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BASIC BRANE MECHANICS

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Abstract.

The basic mechanical properties of classical brane (i.e. submanifold supported) models are presented using a fully covariant approach based on the use of embedding supported background tensors. The application to stationary equilibrium states of elastic string models is given a more particular attention, and a specific illustration is provided by the special case of the non-dispersive (integrable) string model.

1. Introduction.

The following review offers a brief introduction to the essential principles and archetypical applications of a formalism[1,2] that was originally designed for the specific purpose of describing the macroscopic behaviour of current carrying cosmic strings (a subject that will be briefly discussed in the final sections) but that is potentially useful for dealing systematicly with a much wider range of problems involving what have recently come to be known as *branes*. Specifically the term $p-1$ brane has come to be increasingly used[3,4] to denote a dynamical system with support confined to a p (or more specificly $p-1$ space plus one time) dimensional subsurface in a background spacetime of dimension n say, extreme cases being that of a point particle which qualifies as a 0-brane, and a continuous medium which is qualifies as an $n-1$ brane. In higher dimensional contexts, such as Kaluza-Klein type cosmology, many intermediate possibilities can be conceived, but the only non-trivial examples in ordinary space time with $n=4$ are those of a string, which qualifies as a 1-brane, and an ordinary (hypersurface supported) membrane which qualifies as a 2-brane.

The approached described here is designed to facilitate the derivation and expression of general laws by using a formulation that is fully covariant – working as far as possible in terms of quantities that are strictly tensorial – with respect to local *background* coordinates x^μ say ($\mu = 1, ..., n$), and as far as possible avoiding the technical complications that would result from recourse to specific systems of intrinsic coordinates for the various (possibly mutually intersecting) brane submanifolds that may be involved. An important example of a generally valid law that is easy to formulate in the formalism described here but not so simple in other systems is the equation for the characteristic covector χ_μ for extrinsic perturbations of the brane locus, which is given generically by

$$\widetilde{T}^{\mu\nu}\chi_\mu\chi_\nu = 0 \; . \tag{1.1}$$

where $\widetilde{T}^{\mu\nu}$ is the relevant (total) surface stress momentum energy density tensor which must be such that the solutions χ_μ are all real (in order to avoid local instability in infinitesimally small timescales[1]) and which must be also be such that

$$\overline{\overline{g}}^{\mu\nu}\chi_\mu\chi_\nu \geq 0 \tag{1.2}$$

(in order to avoid relativistic causality violation) where $\overline{\overline{g}}^{\mu\nu}$ is what I refer to as the (first) *fundamental tensor* of the imbedding, as derived from the background spacetime metric $g_{\mu\nu}$ by the operation of *tangential projection* which we shall systematicly denote by the use of an overhead parallelism symbol, $^{=}$, while using an overhead perpendicularity symbol $^{\perp}$ for the complementary orthogonal projection operation, so that the identity projection tensor splits up as the sum of a rank p tangential projection tensor $\overline{\overline{g}}^{\mu}{}_{\nu}$ and a rank $n-p$ orthogonal projection tensor $\overset{\perp}{g}{}^{\mu}{}_{\nu}$ in the form

$$g^\mu{}_\nu = \overline{\overline{g}}^{\mu}{}_{\nu} + \overset{\perp}{g}{}^{\mu}{}_{\nu} \; . \tag{1.3}$$

While desirable for reasons of notational simplicity and clarity whenever possible, it is particularly when several intersecting brane subsurfaces are involved that it is advantageous to avoid the use of tensors with indices defined with respect to internal coordinates ξ^i $(i = 1, ..., p)$ say that might be introduced for the explicit expression of a corresponding imbedding mapping, $\xi^i \mapsto x^\mu$. Thus although the background metric tensor $g_{\mu\nu}$ and a p dimensional Lagrangian density $\widetilde{\mathcal{L}}$ may naturally be used for the specification, as an intermediate step, of the correspondings internal metric tensor h_{ij} and internal stress momentum energy tensor t^{ij}, according to prescriptions of the standard forms

$$h_{ij} = g_{\mu\nu}\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j} \; , \qquad t^{ij} = 2\frac{\partial \widetilde{\mathcal{L}}}{\partial h_{ij}} + \widetilde{\mathcal{L}} h^{ij} \; , \tag{1.4}$$

the use of the kind of formalism described and advocated here requires as the next step that such quantities should be pulled back (in contravariant form) to give the corresponding background tensors

$$\overline{\overline{g}}^{\mu\nu} = h^{ij}\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j} \; , \qquad \widetilde{T}^{\mu\nu} = t^{ij}\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j} \; . \tag{1.5}$$

2. Curvature Tensors of the Imbedding.

In so far as we are concerned with tensor fields (such as those appearing in (1.5)) whose support is confined to an imbedded p surface, the effect of Riemannian covariant differentation ∇_μ along an arbitrary directions on the background spacetime will not be well defined, only the corresponding tangentially projected differentiation operation

$$\overline{\overline{\nabla}}_\mu \stackrel{\rm def}{=} \overline{\overline{g}}{}^\nu{}_\mu \nabla_\nu \ , \tag{2.1}$$

being meaningful for them. In particular this operation can be used for the construction of various kinds of purely geometric tensor characterising the curvature of the imbedding, starting with the *second* fundamental tensor, $K_{\mu\nu}{}^\rho$, which may be defined[5,1,2] in terms of the first fundamental tensor $\overline{\overline{g}}{}^\mu{}_\nu$ of the imbedded p-surface under consideration by

$$K_{\mu\nu}{}^\rho \stackrel{\rm def}{=} \overline{\overline{g}}{}^\sigma{}_\nu \overline{\overline{\nabla}}_\mu \overline{\overline{g}}{}^\rho{}_\sigma \ . \tag{2.2}$$

Such a tensor $K_{\mu\nu}{}^\rho$ is of course definable not only for the fundamental projection tensor of a p-surface, but also for any (smooth) field of rank p projection operators $\overline{\overline{g}}{}^\mu{}_\nu$ as specified by a field of arbitrarily orientated p-surface elements. What distinguishes the integrable case, i.e. that in which the elements mesh together to form a well defined p-surface through the point under consideration, is the condition that the tensor defined by (2.5) should also satisfy the *Weingarten identity*

$$K_{[\mu\nu]}{}^\rho = 0 \tag{2.3}$$

(where the square brackets denote antisymmetrisation), this symmetry property of the second fundamental tensor being derivable[2] as a version of the well known Frobenius theorem. In addition to this non-trivial symmetry property, the second fundamental tensor is also obviously tangential on the first two indices and almost as obviously orthogonal on the last, i.e.

$$\perp\!\!\!\!g_\mu{}^\sigma K_{\sigma\nu}{}^\rho = K_{\mu\nu}{}^\sigma \overline{\overline{g}}_\sigma{}^\rho = 0 \ . \tag{2.4}$$

The second fundamental tensor $K_{\mu\nu}{}^\rho$ has the property of fully determining the tangential derivatives of the first fundamental tensor $\overline{\overline{g}}{}^\mu{}_\nu$ by the formula

$$\overline{\overline{\nabla}}_\mu \overline{\overline{g}}_{\nu\rho} = 2K_{\mu(\nu\rho)} \tag{2.5}$$

(using round brackets to denote symmetrisation) and it can be seen to be characterisable by the condition that the orthogonal projection of the acceleration of any tangential vector field u^μ will be given by

$$\perp\!\!\!\!g^\rho{}_\mu u^\nu \nabla_\nu u^\mu = u^\mu u^\nu K_{\mu\nu}{}^\rho \ . \tag{2.6}$$

It is very useful for a great many purposes to introduce the *extrinsic curvature vector* K^μ, defined as the trace of the second fundamental tensor, i.e.

$$K^\mu \stackrel{\text{def}}{=} K^\nu{}_\nu{}^\mu \, . \qquad \overline{\overline{K}}^\mu = 0 \tag{2.7}$$

The specification of this extrinsic curvature vector for a timelike p-surface in a dynamic theory provides what can be taken as the equations of extrinsic motion of the p-surface[1] (the simplest case being the "harmonic" condition $K^\mu = 0$ obtained from a simple surface measure variational principle such as that of the Goto-Nambu string model or the Dirac membrane model). It is also useful for many purposes[2] to introduce the *extrinsic conformation* tensor $C_{\mu\nu}{}^\rho$ defined as the trace free part of the second fundamental tensor by

$$C_{\mu\nu}{}^\rho \stackrel{\text{def}}{=} K_{\mu\nu}{}^\rho - p^{-1}\overline{\overline{g}}_{\mu\nu}K^\rho \, , \qquad C^\nu{}_\nu{}^\mu = 0 \, . \tag{2.8}$$

which (like the Wey tensor of the background metric) has the noteworthy property of being conformally invariant with respect to conformal modifications of $g_{\mu\nu} \mapsto e^{2\sigma} g_{\mu\nu}$ of the background metric.

Going on to higher order we can introduce the *third* fundamental tensor[2] in an analagous manner as

$$\Xi_{\lambda\mu\nu}{}^\rho \stackrel{\text{def}}{=} \overline{\overline{g}}{}^\sigma{}_\mu \overline{\overline{g}}{}^\tau{}_\nu \overset{\perp}{g}{}^\rho{}_\alpha \overline{\overline{\nabla}}_\lambda K_{\sigma\tau}{}^\alpha \, , \tag{2.9}$$

which by construction is obviously symmetric between the second and third indices and tangential on all the first three indices. In a spacetime background that is flat (or of constant curvature as is the case for the DeSitter universe model) this third fundamental tensor is fully symmetric over all the first three indices by what is interpretable as the *generalised Codazzi identity* which is expressible[2] in a background with arbitrary Riemann curvature $R_{\lambda\mu}{}^\rho{}_\sigma$ as

$$\Xi_{\lambda\mu\nu}{}^\rho = \Xi_{(\lambda\mu\nu)}{}^\rho + \tfrac{2}{3}\overline{\overline{g}}{}^\sigma{}_\lambda \overline{\overline{g}}{}^\tau{}_{(\mu} \overline{\overline{g}}{}^\alpha{}_{\nu)} R_{\sigma\tau}{}^\beta{}_\alpha \overset{\perp}{g}{}^\rho{}_\beta \, . \tag{2.10}$$

The condition of preserving the tangent element to an imbedded p-surface at a point breaks down the full n dimensional rotation group preserving the background metric into the product of the restricted p dimensional rotation group preserving the induced metric in the imbedding with the restricted $(n-p)$ dimensional rotation group preserving the induced metric in the orthogonal element. Associated with each of these subgroups there is a corresponding naturally induced connection and covariant differentiation operator acting on the corresponding bundles of tangent vectors and orthogonal vectors respectively, and for each there will be a corresponding, respectively "inner" and "outer" bundle curvature, which will be represented corresponding background tensors,

$\widetilde{R}_{\mu\nu}{}^{\rho}{}_{\sigma}$ and $\Omega_{\mu\nu}{}^{\rho}{}_{\sigma}$ say, the former "inner" curvature tensor being just the pull-back onto the background by the imbedding mapping of the ordinary Riemann curvature of the intrinsic geometry induced by the imbedding. An overhead wide tilde is used here to distinguish the "inner" Riemann curvature tensor $\widetilde{R}_{\mu\nu}{}^{\rho}{}_{\sigma}$ from the ordinary Riemann tensor $R_{\mu\nu}{}^{\rho}{}_{\nu}$ of the background, which (for $n > 2$) will be decomposible in terms of the the corresponding (trace free conformally invariant) Weyl tensor $C_{\mu\nu}{}^{\rho}{}_{\sigma}$, and of the usual background Ricci tensor and Ricci scalar,

$$R_{\mu\nu} = R_{\rho\mu}{}^{\rho}{}_{\nu} \ , \qquad R = R^{\nu}{}_{\nu} \ , \tag{2.12}$$

in the form

$$R_{\mu\nu}{}^{\rho\sigma} = C_{\mu\nu}{}^{\rho\sigma} + \tfrac{4}{n-2} g^{[\rho}{}_{[\mu} R^{\sigma]}{}_{\nu]} - \tfrac{2}{(n-1)(n-2)} R g^{[\rho}{}_{[\mu} g^{\sigma]}{}_{\nu]} \ . \tag{2.13}$$

In terms of the tangential projection of the latter, as denoted by an overhead parallel symbol (indicating contraction with the fundamental tensor $\overline{\overline{g}}_{\mu}{}^{\nu}$), one can evaluate the corresponding *internal* curvature tensor in the form

$$\widetilde{R}_{\mu\nu}{}^{\rho}{}_{\sigma} = 2K^{\rho}{}_{[\mu}{}^{\tau}K_{\nu]\sigma\tau} + \overline{\overline{R}}_{\mu\nu}{}^{\rho}{}_{\sigma} \ , \tag{2.14}$$

which is the translation into the present scheme of what is well known in other schemes as the *generalised Gauss identity.* The much less well known analogue for the (identically trace free and conformally invariant) *outer* curvature, for which the most historically appropriate name is arguably that of Schouten, is given[2] in terms of the corresponding projection of the background Weyl tensor by the expression

$$\Omega_{\mu\nu}{}^{\rho}{}_{\sigma} = 2C_{[\mu}{}^{\tau\rho}C_{\nu]\tau\sigma} + \overline{\overline{g}}{}^{\kappa}{}_{\mu}\overline{\overline{g}}{}^{\lambda}{}_{\nu}C_{\kappa\lambda}{}^{\alpha}{}_{\tau}\,\overset{\perp}{g}{}^{\rho}{}_{\alpha}\,\overset{\perp}{g}{}^{\tau}{}_{\sigma} \ . \tag{2.15}$$

It follows from this last identity that in a flat or conformally flat background (for which it is necessary, and for $n \geq 4$ sufficient, that the Weyl tensor should vanish) the vanishing of the extrinsic conformation tensor $C_{\mu\nu}{}^{\rho}$ will be sufficient (independently of the behaviour of the extrinsic curvature vector K^{μ}) for vanishing of the outer curvature tensor $\Omega_{\mu\nu}{}^{\rho}{}_{\sigma}$, which is the condition for it to be possible to construct fields of vectors λ^{μ} orthogonal to the surface and such as to satisfy the generalised Fermi-Walker propagation condition to the effect that $\overset{\perp}{g}{}^{\rho}{}_{\mu}\overline{\overline{\nabla}}_{\nu}\lambda_{\rho}$ should vanish. It can also be shown[2] (taking special trouble for the case $p = 3$) that in a conformally flat background (of arbitrary dimension n) the vanishing of the conformation tensor $C_{\mu\nu}{}^{\rho}$ is always sufficient (though by no means necessary) for conformal flatness of the induced geometry in the imbedding.

3. Mechanics of a brane-complex.

Employment of brane models of dimension p lower than the background dimnsion n is often useful for providing an approximate description of higher dimensional configurations when the the fields characterising the latter are highly concentrated in the neighbourhood of a lower dimensional world sheet within a distance that is small compared with the scales characteristic of dynamic variations in directions tangential to the world sheet. Thus for example a point particle model might be useful for describing the motion, with respect to a relatively slowly varying background, of a small loop in a string model that might itself be just an approximation for decribing what at a more microscopically accurate level might need the use of a continuum model. The example that has been most important in motivating the development of the relativistic formalism described here is that of the representation (as originally suggested by Kibble[6], Witten[7] and others) of vortex defects (due to spontaneous symmetry breaking) of the vaccuum by ("cosmic") string models as a macroscopic approximation for use in the (cosmologically important) cases in which the vortex thickness can be treated as negligible compared with other relevant length scales. This lead to the introduction of models of variational type in which the action was to be thought of as being derived from the microscopic action of the relevant underlying field theory by integral across the vortex in a local equilibrium state.

The most general variational models involve a compound system with a total action of the form $\sum \mathcal{I}$, in which the action contribution of an individual p brane of the system will be given by a corresponding p surface integral

$$\mathcal{I} = \int \widetilde{\mathcal{L}} d\widetilde{\mathcal{S}} \tag{3.1}$$

where $d\widetilde{\mathcal{S}}$ denotes the induced surface measure and $\widetilde{\mathcal{L}}$ is a Lagrangian scalar function of whatever internal fields on the world sheet are involved and also of any relevant externally induced fields. We shall restrict our attention here to cases in which the only relevant background field is the (flat or gravitationally curved) spacetime metric, but there is no particular difficulty in allowing also for electromagnetic effects[1].

Let us consider the very large class of situations[1] that can be represented by a "well behaved" brane complex (or "rigging system") in which direct action of a lower on a higher dimensional brane occurs only when the former forms a smooth boundary segment of the latter (as when a monopole, treated as a point particle, forms the termination of a string, or when a sail forms the boundary between two external wind volumes), subject to dynamic equations to the effect that the infinitesimal variation of the relevant fields other than the externally determined background metric $g_{\mu\nu}$, gives no contribution to the variation of the combined action $\sum \mathcal{I}$ taken over the various brane constituents of the system, restricting ourselves to cases in which derivatives of the external fields $g_{\mu\nu}$ are not involved in the action. (The exclusion of more general derivative couplings merely avoids the extra technical complications that are present in more elaborate, e.g. polarised systems, but the exclusion of direct action except on a smooth boundary is more essential, being needed to avoid the serious divergence difficulties, exemplified by that of the radiation back reaction on a point particle, which would otherwise be involved.)

Since (using the surface tangentiality condition that the surface stress energy momentum tensor must satisfy by the construction) the variation in the action due to an infinitesimal diffeomorphism $g_{\mu\nu} \mapsto g_{\mu\nu} + \nabla_{(\mu}\zeta_{\nu)}$ of the metric will be given by the surface integral of an adjustment expressible as

$$\widetilde{T}^{\mu\nu}\nabla_\mu\zeta_\nu = \overline{\overline{\nabla}}_\mu(\zeta_\nu\widetilde{T}^{\mu\nu}) - \zeta_\nu\overline{\overline{\nabla}}_\mu\widetilde{T}^{\mu\nu} \ , \tag{3.2}$$

it then follows (by systematicly using (3.2) to convert divergences to boundary contributions) that the requirement that this combined action $\sum\mathcal{I}$ should be identically *invariant under diffeomorphisms* generated by an arbitrary vector field ζ^μ is equivalent to a local energy momentum conservtion law to the effect[1] that for each brane of the system we should have

$$\overline{\overline{\nabla}}_\nu\widetilde{T}^{\mu\nu} = f^\mu \ , \qquad f_\mu = \sum\lambda_\nu T^\nu{}_\mu \tag{3.3}$$

in which the force density on the right is obtained as the sum of contact contributions from the higher dimensional (untilded) stress momentum energy density tensor $T^{\mu\nu}$ of each of the attached p branes (at most two if $p = n$, but arbitrarily many for $p < n$) of which the $(p-1)$ brane under consideration is a boundary segment.

It typically occurs that the approximate macroscopic treatment of a system that is conservative, with a variational formulation, at a microscopic level may require the use of a non conservative macroscopic model involving averaging over microscopic degrees of freedom that are taken into count as entropy. Although it may invalidate the conservative nature of the model as a whole, such an averaging process does not invalidate the local conservation laws obeyed by additive quantities such as energy momentum or electromagnetic charge: what happens is that instead of having the status of *Noether identities* expressing the invariance properties that hold for the underlying variational model, such conservation laws are to be interpreted in the macroscopic model as *constistency conditions* for the existence of a corresponding microscopic variational model. These considerations imply that although the above direct derivation starts from a variational postulate, energy momentum conservation laws of the form (3.3) can still be expected to hold for more general dissipative models such as would be obtained by macroscopic averaging over internal degrees of freedom whose net effect would be taken into account in terms of entropy currents. An alternative (for some tastes more intuitive, but mathematically much more awkward) way of deriving (3.3) in such cases would be to consider the brane system as the infinitely thin limit of a continuous medium model where the stress energy momentum density components $T^{\mu\nu}$ are no longer continuous fields but have become Dirac distributions, whose coefficients are interpretable as the corresponding smooth world sheet supported fields $\widetilde{T}^{\mu\nu}$.

By whatever route they may have been obtained, the ubiquitous generality of (3.3) cannot be overemphasised. As an immediate cosequence (by an integration by parts) one obtains the general equation governing the extrinsic motion of any brane of the complex in terms of the second fundamental tensor $K_{\mu\nu}{}^{\rho}$ of its supporting subsurface, in the form of the " generalised sail equation"[1]

$$\widetilde{T}^{\mu\nu} K_{\mu\nu}{}^{\rho} = \overset{\perp}{f}{}^{\rho} , \tag{3.4}$$

where $\overset{\perp}{f}{}^{\rho}$ is the total orthogonally projected force contribution. It is from this final form that (1.1) is derived.

In the particular case of a free ($f_\rho = 0$) Goto Nambu string model[8] or a Dirac membrane model[9], as characterised by an action of the form (3.1) with $\widetilde{L} = L_0$ for some *fixed* value L_0, one will simply have $\widetilde{T}^{\mu\nu} = L_0 \overline{\overline{g}}{}^{\mu\nu}$, so it can be seen that in this particularly simple case the equation of motion (1.4) reduces simply to the harmonicity condition $K^\rho = 0$.

4. Perfect Brane Models.

For a general brane model, we can always define an energy density scalar, U say, as the negative of the eigenvalue specified by

$$\widetilde{T}^{\mu}{}_{\nu} u^{\nu} = -U u^{\nu} \tag{4.1}$$

where the corresponding eigenvector u^μ is distinguished by the requirement that it be timelike or null. As a widely applicable special case (including the Dirac membrane mentioned above, as well as *all* point particle and string models) a $(p-1)$ brane may be described as "perfect" if its surface stress momentum energy tensor is isotropic with respect to the other orthogonal directions, which in the generic case for which the eigenvector u^μ is strictly timelike (not null) and hence normalisable to unity, one gets[1] the explicit form

$$\widetilde{T}^{\mu}{}_{\nu} = (U-T) u^\mu u_\nu - T\, \overline{\overline{g}}{}^{\mu}{}_{\nu} , \qquad u^\mu u_\mu = -1 , \tag{4.2}$$

where T (the negative of the other $(p-1)$ degenerate eigenvalues) is what is interpretable as the *tension* of the $(p-1)$ brane.

The category of perfect branes includes, as the extreme case $p = n$, the example of an ordinary "perfect fluid" (with $U = \rho$, where ρ is the ordinary volume density of mass-energy, while $T = -P$ where P is the ordinary, positive, pressure). In the other cases, i.e. for a $(p-1)$ brane of lower dimension than the background, $p < n$, for which extrinsic displacements are possible (so that the tension must be non negative in order to avoid local instability[1]) the extrinsic motion will be governed by (3.4) which, on substitution of (4.2) gives the dynamic equations for a free perfect brane world sheet in the form

$$c_{_E}{}^2 K^\mu = (1 - c_{_E}{}^2) \overset{\perp}{g}{}^\mu{}_\nu \dot{u}^\mu \ , \qquad \dot{u}^\mu = u^\nu \nabla_\nu u^\mu \ , \tag{4.3}$$

where $\dot{u}^\mu$ is the acceleration vector of the unit eigenvector u^μ, and the quantity

$$c_{_E} = \sqrt{\frac{T}{U}} \tag{4.4}$$

represents the speed of propagation - relative to the preferred frame specified by u^μ - of extrinsic perturbations, as derived[1,5] from the general characteristic equation (1.1). It is to be noted that in the ultra relativistic case of a Dirac membrane or Goto Nambu string one has $c_{_E} = 1$ which means that the right hand side of (4.3) will vanish. On the other hand the strings and membranes that are commonly used (in violins, drums, etc.) by old fashionned non relativistic (i.e. non electronic) orchestras for music generation, will also be describable to a very good approximation by this *same* equation but with $c_{_E} << 1$, which means that the coefficient $c_{_E}{}^2$ will be able to be neglected on the right though not of course on the left.

The extreme case of a "zero brane" with $p = 1$, i.e. that of an ordinary (massive) point particle, can be considered as being automatically of the perfect type characterised by (4.2) with $U = m$ where m is its mass, and with identically vanishing tension $T = 0$ which is consistent with the obvious necessity of having zero relative speed of propagation of any perturbation in this one dimesional case. For a point particle trajectory the first and second fundamental tensors will be given simply by

$$\overline{\overline{g}}{}^\mu{}_\nu = -u^\mu u_\nu \ , \qquad K_{\mu\nu}{}^\rho = \overline{\overline{g}}_{\mu\nu} K^\rho \ , \qquad K^\mu = -\dot{u}^\mu \ , \tag{4.5}$$

while in terms of the particle mass m say substitution of the appropriate expression

$$\widetilde{T}^\mu{}_\nu = -m \, \overline{\overline{g}}{}^\mu{}_\nu \ , \tag{4.6}$$

into the general expressions (3.3) and (3.4) gives the dynamical equations for free motion in the familiar form

$$u^\mu \nabla_\mu m = 0 \ , \qquad -m K_\mu = 0 \ . \tag{4.7}$$

The case of a membrane in 4-dimensions (or more generally of an $(n-2)$ brane in n dimensions) shares with the opposite extreme case of a point particle the property of having comparatively simple kinematic properties, since any timelike hypersurface has first and second fundamental tensors that are expressible in terms of its unit normal λ_μ (as specified by an arbitrary choice of orientation) in the form

$$\overline{\overline{g}}{}^\mu{}_\nu = g^\mu{}_\nu - \lambda^\mu \lambda_\nu \ , \qquad K_{\mu\nu}{}^\rho = K_{\mu\nu}\lambda^\rho \ , \qquad \lambda_\mu \lambda^\mu = 1 \ . \tag{4.8}$$

Analogously to the way the first fundamental tensor $\overline{\overline{g}}{}^{\mu\nu}$ is specifiable (by (1.5)) as the pull back of the contravariant version of the induced metric, i.e. of what is commonly known as the first fundamental form of the imbedding, so analogously the symmetric tensor $K^{\mu\nu}$ is the pull back of the contravariant version of what is commonly known as the *second fundamental form* on the hypersurface, a quantity whose specification, like that of the unit normal λ_μ involves an arbitrary choice of sign. (In addition to its principle advantage of being applicable to imbeddings of arbitrary dimension, not just hypersurfaces, a convenient feature of the three index second fundamental *tensor*, as compared with the two index second fundamental *form* even in the hypersurface case where the latter is available, is that unlike that of $K_{\mu\nu}$ the specification of $K_{\mu\nu}{}^\rho$ is quite unambiguous.) Whereas the kinematic specifications (4.8) are simpler than their analogues for the lower dimensional case of a string, on the other hand the dynamics of a membrane are generally more complicated. Unlike the case of a string model which must always, trivially, be perfect in the sense of (4.2) (or of its null limit[1]) the postulate of "perfection" in this sense is a serious restriction in the case of a membrane, being satisfied for a Dirac membrane or an ordinary soap bubble type membrane, (and even as a reasonable approximation to the way musical drum membranes are most commonly tuned), but it will not be at all valid for such applications as to a typical ship's sail.

5. Strings.

Between the hypersurface supported case of a membrane and the curve supported case of a point particle the only intermediate kind of brane that can exist in 4-dimensions is that of 1-brane, i.e. a string model, which (for any background dimension n) will have a first fundamental tensor that is expressible as the square of the antisymmetric tangential tensor $\mathcal{E}^{\mu\nu}$ that is defineable[10] as the pullback of the contravariant version of the induced measure tensor that is specified modulo a choice of orientation by the imbedding, i.e. we shall have

$$\overline{\overline{g}}{}^\mu{}_\nu = \mathcal{E}^\mu{}_\rho \mathcal{E}^\rho{}_\nu \ , \qquad \mathcal{E}^{\mu\nu} = \overline{\overline{\mathcal{E}}}{}^{[\mu\nu]} \ . \tag{5.1}$$

A special feature distinguishing string models from point particle models on one hand and from higher dimensional brane models on the other is the *dual symmetry*[11] that exists at a formal level between the spacelike and timelike *unit eigenvectors* u^μ (as already introduced) and v^μ that for a generic case (excluding the null state limit[1]) are characterised modulo a choice of orientation by the expressions

$$\widetilde{T}^{\mu\nu} = U u^\mu u^\nu - T v^\mu v^\nu \ , \qquad \mathcal{E}^{\mu\nu} = u^\mu v^\nu - u^\nu v^\mu \ , \tag{5.2}$$

in which the tension T appears as the dual analogue of the "rest frame" energy per unit length U. This formal duality can also be made apparent in the expression for the extrinsic curvature vector of the string, which can be expressed as

$$K^\mu = \overset{\perp}{g}{}^\mu{}_\nu (v'^\nu - \dot{u}^\nu) \ , \qquad v'^\mu = v^\nu \nabla_\nu v^\mu \ , \qquad \dot{u}^\mu = u^\nu \nabla_\nu u^\mu \tag{5.3}$$

whose substitution in (4.3) enables the equation of extrinsic motion of a free string to be expressed in the manifestly self dual form

$$U \, \overset{\perp}{g}{}^\mu{}_\nu \dot{u}^\nu = T \, \overset{\perp}{g}{}^\mu{}_\nu v'^\nu \ . \tag{5.4}$$

Of course the extrinsic equation of motion, whether of the general form (3.4) or the free string specialised form (5.4), cannot actually be used to determine the evolution of the world sheet until the appropriate prescription has been given for evaluating the necessary stress momentum energy tensor components, which in the string case can be taken to be just T and U. In the simple Goto-Nambu case, for which these eigenvalues are specified in advance to have constant values, $U = T = -L_0$, no further preparation is needed for the integration of (5.4) but in general, for a string model with non trivial intrinsic structure the completion of the system of equations of motion will involve the specification of other differential equations. The simplest non trivial possibility, which is applicable to higher dimensionsional perfect brane models as well as to strings, is what is known in the specific context of perfect fluid theory as the "barotropic" case, meaning the case in which T is specified (directly or parametrically) as a function only of U by a *single equation of state*. In this barytropic case (which includes the Witten type conducting cosmic string models[7] whose investigation provided the original motivation for this work) the only differential equations that are needed to supplement the extrinsic equation of motion (4.3) or (5.4) are those that are obtained from the projection into the world sheet of the full local momentum energy conservation equation (3.3), which in the force free case simply gives

$$\overline{\overline{\nabla}}_\rho \widetilde{T}^{\rho\sigma} = 0 \tag{5.5}$$

whose two independent components can be conveniently expressed as a pair of mutually dual surface current conservation laws given by

$$\overline{\overline{\nabla}}_\rho (\nu u^\rho) = 0 \ , \qquad \overline{\overline{\nabla}}_\rho (\mu v^\rho) = 0 \ , \tag{5.6}$$

in terms of an effective number density ν and an associated effective mass density μ that are obtained from the equation of state as functions of U or equivalently of T by a pair of(mutually dual) integral relations of the form

$$\ln \nu = \int \frac{dU}{U - T} \ , \qquad \ln \mu = \int \frac{dT}{T - U} , \tag{5.7}$$

which fix them modulo a pair of constants of integration of which one is conventionally fixed by imposing the (self dual) restraint condition

$$\mu\nu = U - T \ . \tag{5.8}$$

Appart from the extrinsic perturbations of the world sheet location itself, which propagate with the "brane wave" speed c_E (relative to the frame deterined by u^μ) as already discused, the only other kind of perturbation mode that can occur in a barytropic string are longitudinal modes specified by the variation of U or equivalently of T within the world sheet. Such longitudinal perturbations (the analogue of ordinary sound waves in a perfect fluid) can easly be seen[1,5] to have a relative propagation velocity given by

$$c_L = \sqrt{\frac{\nu d\mu}{\mu d\nu}} = \sqrt{\frac{-dT}{dU}} \tag{5.9}$$

which must be real in order for the string to be locally stable. Knowledge of whether the longitudinal perturbation speed c_L is greater or less than the extrinsic speed c_E may be critically significant for questions such as the stability of stationary rotating ring equilibrium states[1,12,13,14] and their deformed generalisations[15] which will be discussed in the final section and which for Witten type cosmic strings (as opposed to the ordinary Goto Nambu type for which no such equilibrium states exist) may be cosmologically important[12,16,17]. Most early, and many more recent discussions[18,19,20,21] of Witten type strings were implicitly based on the use of an equation of state for which the sum $U+T$ remains constant, which implies longitudinal propagation at a speed equal to that of light, $c_L = 1$ which thus necessarily exceeds c_E, but more accurate investigations[22,23] have recently been developed[24] to a stage at which it is becoming increasingly clear that in general the opposite the case, i.e. Witten type models would seem to be typified by $c_L < c_E$.

A very special interest attaches to the intermediate, *non-dispersive* case characterised by $c_E = c_L$, which corresponds to an equation of state specified by either of the mutually dual relations

$$U = \tilde{m}\sqrt{m^2 + \nu^2} \ , \qquad T = m\sqrt{\tilde{m}^2 - \mu^2} \tag{5.10}$$

where m and $\tilde{m}$ are constant mass parameters, so that the eigenvalue product is constant:

$$TU = m\tilde{m} \ . \tag{5.11}$$

This leads to dynamic equations that I have shown[25] to be explicitly integrable (like those of the degenerate Goto Nambu case) in a flat spacetime background, the general form in an arbitrary curved background being expressible as

$$L_\pm{}^\nu \nabla_\nu L_\mp{}^\mu = 0 \ , \qquad L_\pm{}^\mu = \frac{\sqrt{U}u^\mu \pm \sqrt{T}v^\mu}{\sqrt{U - T}} \ , \tag{5.12}$$

where the (timelike) unit vectors $L_{\pm}{}^{\mu}$ are directed along the "left" and "right" moving unit characteristic directions, the former being parallel propagated by the latter and vice versa.

This special "constant product" string model can be recognised[1] as turning up spontaneously by the Nielsen mechanism[26] in the context of Kaluza Klein theory[27,28,29] (an observation which incidentally invalidates the uncautious claim[27] that the Nielsen mechanism[26] leads to dynamical behaviour equivalent to that resulting from the more complicated but more physically realistic Witten mechanism[7], since, as remarked above, the latter gives rise to more complicated dispersive equations of state[24] that are generically characterised by the strict inequality $c_{_L} < c_{_E}$.) The special integrable equation of state with $c_{_L} = c_{_E}$ is not just of purely mathematical interest: my heuristic argument[25] to the effect that that it should provide a good description of the averaged effect of random noise perturbations on an "ordinary" Goto-Nambu type cosmic string (on the grounds that their presence should not introduce dispersion) has been confirmed by Vilenkin's more detailed "wiggly string "calculations[30].

6. Stationary Applications.

Whenever the background is stationary in the sense of having a metric that is invariant under the action generated by a timelike vector field k^ρ, which must therefore satisfy the Killing equations

$$\nabla_{(\rho} k_{\sigma)} = 0 \ , \tag{6.1}$$

it will be of interest to consider the corresponding subclass of *equilibrium solutions*[15], i.e. the class of solutions that are themselves stationary with respect to the action generated by the same vector field. This condition of stationarity means not only that the imbedded 2-surface of support of the string should include the Killing vector as a tangent vector, i.e.

$$k^{[\rho} \mathcal{E}^{\sigma\tau]} = 0 \tag{6.2}$$

where $\mathcal{E}^{\rho\sigma}$ is the tangent element introduced in (5.1), but also that the the corresponding invariance conditions should be satisfied by the physical fields characterising the internal state of the string, and in particular by its surface stress momentum energy tensor $\widetilde{T}^{\rho\sigma}$ for which the required stationarity condition takes the form

$$k^\rho \overline{\overline{\nabla}}_\rho \widetilde{T}^{\sigma\tau} = 2\widetilde{T}^{\rho(\sigma} \overline{\overline{\nabla}}_\rho k^{\tau)} \ . \tag{6.3}$$

When these stationarity conditions are satisfied it is easy to see that the internal equations of motion (5.6) imply the existence of a corresponding pair of Bernouilli type constants of the motion,

$$\omega = \mu u_\rho k^\rho \ , \quad \beta = \nu v_\rho k^\rho \ , \tag{6.4}$$

which can be extrapolated off the string world sheet as a pair of spacetime fields satisfying the uniformity conditions

$$\nabla_\rho \omega = 0 \ , \qquad \nabla_\rho \beta = 0 \ . \tag{6.5}$$

The ratio

$$v = \frac{\beta \mu}{\omega \nu} \tag{6.6}$$

is interpretable as the longitudinal running velocity of the preferred rest frame of the string relative to that of the stationary background. The special case $\beta = 0$ for which this running velocity vanishes is thus interpretable as an equilibrium that is not just statonary but actually static with respect to the background rest frame determined by the Killing field. It evidently follows from (6.4) that on the world sheet the norm of the Killing vector will be given in terms of the "tuning constants" ω and β by the relation

$$k_\rho k^\rho = \frac{\beta^2}{\nu^2} - \frac{\omega^2}{\mu^2} \ , \tag{6.7}$$

while conversely, using the functional relation between the effective mass μ and the number density ν that is specified by the equation of state, the relation (6.7) can be thought of as implicitly determining the quantities μ and ν, and thus also the quantities T and U, as functions of $k_\rho k^\rho$ not only on the particular stationary string world sheet under consideration but by extension via (6.5) as scalar fields over the entire stationary background.

A systematic variational approach[15] to the stationary string problem has drawn attention to the special interest of a particular string worldsheet generating vector given by

$$\sigma^\rho = k^\mu \mathcal{E}_{\mu\nu} \widetilde{T}^{\nu\rho} \ , \tag{6.8}$$

whose norm σ, as given by

$$\sigma^2 = \sigma_\rho \sigma^\rho = \frac{\omega^2 T^2}{\mu^2} - \frac{\beta^2 U^2}{\nu^2} \tag{6.9}$$

is also extensible off the string world sheet as a field over the spacetime background via the implicit relation (6.7). Using the relations (6.4) and (6.5), which are such as to automatically ensure the satisfaction of the internal equations of motion (5.6), it can be verified directly, without reference to the variational analysis that motivated the introduction of this vector σ^ρ, that it must identically satisfy the relation

$$\sigma^\nu \nabla_\nu \sigma^\rho - \sigma \nabla^\rho \sigma = (k^\sigma \mathcal{E}_{\sigma\tau} \sigma^\tau)(\widetilde{T}^{\mu\nu} K_{\mu\nu}{}^\rho) \ , \tag{6.10}$$

where σ is the field specified by (6.9).

It obviously follows from the identity (6.9) that the the dynamic equation for free motion of the string automatically requires that the vector field σ^μ should satisfy an equation of motion of the simple conformal geodesic form

$$\sigma^\nu \nabla_\nu \sigma^\rho = \sigma \nabla^\rho \sigma \tag{6.11}$$

and conversely that this equation (6.11) is sufficient to ensure satisfaction of the extrinsic equation of free motion (the intrinsic equations of motion (5.6) being satisfied in any case by (6.4) and (6.5)) to the effect that the combination

$$\widetilde{T}^{\mu\nu} K_{\mu\nu}{}^\rho = \overset{\perp}{g}{}^\rho{}_\mu (U \dot{u}^\mu - T v'^\mu) \tag{6.12}$$

should vanish, provided the world sheet generating vector σ^μ is independent of the Killing vector k^μ itself. The exceptional case excluded by this independence requirement is that for which the factor

$$k^\sigma \mathcal{E}_{\sigma\tau} \sigma^\tau = \frac{\beta^2 U}{\nu^2} - \frac{\omega^2 T}{\mu^2} \tag{6.13}$$

also vanishes, so that (6.9) leaves (6.10) indeterminate. This special case is evidently is describable as *transcharacteristic* since the vanishing of the factor given by (6.13) is equivalent to the condition

$$v^2 = c_E{}^2 \tag{6.14}$$

where v is the running velocity given by (6.6) and c_E is the extrinsic perturbation speed as given by (4.4).

In the special case of a string with the non-dispersive equation of state specified by (5.10) it is easy to solve (6.7) and substitute the result in (6.9) so as to obtain the explicit expression

$$\sigma^2 = -m^2 \tilde{m}^2 k^\rho k_\rho - m^2 \omega^2 - \tilde{m}^2 \beta^2 \tag{6.15}$$

A recent addition to the long list of rather miraculous special properties of the Kerr black hole solutions is the discovery[15] that, for a background of this particular form, with k^μ taken as the unique asymptoticly timelike ("primary") Killing vector, the substitution of (6.15) in (6.11) gives an equation for σ^μ that is explicitly soluble by separation of variables in the corresponding Hamilton Jacobi equation.

The transcharacteristic case specified by (6.14) is of particular importance in the context of stationarity of the ordinary kind, as determined with respect to a *non-rotating* time translation generator in a *flat* (i.e. Minkowski) background space-time, for which the generic Killing equations (6.1) may be replaced by the stronger (unsymmetrised) condition

$$\nabla_\mu k_\nu = 0 \ . \tag{6.16}$$

This condition (6.16) implies in particular that the magnitude $k_\rho k^\rho$ is constant and thus that all the corresponding string equilibrium states must be intrinsicly uniform, since (6.7) will then give correspondingly constant values for the quantities μ, ν, T, U and hence also for the running velocity v as given by (6.6), and for the scalar σ that is given by (6.9). The constancy of the latter implies that in these circumstances the right hand side of (6.11) will vanish, so that the equation of motion for the worldsheet generating vector σ^ρ reduces to that of an ordinary geodesic. This means that all the equilibrium configurations will simply be *straight*, except in the transcharacteristic case, for which there is no restriction at all on the geometric configuration, since it can be seen that subject to (6.16) (though not in the more general conditions that might apply in a curved background) the condition (6.14) is sufficient by itself to ensure that the equilibrium equation

$$\widetilde{T}^{\mu\nu} K_{\mu\nu}{}^{\rho} = 0 \tag{6.17}$$

will be satisfied.

A particularly important application is to the case of closed string loops, for which the locally conserved currents appearing in (5.6) give rise to corresponding globally conserved integrals

$$B = \oint dx^\rho \mathcal{E}_{\rho\sigma}(\nu u^\sigma) \ , \qquad C = \oint dx^\rho \mathcal{E}_{\rho\sigma}(\mu v^\sigma) \ , \tag{6.18}$$

and for which straight solutions are obviously excluded topologically, which implies that their stationary equilibrium states must *necessarily* be of the special transcharacteristic type characterised by (6.14). This class of topologically compact transcharacteristic equilibrium solutions includes, as the geometrically simplest possibility, the subclass of *circular* ring configurations that have been the subject of several previous discussions[1,12,13,14]: among the geometrically arbitrary equilibrium configurations that are in principle possible, the circular configurations can be singled out as those whose angular momentum has the maximum value, $J = BC/2\pi$ that is compatible with the given values of the conserved integrals B and C.

The physical interest of any such equilibrium states is of course dependent on the extent to which they are stable. No thorough stability analysis has yet been carried out, but the most obvious and easily verifiable requirement, namely that for given values of the integrals (6.18) the total energy should not be able to be reduced by uniform extension or contraction, leads[1,14] to the requirement that the extrinsic and longitudinal perturbation velocities c_E and c_L defined by (4.4) and (5.9) should satisfy the inequality

$$\frac{c_E{}^2}{c_L{}^2} \geq \frac{3c_E{}^2 - 1}{3 - c_E{}^2} \tag{6.19}$$

whose validity is guaranteed by the negativity of the right hand side whenever $c_E^2 < 1/3$, as will be the case for ordinary non relativistic applications (as exemplified by a cowboy's lasso loop) for which one will have $T << U$.

Another reqirement that one would expect to be necessary for strict stability is based on the argument[31] developed by Friedman and Schutz to the effect that under rather general conditions a perturbation mode whose propagation velocity is slower than that of the unperturbed background is likely to be unstable with respect to any kind of radiation to which it may be coupled. Since there will always be at least a very weak coupling to gravitational radiation, a corresponding very long timescale instability is to be expected whenever the longitudinal perturbation velocity is smaller than the extrinsic perturbation velocity that determines the equilibrium running speed v, i.e. whenever

$$\frac{{c_E}^2}{{c_L}^2} \geq 1 \ . \tag{6.20}$$

Although this does not occur in the approximate linearised model implicitly used in most early and many more recent discussions[18,19,20,21] of Witten type strings, it has recently been found by Peter[24] that, on the contrary, in a more exact treatment the inequality (6.20) is always satisfied. Since the right hand side of (6.19) is always less than unity, this result of Peter establishes that the Witten string loops will always satisfy this short-timescale dynamical stability requirement, but it also indicates that they will in principle be subject to a long-timescale radiative instability. In practice however, the analogy with the familiar case of the Friedman Schutz instability of rotating stars suggests that, if the only relevant radiation is gravitational, such an instability is likely to be far too weak to be significant on cosmological timescales (whereas if electromagnetic coupling is involved, the consequences might be more important).

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