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Hamiltonian dynamics on pseudodifferential symbols

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The talk is based on the joint work with Ilya ZAKHAREVICH (Math. Dept., MIT) on the Lie-Poisson group of pseudodifferential operators. We present here a short summary of the results and refer to [1,2] for all the details.

The ring $\mathfrak{G}$ of pseudodifferential symbols on the circle by definition consists of formal series $A(x,D) = \sum_{-\infty}^{n} a_j(x)D^j$ with respect to D $(= d/dx)$ where $a_j \in C^\infty(S^1, \mathbb{R} \text{ or } \mathbb{C})$. The multiplication law in $\mathfrak{G}$ is given by the Leibnitz rule for multiplication of symbols: $A(x,\xi) \circ B(x,\xi) = \sum_{n\geq 0} \frac{1}{n!} A_\xi^{(n)}(x,\xi)\, B_x^{(n)}(x,\xi)$ where $A_\xi^{(n)} = d^n/d\xi^n A(x,\xi)$, $B_x^{(n)} = d^n/dx^n B(x,\xi)$. The Lie algebra structure on $\mathfrak{G}$ is natural: $[A,B] = A \circ B - B \circ A$, and the operator $\operatorname{res} : \mathfrak{G} \to C^\infty(S^1)$ is defined by $\operatorname{res}\left(\sum a_i(x)D^i\right) = a_{-1}(x)$.

The formal expression $\log D$ defines an outer derivation of $\mathfrak{G}$ by: $[\log D, A] = \log D \circ A - A \circ \log D$ where the r.h.s is understood in the sense of the Leibnitz law above and belongs to $\mathfrak{G}$ for any $A \in \mathfrak{G}$. Then the ($\mathbb{R}$ or $\mathbb{C}$-valued) 2-cocycle

$$c(A,B) = \int \operatorname{res}\left([\log D, A] \circ B\right) = \int \operatorname{res}\left(\sum_{k\geq 1} \frac{(-1)^{k+1}}{k}\, A_x^{(k)} D^{-k} \circ B\right)$$

gives a nontrivial central extension of the Lie algebra $\mathfrak{G}$ (here A and B are arbitrary pseudodifferential symbols on S^1) (see [3]). The restrictions of this cocycle to the subalgebra $\mathfrak{G}_{DO}$ of purely differential operators $\{\sum_0^n a_j(x)D^j\}$ gives the Kac-Peterson cocycle [4], while restricting it to the Lie algebra of vector fields $\{f(x)D\}$ one obtains the Gelfand-Fuchs cocycle of the Virasoro algebra.

Let the algebra $\widetilde{\mathfrak{G}} = \left\{ \left(\sum_{j=-\infty}^{n} a_j(x)D^j + \lambda \log D, c \right) \right\}$ be the extension of $\mathfrak{G}$ by the 2-cocycle $c(A,B)$ and by the cocentral element $\log D$. The algebra $\widetilde{\mathfrak{G}}$ has a natural *ad*-invariant nondegenerate inner product ("Killing form") and two remarkable isotropic (with respect to this inner product) subalgebras: 1) $\widetilde{\mathfrak{G}}_{DO}$ which is the algebra of centrally extended differential operators $\{(\sum\limits_{j\geq 0} a_j(x)D^j, c)\}$, and 2) $\widetilde{\mathfrak{G}}_{\mathrm{Int}}$ which is the algebra of integral symbols together with $\log D$: $\{\sum\limits_{j=-\infty}^{-1} a_j(x)D^j + \lambda \log D\}$. The triple $\left(\widetilde{\mathfrak{G}}, \widetilde{\mathfrak{G}}_{DO}, \widetilde{\mathfrak{G}}_{\mathrm{Int}}\right)$ is a Manin triple (or, equivalently, $\widetilde{\mathfrak{G}}_{\mathrm{Int}}$ is a Lie bialgebra).

This implies that the Lie group $\widetilde{G}_{\mathrm{Int}} = \{(1+\sum\limits_{k=-\infty}^{-1} u_k(x)D^k) \circ D^\alpha | \alpha \in \mathbb{R} \text{ ou } \mathbb{C}\}$ corresponding to the Lie bialgebra $\widetilde{\mathfrak{G}}_{\mathrm{Int}}$ has a natural Poisson-Lie structure. This structure generalises the second Adler-Gelfand-Dickey structure [5,6] on fixed order differential operators.

This Lie group is quasi-nilpotent, and its the exponential map $\exp : \widetilde{\mathfrak{G}}_{\mathrm{Int}} \to \widetilde{G}_{\mathrm{Int}}$ is one-to-one. It allows one to define arbitrary complex powers of an operator $L \in \widetilde{G}_{\mathrm{Int}}$. Then the following infinite sequence of evolution equations on the coefficients of L: $\frac{\partial L}{\partial t_m} = \left[L, (L^{m/\alpha})_+\right]$, $m = 1, 2, \ldots$ is well defined ($\deg L = \alpha$, $\deg L^{m/\alpha} = m$, and the operation $+$, taking the differential part, makes sense). For any $\alpha \neq 0$ and any integral m these equations are Hamiltonian relative to the Poisson structure on $\widetilde{G}_{\mathrm{Int}}$ with Hamiltonian functions $H_m(L) = \frac{\alpha}{m} \int \mathrm{res}(L^{m/\alpha})$. The set $(\{H_m\}, m = 1, 2, \ldots)$ is the set of integrals in involution, and this family interpolates between the KP hierarchy ($\alpha = 1$) and n-KdV hierarchies ($\alpha = n$).

The same system of evolution equations on the subspace $\{L = D + \psi(x)D^{-1}\psi^*(x)\}$ (where $\psi(x)$ is a complex-valued function on the circle, and Hamiltonians $\widetilde{H}_m = i^m H_m$) generates NLS hierarchy. The classical NLS-equation corresponds to the Hamiltonian function $\widetilde{H}_2$, cf.[7,8].

This Poisson-Lie point of view suggests a geometric interpretation of relation between W_n and W_∞ - algebras appeared recently in theoretical physics.

REFERENCES

1. B.A. Khesin and I.S. Zakharevich, Poisson-Lie group of pseudodifferential symbols and fractional $KP - KdV$ hierarchies, C.R.Acad.Sci. **316** (1993), pp.621-626.

2. B.A. Khesin and I.S. Zakharevich, Poisson-Lie group of pseudodifferential symbols, preprint IHES (Oct. 1993), pp.64.

3. O.S. Kravchenko and B.A. Khesin, Central extension of the Lie algebra of pseudodifferential operators, Funct. Anal. Appl., **25** (1991) , n^{0}2, pp.83-85.

4. V.G. Kac and D.H. Peterson, Spin and wedge representations of infinite-dimensional Lie algebras and groups, Proc. Nat. Acad. Sci. USA, **78** (1981), pp.3308-3312.

5. M. Adler, On a trace functional for formal pseudodifferential operators and the symplectic structure of the Korteweg-deVries type equations, Inven. Math., **50** (1979), pp.219-248.

6. I.M.Gelfand and L.A.Dickey, A family of Hamiltonian structures associated with non-linear integrable differential equations. Preprint IPM AN SSSR - 139 (1978).

7. M.D. Freeman and P. West On the relation between integrability and infinite-dimensional algebras, preprint hep-th/9303119 (March 1993), pp.22.

8. L. Bonora and C.S. Xiong, Matrix models without scaling limit, preprint SISSA-ISAS 161/92/EP.