)/2} = \frac{(2d)!}{2^d d!} = (2d - 1)!! \qquad (4.17)$$

the number of pairings as in Wick's theorem. From section 2

$$Z = \sum_{d \geq 0} \frac{y_2^{d(d+3)/2}}{(d(d+3)/2)!} e^{dy_1} + \sum_{d \geq 2} 3(d-1)^2 \frac{y_2^{d(d+3)/2-1}}{(d(d+3)/2 - 1)!} e^{dy_1}$$

$$+ \sum_{d \geq 3} \frac{3}{2} d(d-1)(d-2)(3d^2 - 3d - 11) \frac{y_2^{d(d+3)/2-2}}{(d(d+3)/2 - 2)!} e^{dy_1} \qquad (4.18)$$

$$+ \cdots + \sum_{2\delta \leq d(d-1)} \left(\frac{(3d^2)^\delta}{\delta!} + \cdots\right) \frac{y_2^{d(d+3)/2-\delta}}{(d(d+3)/2 - \delta)!} e^{dy_1} + \cdots$$

4.4. Higher genus

As already stressed the challenge is to find a compact means – at least as a conjecture – to generate the contributions of higher genera. Focussing on the $\mathbb{P}_2$ case, we have unsuccessfully tried a number of possibilities with disappointing results. Although this is not the usual practice, let us briefly mention some.

Perhaps too naively one might think that equation (2.30) is some genus zero restriction of an equation valid for the full free energy. To check this idea we have once more tried to imitate the reasoning of section 2.3 by looking at a one dimensional family of elliptic curves C_λ of degree $d \geq 3$ through $3d - 1$ points using a projection

$$C_\lambda \rightarrow \mathbb{P}_1$$

where $\mathbb{P}_1$ is the linear pencil of degree d smooth adjoints of degree $d - 2$ (i.e. curves through the nodes of C_λ) through $d - 2$ of the fixed points of the family C_λ. Such adjoints cut residually C_λ in two points. The projection therefore gives a presentation of the elliptic curves as double covers of $\mathbb{P}_1$ ramified at 4 points (corresponding to the 4 tangent adjoints in the family). Then repeating the argument of section 2.3 one obtains, with $N_d^{(0)}$ ($N_d^{(1)}$) the number of rational (elliptic) degree d curves through $3d - 1$ ($3d$) points we find a relation

$$
\begin{aligned}
N_d^{(1)} = \sum_{d_1+d_2=d} &\left[N_{d_1}^{(1)} N_{d_2}^{(0)} \left[2d_1^2 d_2^2 \binom{3d-3}{3d_1-1} - d_1 d_2^3 \binom{3d-3}{3d_1-2} \right] \right. \\
&\left. - N_{d_1}^{(0)} N_{d_2}^{(1)} d_1 d_2^3 \binom{3d-3}{3d_1-3} \right] + 2\nu_d^{(0,1)} - \nu_d^{(0,0)} - \nu_d^{(1,1)}
\end{aligned}
\tag{4.19}
$$

where the numbers $\nu_d^{(0,0)}$ stand for the numbers of elliptic curves through $3d - 1$ points which have two extra assigned points with equal projections (i.e. the number of elliptic curves in such a family such that an adjoint goes through d assigned points), $\nu_d^{(0,1)}$ a similar number when one of the extra assigned points and a point of $C_\lambda \cap l$ (where l is a fixed line) have equal projection. Finally $\nu_d^{(1,1)}$ counts the curves when two points, one from $C_\lambda \cap l$ and one from $C_\lambda \cap l'$ have equal projections, l and l' two fixed lines.

Unfortunately we do not know how to evaluate the $\nu_d^{(i,j)}$ but at least it gives a feeling, and a geometric interpretation, of the corrections to a polarized form of equation (2.30) to evaluate the contributions in higher genus.

Another direction is to try to generalize the topological relations in a way similar to those resulting from Fig.1, including 1 loop for genus 1 as an example, in this way yielding several equations. Apparently we failed, perhaps for not having introduced "gravitational descendants" in the calculation.

4.5. Miscellany

(i) We have tentatively attributed the simplifications that seem to occur for quadrics (isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$) to the fact that the genus 0 free energy for $\mathbb{P}_1$ is very simple, assuming that some relations exist between F_{M_1}, F_{M_2} and $F_{M_1 \times M_2}$. What is such a relation?

(ii) There are some analogies between counting rational curves over $\mathbb{C}$ and counting rational points for varieties over $\mathbb{Q}$ and even counting the integral points (i.e. points with integer coordinates) inside a dilated integral polytope (i.e. a polytope with vertices at integral points). It would be highly interesting to discover a common framework.

(iii) Last but not least it remains of course to ascertain properly the status of the various relations in this paper.

In short, the subject remains wide open and the above presentation had no other pretence than to encourage further investigations.

Appendix A.

Following Dubrovin [4] let us show that the differential equation (2.55), taking into account homogeneity, and governing the quantum ring for $\mathbb{P}_2$, is equivalent to a Painlevé VI equation. The argument amounts to recognize that for generic values of the deformation parameters the corresponding commutative algebra over $\mathbb{C}$ is semi–simple. A reparametrization expresses it in the form

$$\mathcal{T}_\alpha \circ \mathcal{T}_\beta = \delta_{\alpha,\beta} \, \mathcal{T}_\alpha \tag{A.1}$$

as noticed in (2.90) (we use the symbol $\circ$ for the ring multiplication to avoid confusion in the sequel). In this basis the equivalent intersection form is interpreted as a metric on the tangent space to the parameter space H^*. Expressing that the metric is flat (constant in the original coordinates) and taking into account homogeneity, one obtains, after some tedious calculations, the required Painlevé VI equation. As far as possible we try to keep general notations for an expected generalization to other quantum rings.

1. We revert to upper indices for the variables y^i, $0 \le i \le 2$, and identify the ring structure as one on the tangent space to H^* through the correspondence

$$T_i \leftrightarrow \frac{\partial}{\partial y^i} \tag{A.2}$$

The "metric"

$$\eta_{ij} = \langle T_i, T_j \rangle \tag{A.3}$$

is constant in these coordinates (the only non vanishing components being $\eta_{02} = \eta_{11} = 1$) and numerically equal to its inverse

$$\eta^{ij} = (\eta^{-1})_{ij} \tag{A.4}$$

It enjoys the fundamental property

$$\langle T_i \circ T_j, T_k \rangle = \langle T_i, T_j \circ T_k \rangle = \frac{\partial^3 F}{\partial y^i \partial y^j \partial y^k} \tag{A.5}$$

Latin indices will label the initial *flat coordinates* y^i, while greek indices will be used for canonical ones to be introduced below.

As noticed at the end of section 2, eqns (2.88)-(2.89), the ring for $\mathbb{P}_2$ is identified with $\mathbb{C}[x]/P(x)$, with

$$P(x) = x^3 - f_{111}x^2 - 2f_{112}x - f_{122} = \prod_{\alpha=1}^{3}(x - q^\alpha) \qquad (A.6)$$

through

$$T_0 \leftrightarrow 1 \quad T_1 \leftrightarrow x \quad T_2 \leftrightarrow x^2 - f_{111}x - f_{112} \qquad (A.7)$$

where q^α denote the roots of the polynomial $P(x)$. For generic y's one introduces new *canonical coordinates* u^α and a corresponding basis $T_\alpha \leftrightarrow \partial/\partial u^\alpha$ such that the ring structure takes its canonical form (A.1). Using the identification with the tangent space (and summation over dummy indices) we have on the one hand

$$T_\alpha = \frac{\partial y^i}{\partial u^\alpha}\, T_i \qquad (A.8)$$

while on the other hand in $\mathbb{C}[x]/P(x)$

$$T_\alpha \leftrightarrow \prod_{\beta,\beta\neq\alpha} \frac{x - q^\alpha}{q^\alpha - q^\beta} \qquad (A.9)$$

Hence

$$T_0 = \sum_\alpha T_\alpha$$

$$T_1 = \sum_\alpha q^\alpha\, T_\alpha \qquad (A.10)$$

$$T_2 = \sum_\alpha \left[(q^\alpha)^2 - f_{111}q^\alpha - f_{112}\right] T_\alpha$$

2. In canonical coordinates the intersection form, or metric, reads

$$\eta_{\alpha\beta} = \langle T_\alpha, T_\beta \rangle = \frac{\partial y^i}{\partial u^\alpha}\eta_{ij}\frac{\partial y^j}{\partial u^\beta} \qquad (A.11)$$

The unit being $T_0 = \sum_\alpha T_\alpha$, we have from (A.5)

$$\eta_{\alpha\beta} = \langle T_\alpha \circ \sum_\gamma T_\gamma, T_\beta \rangle = \langle \sum_\gamma T_\gamma, T_\alpha \circ T_\beta \rangle = \delta_{\alpha,\beta}\langle \sum_\gamma T_\gamma, T_\alpha \rangle \qquad (A.12)$$

and the metric reduces to its diagonal elements $\eta_{\alpha\alpha}$. With the notation (,) for the duality between tangent and cotangent spaces

$$(dy^i, \frac{\partial}{\partial y^j}) = \delta^i_j \qquad (A.13)$$

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we have

$$\langle T_i, T_j \rangle = \eta_{ik}(dy^k, \frac{\partial}{\partial y^j}) \tag{A.14}$$

Correspondingly

$$\eta_{\alpha\alpha} = \langle \sum_\beta T_\beta, T_\alpha \rangle = \langle T_0, T_\alpha \rangle = \eta_{0k}(dy^k, \frac{\partial}{\partial u^\alpha}) \tag{A.15}$$

$$\eta_{\alpha\alpha} = \frac{\partial y_0}{\partial u^\alpha} \qquad y_0 = \eta_{0k} y^k \text{ (here } y^2). $$

3. Inserting in the structure equations $T_i \circ T_j = F_{ijl}\eta^{lk}T_k$ the expressions $T_i = (\partial u^\alpha/\partial y^i)T_\alpha$, and taking into account (A.1), one finds for any u^α, denoted u, the differential equations

$$\frac{\partial u}{\partial y^0} = 1 \qquad \left(\frac{\partial u}{\partial y^1}\right)^2 = \frac{\partial u}{\partial y^2} + f_{111}\frac{\partial u}{\partial y^1} + f_{112}$$

$$\frac{\partial u}{\partial y^1}\frac{\partial u}{\partial y^2} = f_{112}\frac{\partial u}{\partial y^1} + f_{122} \qquad \left(\frac{\partial u}{\partial y^2}\right)^2 = f_{122}\frac{\partial u}{\partial y^1} + f_{222} \tag{A.16}$$

Therefore each $u^\alpha - y^0$ is a function of y^1, y^2 only. In terms of the roots q^α we can identify the derivatives as

$$\frac{\partial u^\alpha}{\partial y^0} = 1$$

$$\frac{\partial u^\alpha}{\partial y^1} = q^\alpha \tag{A.17}$$

$$\frac{\partial u^\alpha}{\partial y^2} = \frac{P'(q^\alpha) - (q^\alpha)^2}{2} = (q^\alpha)^2 - f_{111}q^\alpha - f_{112}.$$

Changing coordinates from flat (y) to canonical (u) is highly non trivial, it not only requires the knowledge of the function f, but also of the roots of $P(x)$.

4. Using homogeneity properties it is possible to find expressions for the coordinates u^α rather than their derivatives. Recall that the free energy satisfies

$$F = f_{cl} + f \tag{A.18}$$

$$E f_{cl} = 3y^0 y^1 + f_{cl}$$

$$E f = f \tag{A.19}$$

$$\frac{\partial f}{\partial y^0} = 0$$

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where E is the vector field

$$E = y^0 \frac{\partial}{\partial y^0} + 3\frac{\partial}{\partial y^1} - y^2 \frac{\partial}{\partial y^2} = \sum_\alpha u^\alpha \frac{\partial}{\partial u^\alpha} \tag{A.20}$$

The second equality stems from the fact that the roots q^α as well as the canonical coordinates are of weight 1

$$Eq^\alpha = \frac{\partial q^\alpha}{\partial y^0} = q^\alpha \qquad Eu^\alpha = u^\alpha \tag{A.21}$$

and the vector field E can be interpreted as an element of the ring[4]

$$E \leftrightarrow y^0 T_0 + 3T_1 - y^2 T_2 = \sum_\alpha u^\alpha T_\alpha \tag{A.22}$$

Since the metric identifies the tangent space with its dual, the product structure can be transfered to the cotangent space. With $\eta^{\alpha\beta} = (\eta^{-1})_{\alpha\beta}$, we have

$$\sum_\gamma \eta^{\alpha\gamma} \langle \frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta} \rangle = \delta^\alpha_\beta = (du^\alpha, \frac{\partial}{\partial u^\beta}) \tag{A.23}$$

Moreover $\eta_{\alpha\beta}$ is diagonal, so is $\eta^{\alpha\beta}$, hence the differentials du^α behave under multiplication as $\eta^{\alpha\alpha}\partial/\partial u^\alpha$ (no summation), i.e.

$$du^\alpha \circ du^\beta \leftrightarrow \eta^{\alpha\alpha} \frac{\partial}{\partial u^\alpha} \circ \eta^{\beta\beta} \frac{\partial}{\partial u^\beta} = \eta^{\alpha\alpha} \eta^{\beta\beta} \delta_{\alpha,\beta} \frac{\partial}{\partial u^\alpha}$$

$$du^\alpha \circ du^\beta = \eta^{\alpha\alpha} \delta_{\alpha,\beta} du^\alpha = \frac{\delta_{\alpha,\beta}}{\eta_{\alpha\alpha}} du^\alpha \tag{A.24}$$

Define

$$g^{\alpha\beta} = \ll du^\alpha, du^\beta \gg \equiv (du^\alpha \circ du^\beta, E)$$

$$= \sum_\gamma u^\gamma (du^\alpha \circ du^\beta, \frac{\partial}{\partial u^\gamma}) = \frac{\delta_{\alpha,\beta}}{\eta_{\alpha\alpha}} u^\alpha \tag{A.25}$$

i.e.

$$\sum_\gamma \eta_{\alpha\gamma} g^{\gamma\beta} = u^\alpha \, \delta_{\alpha,\beta} \tag{A.26}$$

---

[4] From eqs.(A.10) and (A.22) it follows that setting $L_{-1} = T_0$, $L_n = E^{\circ(n+1)}$, $n \geq 0$, and considering the L's as vector fields, they form a Lie algebra $\{L_n, L_m\} = (n-m)L_{n+m}$, isomorphic to the Lie algebra on the affine line (part of a Virasoro algebra) as noticed by Kontsevich. It is not known at present what to do with this remark.

and

$$g^{ij} = \ll dy^i, dy^j \gg = \sum_\alpha \frac{\partial y^i}{\partial u^\alpha} \frac{\partial y^j}{\partial u^\alpha} \frac{u^\alpha}{\eta_{\alpha\alpha}} \tag{A.27}$$

The canonical coordinates u^α appear as eigenvalues of the matrix $\sum_\gamma \eta_{\alpha\gamma} g^{\gamma\beta}$ or equivalently of $\sum_k \eta_{ik} g^{kj}$, thus as solutions of

$$\det(g^{ij} - u\eta^{ij}) = 0 \tag{A.28}$$

Now $g^{ij} = (dy^i \circ dy^j, E)$, and since the differentials dy^i behave under multiplication as $\eta^{ik} \partial/\partial y^k$ we have

$$g^{ij} = \eta^{ik} \eta^{jl} E F_{kl} \qquad g_{ij} = E F_{ij} \tag{A.29}$$

From (A.19) (A.20)

$$g_{ij} = F_{ij} - (\delta_{0i} F_{0j} - \delta_{2i} F_{2j}) - (\delta_{0j} F_{0i} - \delta_{2j} F_{2i}) + 3(\delta_{0i}\delta_{1j} + \delta_{0j}\delta_{1i}) \tag{A.30}$$

In a more transparent notation, denote the weights as

$$d_F = d_f = 1 \quad , \quad d_i = \begin{cases} 1 & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i = 1 \\ -1 & \text{if} \quad i = 2 \end{cases} \tag{A.31}$$

then

$$g_{ij} = (d_F - d_i - d_j) F_{ij} + 3(\delta_{0i}\delta_{1j} + \delta_{0j}\delta_{1i}) \tag{A.32}$$

The equation determining the coordinates u^α (rather than their derivatives) takes the form

$$\det(g_{ij} - u\eta_{ij}) = 0 \tag{A.33}$$

This is again given in terms of roots of a cubic polynomial depending on the second derivatives of F. Explicitly this is

$$\begin{vmatrix} -y^2 & 3 & y^0 - u \\ 3 & f_{11} + y^0 - u & 2f_{12} \\ y^0 - u & 2f_{12} & 3f_{22} \end{vmatrix}$$

$$= (u - y^0)^3 - f_{11}(u - y^0)^2 + 3(y^2 f_{22} - 4f_{12})(u - y^0) + y^2(4f_{12}^2 - 3f_{11}f_{22}) - 27f_{22} = 0 \tag{A.34}$$

Once more we see that $(u^\alpha - y^0)$ are functions of y^1, y^2 only.

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5. The metric is flat and the curvature vanishes. In the coordinates u^α, the metric is diagonal. This simplifies the corresponding (Darboux) equations which in terms of the (symmetric) "rotation coefficients", defined for $\alpha \neq \beta$ through

$$\Gamma_{\alpha\beta} = \frac{1}{\sqrt{\eta_{\beta\beta}}} \frac{\partial \sqrt{\eta_{\alpha\alpha}}}{\partial u^\beta} = \frac{1}{2\sqrt{\eta_{\alpha\alpha}\eta_{\beta\beta}}} \frac{\partial \eta_{\alpha\alpha}}{\partial u^\beta}$$

$$= \frac{1}{2\sqrt{\eta_{\alpha\alpha}\eta_{\beta\beta}}} \frac{\partial^2 y_0}{\partial u^\alpha \partial u^\beta} \tag{A.35}$$

where we used (A.15), take the form

$$\boxed{\begin{array}{ll} \alpha \neq \beta \neq \gamma & \dfrac{\partial \Gamma_{\beta\gamma}}{\partial u^\alpha} = \Gamma_{\beta\alpha}\,\Gamma_{\gamma\alpha} \\[2ex] \alpha \neq \beta & \displaystyle\sum_\gamma \dfrac{\partial \Gamma_{\alpha\beta}}{\partial u^\gamma} = 0 \end{array}} \tag{A.36}$$

From their definition the Γ's are of weight -1, hence

$$\boxed{\left(\sum_\gamma u^\gamma \frac{\partial}{\partial u^\gamma} + 1\right)\Gamma_{\alpha\beta} = 0} \tag{A.37}$$

In deriving (A.36) we shall also obtain translation invariance of the metric, by an equal shift on all u's

$$\sum_\gamma \frac{\partial}{\partial u^\gamma} \eta_{\alpha\alpha} = 0 \tag{A.38}$$

Proof: With a metric of the form $\sum_\alpha \eta_{\alpha\alpha}(du_\alpha)^2$, the covariant derivative of a vector field $\sum_\alpha w^\alpha \partial/\partial u^\alpha$ reads

$$D_\beta w^\alpha = \frac{\partial w^\alpha}{\partial u^\beta} + \frac{1}{2\eta_{\alpha\alpha}}\left(\frac{\partial \eta_{\alpha\alpha}}{\partial u^\beta} w^\alpha - \frac{\partial \eta_{\beta\beta}}{\partial u^\alpha} w^\beta + \frac{1}{2}\delta^\alpha_\beta \sum_\gamma \frac{\partial \eta_{\alpha\alpha}}{2\eta_{\alpha\alpha}\partial u^\gamma} w^\gamma\right)$$

$$= \frac{1}{\sqrt{\eta_{\alpha\alpha}}}\left(\frac{\partial \psi^\alpha}{\partial u^\beta} - \Gamma_{\beta\alpha}\psi^\gamma + \delta^\alpha_\beta \sum_\gamma \Gamma_{\alpha\gamma}\psi^\gamma\right) \tag{A.39}$$

where we have set

$$\psi^\alpha = \sqrt{\eta_{\alpha\alpha}} w^\alpha \tag{A.40}$$

and $\Gamma_{\alpha\beta}$ is as above (note that only $\Gamma_{\alpha\beta}$ for $\alpha \neq \beta$ enters in (A.39)).

In particular the vector $\sum_\alpha \partial/\partial u^\alpha = \partial/\partial y^0$ with constant components $w^\alpha = 1$ for each α, is covariantly constant, i.e.

$$0 = \left(\frac{\partial \eta_{\alpha\alpha}}{\partial u^\beta} - \frac{\partial \eta_{\beta\beta}}{\partial u^\alpha}\right) + \sum_\gamma \frac{\partial \eta_{\alpha\alpha}}{\partial u^\gamma} \qquad (A.41)$$

The first bracket vanishes by virtue of (A.15) i.e. $\eta_{\alpha\alpha}$ is a gradient, proving (A.38). From (2.69) it follows that the zero curvature conditions are equivalent to the integrability conditions of the system

$$\frac{\partial \psi^\alpha}{\partial u^\beta} = \Gamma_{\alpha\beta}\psi^\beta - \delta^\alpha_\beta \sum_\gamma \Gamma_{\alpha\gamma}\psi^\gamma \qquad (A.42)$$

In matrix notation, with ψ a vector of component ψ^α, $[\Gamma]_{\alpha\beta} = \Gamma_{\alpha\beta} = \Gamma_{\beta\alpha}$ for $\alpha \neq \beta$ and $[\Gamma]_{\alpha\alpha} = 0$, and E_α the matrix with elements $[E_\alpha]_{\beta\gamma} = \delta_{\alpha\beta}\delta_{\alpha\gamma}$, this reads

$$\frac{\partial \psi}{\partial u^\beta} = [\Gamma, E_\beta]\psi \qquad (A.43)$$

From the definition of Γ and (A.38) a first natural solution of this system is

$$\psi^\alpha_0 = \sqrt{\eta_{\alpha\alpha}} = \frac{1}{\sqrt{\eta_{\alpha\alpha}}}\frac{\partial y_0}{\partial u^\alpha} \qquad (A.44)$$

One readily sees that the compatibility conditions of (A.43) are the Darboux equations (A.36) as claimed. The latter together with the homogeneity constraint (A.37) can be summarized in a single equation as follows. Define the antisymmetric matrix V through

$$V = [\Gamma, U] \qquad V^{\alpha\beta} = -V^{\beta\alpha} = \Gamma_{\alpha\beta}(u^\beta - u^\alpha) \qquad (A.45)$$

where

$$U = \text{diag}\{u^\alpha\} \qquad (A.46)$$

Then a little calculation shows that (A.36)-(A.37) are equivalent to

$$\boxed{\frac{\partial V}{\partial u^\alpha} + \left[V, [\Gamma, E_\alpha]\right] = 0}$$

$$(A.47)$$

The matrix V enjoys the following properties

(i) If ψ is a solution of (A.43) it follows from (A.47) that so is $V\psi$, and generically V can be diagonalized in the space of solutions of (A.43) with constant eigenvalues. Indeed let ψ be an eigenvector of V with eigenvalue μ

$$V\psi = \mu\psi \qquad (A.48)$$

then

$$\begin{aligned}
\frac{\partial(V\psi)}{\partial u^\alpha} &= \frac{\partial\mu}{u^\alpha}\psi + \mu[\Gamma, E_\alpha]\psi \\
&= \frac{\partial\mu}{u^\alpha}\psi + [\Gamma, E_\alpha](V\psi)
\end{aligned} \qquad (A.49)$$

and from $\partial(V\psi)/\partial u^\alpha = [\Gamma, E_\alpha](V\psi)$ we conclude that

$$\frac{\partial\mu}{\partial u^\alpha} = 0 \qquad (A.50)$$

(ii) The eigenvalues of V are the weights of the corresponding eigenvectors, indeed from $\sum_\alpha u^\alpha E_\alpha = U$ and the fact that ψ is an eigenvector of V, we get

$$\sum_\alpha u^\alpha \frac{\partial\psi}{\partial u^\alpha} = \sum_\alpha u^\alpha[\Gamma, E_\alpha]\psi = [\Gamma, U]\psi = V\psi = \mu\psi \qquad (A.51)$$

(iii) We noted in (A.44) that $\psi_0^\alpha = \partial y_0/\sqrt{\eta_{\alpha\alpha}}\partial u^\alpha$ is a solution of (A.43). It is also an eigenvector of V with eigenvalue $\mu_0 = -1$ since $\psi_0^\alpha = \sqrt{\eta_{\alpha\alpha}}$, and the weight of $\eta_{\alpha\alpha}$ is $[\eta_{\alpha\alpha}] = [y_0] - [u^\alpha] = [y^2] - [u^\alpha] = -2$. More generally if ψ^α is of the form $\sqrt{\eta_{\alpha\alpha}}w^\alpha$, where w^α are the components of a covariantly constant vector field, it satisfies eq.(A.42) from its very derivation. Now for each index i the vector field $\partial/\partial y^i$ is covariantly constant, hence yields a solution

$$\begin{aligned}
\psi_i^\alpha &= \sqrt{\eta_{\alpha\alpha}}\frac{\partial u^\alpha}{\partial y^i} = \sum_\beta \frac{\eta_{\alpha\beta}}{\sqrt{\eta_{\alpha\alpha}}}\frac{\partial u^\beta}{\partial y^i} \\
&= \sum_j \frac{\eta_{ij}}{\sqrt{\eta_{\alpha\alpha}}}\frac{\partial y^j}{\partial u^\alpha} = \frac{1}{\sqrt{\eta_{\alpha\alpha}}}\frac{\partial y_i}{\partial u^\alpha}
\end{aligned} \qquad (A.52)$$

In this derivation we have used the fact that $\eta_{\alpha\beta}$ is diagonal and equal to $(\partial y^i/\partial u^\alpha)\eta_{ij}(\partial y^j/\partial u^\beta)$, with η_{ij} constant, and we have set

$$y_i = \sum_j \eta_{ij}y^j \qquad (A.53)$$

The solutions ψ_i are a complete set of homogeneous eigenvectors of V and in the present case their weights are

$$\mu_0 = -1 \qquad \mu_1 = 0 \qquad \mu_2 = 1 \qquad (A.54)$$

Seen as a matrix, ψ_i^α is, up to a multiplicative constant, nothing but the Jacobian of the change of coordinates. It follows from (A.52) that the ψ's satisfy the orthogonality relations

$$\eta_{ij} = \sum_\alpha \frac{\partial u^\alpha}{\partial y^i} \eta_{\alpha\alpha} \frac{\partial u^\alpha}{\partial y^j} = \sum_\alpha \psi_i^\alpha \psi_j^\alpha \tag{A.55}$$

and the completeness relation

$$\delta^{\alpha\beta} = \sum_{i,j} \psi_i^\alpha \eta^{ij} \psi_j^\beta \tag{A.56}$$

Furthermore from (A.44) and (A.52) we have

$$\frac{\partial y_i}{\partial u^\alpha} = \psi_0^\alpha \psi_i^\alpha \tag{A.57}$$

Finally we obtain the structure constants of the ring as follows. Since

$$\frac{\partial u^\alpha}{\partial y^i} = \frac{1}{\sqrt{\eta_{\alpha\alpha}}} \psi_i^\alpha = \frac{\psi_i^\alpha}{\psi_0^\alpha}$$

and

$$\mathcal{T}_\alpha \circ \mathcal{T}_\beta = \delta_{\alpha\beta} \mathcal{T}_\alpha$$

$$T_i = \sum_\alpha \frac{\partial u^\alpha}{\partial y^i} \mathcal{T}_\alpha$$

$$\mathcal{T}_\alpha = \sum_\alpha \frac{\partial y^i}{\partial u^\alpha} T_i$$

we have

$$T_i \circ T_j = \sum_{\alpha\beta} \frac{\partial u^\alpha}{\partial y^i} \frac{\partial u^\beta}{\partial y^j} \mathcal{T}_\alpha \circ \mathcal{T}_\beta$$

$$= \sum_\alpha \frac{\partial u^\alpha}{\partial y^i} \frac{\partial u^\alpha}{\partial y^j} \mathcal{T}_\alpha$$

$$= \sum_{\alpha,l} \frac{\partial u^\alpha}{\partial y^i} \frac{\partial u^\alpha}{\partial y^j} \frac{\partial y^l}{\partial u^\alpha} T_l$$

$$= \sum_{\alpha,k,l} \frac{\partial u^\alpha}{\partial y^i} \frac{\partial u^\alpha}{\partial y^j} \frac{\partial y_k}{\partial u^\alpha} \eta^{kl} T_l$$

$$= \sum_{\alpha,k,l} \frac{\psi_i^\alpha}{\psi_0^\alpha} \frac{\psi_j^\alpha}{\psi_0^\alpha} \psi_0^\alpha \psi_k^\alpha \eta^{kl} T_l$$

$$= \sum_{k,l} F_{ijk} \eta^{kl} T_l$$

We finally deduce that

$$F_{ijk} \equiv \frac{\partial^3 F}{\partial y^i \partial y^j \partial y^k} = \sum_\alpha \frac{\psi_i^\alpha \psi_j^\alpha \psi_k^\alpha}{\psi_0^\alpha}$$

(A.58)

a formula reminiscent of similar ones in conformal or integrable field theories (or even finite group theory). Finally from (A.56)

$$V^{\alpha\beta} = \sum_\gamma V^{\alpha\gamma} \delta^{\gamma\beta} = \sum_\gamma V^{\alpha\gamma} \psi_i^\gamma \eta^{ij} \psi_j^\beta = \sum_{ij} \mu_i \psi_i^\alpha \eta^{ij} \psi_j^\beta$$

(A.59)

6. The matrix $V = [\Gamma, U]$ fulfills the master equation (A.47). Since $\mathbf{I} = \sum_\alpha E_\alpha$ it follows that

$$\sum_\alpha \frac{\partial V}{\partial u^\alpha} = \sum_\alpha u^\alpha \frac{\partial V}{\partial \alpha} = 0$$

(A.60)

Now

$$\Gamma_{\alpha\beta} = \frac{V^{\alpha\beta}}{u^\beta - u^\alpha} = -(\mathrm{ad}_U^{-1} V)_{\alpha\beta}$$

(A.61)

where we recall that Γ is symmetric with vanishing diagonal. This allows to rewrite (A.47) as

$$\frac{\partial V}{\partial u^\alpha} = [\mathrm{ad}_{E_k} \mathrm{ad}_U^{-1} V, V]$$

(A.62)

For the $\mathbb{P}_2$ quantum ring the index α runs from 1 to 3. One introduces a simplified notation by setting for $(\alpha\beta\gamma)$ a cyclic permutation of (123)

$$V^{\alpha\beta} = \Omega_\gamma$$

(A.63)

so that (A.60) becomes

$$\left(\frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^2} + \frac{\partial}{\partial u^3}\right)\Omega_\alpha = \left(u^1 \frac{\partial}{\partial u^1} + u^2 \frac{\partial}{\partial u^2} + u^3 \frac{\partial}{\partial u^3}\right)\Omega_\alpha = 0$$

(A.64)

Thus the Ω's, which are a priori functions of three (complex) numbers, are invariant under a common translation or dilation of their arguments, so they are only functions of a single variable

$$s = \frac{u^3 - u^1}{u^2 - u^1}$$

(A.65)

which is the value of u^3 when the coordinate system is chosen so that $u^1 =, u^2 = 1$. As a result the master equation (A.47) reduces to

$$\frac{d\Omega_1}{ds} = -\frac{\Omega_2 \Omega_3}{s} \qquad \frac{d\Omega_2}{ds} = -\frac{\Omega_3 \Omega_1}{1-s} \qquad \frac{d\Omega_3}{ds} = \frac{\Omega_1 \Omega_2}{s(1-s)}$$

(A.66)

This is an evolution on a symplectic leaf

$$\Omega_1^2 + \Omega_2^2 + \Omega_3^2 = \text{cst.} = -R^2$$

(A.67)

isomorphic to a sphere. Indeed with a Poisson structure on the (dual of the) Lie algebra $so(3)$

$$\{\Omega_\alpha, \Omega_\beta\} = \Omega_\gamma \quad (\alpha\beta\gamma) \quad \text{cyclic permutation of (123)}$$

(A.68)

the equations (A.66) read

$$\frac{d\Omega_\alpha}{ds} = \{H, \Omega_\alpha\} \qquad H = \frac{1}{2}\left(\frac{\Omega_1^2}{s-1} + \frac{\Omega_2^2}{s}\right)$$

(A.69)

To find the constant in (A.67) we compute the characteristic polynomial of the matrix V with eigenvalues $\{\mu_i\} = \{0, \pm 1\}$. The trace and determinant vanish, while

$$\text{tr}V \wedge V = \sum_{i<j} \mu_i \mu_j = -1 = \sum_\alpha \Omega_\alpha^2 = -R^2$$

(A.70)

Following Dubrovin we keep the general notation R in (A.67) which in our case can be taken equal to 1.

7. The final step is to replace the system (A.66) by a single non linear differential equation for Ω_3 as follows. We take the s–derivative of the third equation

$$\frac{d^2\Omega_3}{ds^2} = \frac{2s-1}{s(1-s)}\frac{d\Omega_3}{ds} - \frac{1}{s(1-s)}\Omega_3\left(\frac{\Omega_1^2}{1-s} + \frac{\Omega_2^2}{s}\right)$$

while Ω_1^2 and Ω_2^2 can be eliminated through

$$\Omega_1^2 + \Omega_2^2 = -R^2 - \Omega_3^2 \qquad (\Omega_1\Omega_2)^2 = \left(s(1-s)\frac{d\Omega_3}{ds}\right)^2$$

With

$$\Omega_3(s) = i\phi(s)$$

(A.71)

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this yields

$$\left[\frac{d^2\phi}{ds^2} + \frac{1-2s}{s(1-s)}\frac{d\phi}{ds} + \frac{\phi(\phi^2-R^2)}{2s^2(1-s)^2}\right]^2 =$$

$$= \frac{(1-2s)^2}{4s^4(1-s)^4}\phi^2\left[(\phi^2-R^2)^2 + 4s^2(1-s)^2(\frac{d\phi}{ds})^2\right] \tag{A.72}$$

To recognize that this is (a particular case of) a Painlevé VI equation, one has to change both the argument and the function as follows. First one trades s for z defined through[5]

$$1 - 2s = \frac{z^{1/2} + z^{-1/2}}{2} \tag{A.73}$$

so that

$$\left[\frac{d^2\phi}{dz^2} + \frac{3z-1}{2z(z-1)}\frac{d\phi}{dz} + \frac{2\phi(\phi^2-R^2)}{z(z-1)^2}\right]^2 =$$

$$= \left(\frac{(z+1)\phi}{z(z-1)}\right)^2\left((\frac{d\phi}{dz})^2 + \frac{(\phi^2-R^2)^2}{z(z-1)^2}\right) \tag{A.74}$$

The final change amounts to replace the function $\phi(z)$ by $v(z)$ according to the following

Proposition A For $R \neq 0, -1/2$, the equation (A.74), which summarizes the associativity and homogeneity conditions for the quantum ring of $\mathbb{P}_2$ is in one to one correspondence with the Painlevé equation

$$\frac{d^2v}{dz^2} = \frac{1}{2}(\frac{1}{v} + \frac{1}{v-1} + \frac{1}{v-z})(\frac{dv}{dz})^2 - (\frac{1}{z} + \frac{1}{z-1} + \frac{1}{v-z})(\frac{dv}{dz})$$

$$+ \frac{v(v-1)(v-z)}{z^2(z-1)^2}\left[\frac{(2R+1)^2}{8}(1 - \frac{z}{v^2}) + \frac{z(z-1)}{2(v-z)^2}\right] \tag{A.75}$$

through the field redefinition

$$2\phi + 1 = 2\frac{z}{v}\frac{dv}{dz} + \frac{2R+1}{z-1}(\frac{z}{v} - v) \tag{A.76}$$

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[5] Here we seem to disagree with Dubrovin [4].

and conversely

$$v = \frac{A}{B}$$

$$A = (z+1)\frac{d\phi}{dz} - \frac{4R}{z-1}\phi$$

$$+ \frac{z(z-1)^2}{z+1}\phi\left[\frac{d^2\phi}{dz^2} + \frac{3z-1}{2z(z-1)}\frac{d\phi}{dz} + \frac{2\phi(\phi^2 - R^2)}{z(z-1)^2}\right] \quad \text{(A.77)}$$

$$B = 2\frac{d\phi}{dz} - \frac{1}{z}\left[\phi^2 + 2\phi R\frac{z+1}{z-1} + R^2\right]$$

The proof, due to the authors of ref.[18], will not be reproduced here. It follows from a very painful but straightforward calculation.

Some final remarks are in order.

(i) Our interest here is for $R = 1$. For the exceptional cases $R = 0, -1/2$ see Dubrovin [4].

(ii) The most general Painlevé VI equation depends on four parameters a, b, c and d, and reads

$$\frac{d^2v}{dz^2} = \frac{1}{2}(\frac{1}{v} + \frac{1}{v-1} + \frac{1}{v-z})(\frac{dv}{dz})^2 - (\frac{1}{z} + \frac{1}{z-1} + \frac{1}{v-z})(\frac{dv}{dz})$$

$$+ \frac{v(v-1)(v-z)}{z^2(z-1)^2}\left[a + b\frac{z}{v^2} + c\frac{z-1}{(v-1)^2} + d\frac{z(z-1)}{(v-z)^2}\right] \quad \text{(A.78)}$$

The special case occuring here corresponds to

$$a = -b = \frac{(2R+1)^2}{8} \rightarrow \frac{9}{8} \quad c = 0 \quad d = \frac{1}{2} \quad \text{(A.79)}$$

(iii) It would seem futile to reconstruct from an appropriate solution of the Painlevé equation the (genus zero) free energy F. The whole point of the exercise is to prepare the way for a future generalization in the hope of including the badly missing contributions of higher genera.

(iv) Courageous readers are invited to describe quantum deformations of other cohomology rings in the language of this appendix. This should be rather straightforward for higher dimensional projective spaces.

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