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ANYONIC GROUPS (*)

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ABSTRACT We introduce non-standard quasitriangular structures on the finite groups $\mathbf{Z}_n$. These determine non-trivial braidings Ψ in the category of $\mathbf{Z}_n$-graded vector spaces. The braiding is an anyonic one, $\Psi(v\otimes w) = e^{\frac{2\pi i|v||w|}{n}} w\otimes v$ for homogeneous elements of degree $|v|$, $|w|$. This category of anyonic vector spaces generalizes that of super vector spaces, which are recovered as $n=2$. We give examples of anyonic quantum groups. These are like super quantum groups with ± 1 statistics generalised to anyonic ones. They include examples obtained by transmutation of $U_q(sl(2))$ at a root of unity.

1 Introduction

Quasitensor or "braided monoidal" categories occur naturally in a variety of contexts, notably in low-dimensional algebraic quantum field theory[1][2][3] and conformal field theory, and of course in the context of low dimensional topology as in [4]. They also occur in the abstract representation theory of quantum groups[5, Sec. 7] and were introduced on purely category-theoretical grounds in [6]. The quantum groups connection is now well-known and leads to the construction of link and three-manifold invariants. See notably [7]. In [8][9][10][11][12] we have developed a different point of view in which such quasitensor categories are braided generalizations of the category SuperVec of super vector spaces. From this point of view we introduced the notion of groups and Hopf algebras living in such braided categories, with the role of ± 1 statistics in the super case now played by the braiding or quasisymmetry Ψ. These braided groups and quantum braided groups

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(*) Ce texte reprend une prépublication du D.A.M.T.P. de Mars 1992 - DAMTP/91-16 are modelled on super groups and super quantum groups respectively. An interesting feature now, however, is that since the role of transposition is played by a braiding, many algebraic computations inevitably reduce to braid and tangle diagrams. In spite of this, many properties of braided groups and quantum braided groups parallel those of groups and quantum groups. Hence the subject lies on the boundary between low-dimensional topology and abstract algebra, and uses ideas from both areas.

Our goal in the present note is to illustrate this abstract theory of braided groups and quantum braided groups by means of some concrete examples. It is hoped in this way to provide an introduction to the more technical works [11][12]. For our illustration we have chosen the braided monoidal category of anyonic vector spaces.

An anyonic vector space for us is nothing other than a vector space in which the finite group of order n, $\mathbb{Z}_n$, acts. This is related to more sophisticated notions of group-graded spaces though we do not develop this point of view in the present paper. The braiding Ψ in the category of anyonic vector spaces is the one familiar to physicists in the context of anyonic statistics, e.g. [13][14]. Hence the name. The modest result of Section 2 is to identify this category of $\mathbb{Z}_n$-representations with this braiding Ψ, as the category of representations of a certain quantum group $\mathbb{C}\mathbb{Z}'_n$. This is the group algebra $\mathbb{C}\mathbb{Z}_n$ of $\mathbb{Z}_n$ with a certain non-standard quasitriangular structure or "universal R matrix", which we introduce. The case $\mathbb{C}\mathbb{Z}'_2$ (giving the category of super vector spaces) was studied in [10][11]. We also give formulae for the anyonic dimension of objects in the category, and the anyonic trace.

We then proceed in Section 3 to construct examples of quantum groups in such anyonic categories, i.e. anyonic quantum groups. Working with anyonic quantum groups and algebras is like working with super quantum groups and super algebras except that the $\Psi = \pm 1$ statistics are replaced by anyonic ones, namely by complex phase factors. Our first example is an anyonic version of the symmetry group of an equilateral triangle. Our second is an anyonic version of the quantum group $U_q(sl(2))$ at a root of unity.

In Section 4 we give the general construction for braided categories that was used to obtain the anyonic ones. The construction is based on an arbitrary self-dual Hopf algebra H and associates to it a commutative quasitriangular one, H'. Here $H = \mathbb{C}\mathbb{Z}_n$ is an example where $(\mathbb{C}\boldsymbol{Z}_n)^* \cong \mathbb{C}\hat{\boldsymbol{Z}}_n \cong \mathbb{C}\boldsymbol{Z}_n$ is self-dual because $\boldsymbol{Z}_n$ is self dual (i.e. $\hat{\boldsymbol{Z}}_n$, the set of irreducible representations, can be identified with $\boldsymbol{Z}_n$). The first isomorphism here is Fourier's convolution theorem. We note that other generalizations of $\boldsymbol{Z}_n$-graded spaces have been considered in the literature, for example to G-graded spaces (where G may be non-Abelian), see e.g.[15]. By contrast, our generalization by means of self-dual Hopf algebras appears to be in a different direction.

Preliminaries

For an informal introduction to quasitensor or "braided monoidal" categories as generalizing supersymmetry, we refer to [9][10]. A pure mathematical treatment is in [11][12] while a general introduction in the context of quantum groups is in [5, Sec. 7]. Briefly, a quasitensor category is $(\mathcal{C}, \otimes, \underline{1}, \Phi, \Psi)$ where $\mathcal{C}$ is a category (a collection of objects $X, Y, \cdots$ and morphisms or "maps" between them) and $\otimes$ is a tensor product with unit object $\underline{1}$. $\Phi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ are associativity isomorphisms for any three objects and $\Psi_{X,Y} : X \otimes Y \to Y \otimes X$, the *braiding* or "quasisymmetry" between any two. Such categories were first introduced in [6]. We will usually suppress Φ, as well as isomorphisms associated with the unit object. These are all trivial in the examples of the present paper. Then Ψ obeys

$$\Psi_{X,Y\otimes Z} = \Psi_{X,Z} \circ \Psi_{X,Y}, \quad \Psi_{X\otimes Y,Z} = \Psi_{X,Z} \circ \Psi_{Y,Z}, \quad \Psi_{X,\underline{1}} = \Psi_{\underline{1},X} = \mathrm{id}. \tag{1}$$

We work over a commutative field k. Our examples are over $\mathbb{C}$. A quantum group over k in the usual sense means for us a quasitriangular Hopf algebra $(H, \Delta, \epsilon, S, \mathcal{R})$ where H is an algebra over k, $\Delta : H \to H \otimes H$ the coproduct, $\epsilon : H \to k$ the antipode, $S : H \to H$ the antipode and $\mathcal{R}$ the quasitriangular structure or "universal R-matrix". For (H, Δ, ϵ, S) (a Hopf algebra) we use the notation of [16]. The notation for Δ from [16] is $\Delta h = \sum h_{(1)} \otimes h_{(2)}$ for $h \in H$. It is required to be an algebra homomorphism $H \to H \otimes H$ where $H \otimes H$ has the tensor product algebra structure. For the axioms of $\mathcal{R}$ we use the ones introduced by Drinfeld[17], namely

$$(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \quad (\Delta^{\mathrm{op}} \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{23}\mathcal{R}_{13}, \quad \mathcal{R}(\Delta h) = (\Delta^{\mathrm{op}} h)\mathcal{R} \tag{2}$$

for all $h \in H$. Here $\mathcal{R}$ is assumed invertible, $\mathcal{R}_{12} = \mathcal{R} \otimes 1$ etc, and $\Delta^{\mathrm{op}} h = \sum h_{(2)} \otimes h_{(1)}$

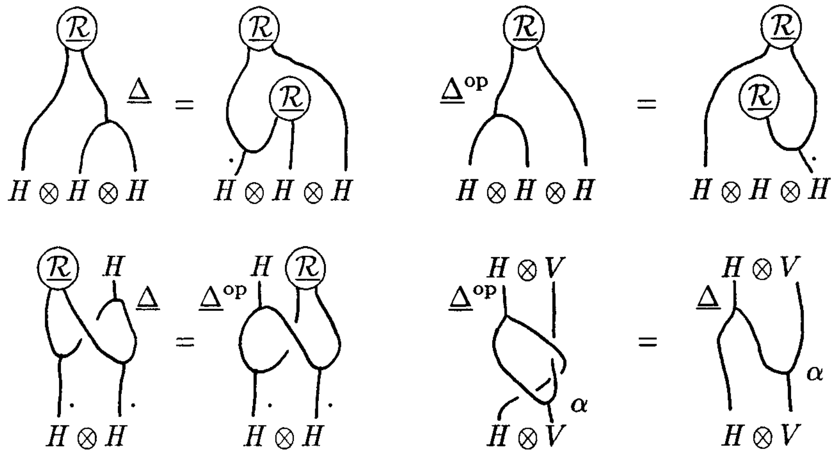

Figure 1: Axioms for $\underline{\mathcal{R}}$ and $\underline{\Delta}^{\rm op}$

is the opposite coproduct. We have written the middle axiom in a slightly unconventional form but one that generalizes immediately to quantum groups in braided categories. It is easy to deduce that $(\epsilon\otimes{\rm id})(\mathcal{R}) = 1 = ({\rm id}\otimes\epsilon)(\mathcal{R})$ and $(S\otimes{\rm id})(\mathcal{R}) = \mathcal{R}^{-1}$, $(S\otimes S)(\mathcal{R}) = \mathcal{R}$. For an introduction to quantum groups see [5], or the fundamental work [17].

The axioms of a quasitriangular Hopf algebra $(\underline{H}, \underline{\Delta}, \underline{\Delta}^{\rm op}, \underline{\epsilon}, \underline{S}, \underline{\mathcal{R}})$ in a quasitensor category $\mathcal{C}$ (i.e. a quantum braided group or "braided quantum group") are just the same except that $\underline{\Delta}$ and $\underline{\mathcal{R}}$ are defined with respect to the braided tensor product algebra structure[11]. In the concrete cases below, this is

$$(a\otimes b)(c\otimes d) = a\Psi(b\otimes c)d, \qquad a,b,c,d \in \underline{H}\otimes\underline{H}. \tag{3}$$

By this we mean to first apply $\Psi_{\underline{H},\underline{H}}$ to $b\otimes c$ and then multiply the result on the left by a and on the right by d. The axioms for $\underline{\mathcal{R}}$ are depicted in diagrammatic form in Figure 1. Morphisms are written pointing downwards. Binary or cobinary ones are written as vertices and $\Psi =$, $\Psi^{-1} =$. Functoriality of Ψ under morphisms means that branches and nodes can be translated past braidings in an obvious way. For our purposes, this is a useful (and well known) shorthand. In the figure, we see the use of the braided tensor product algebra structure of $\underline{H}\otimes\underline{H}$ in defining $\underline{\mathcal{R}}(\underline{\Delta}\ ) = (\underline{\Delta}^{\rm op}\ )\underline{\mathcal{R}}$. Likewise for the axioms of the coproduct $\underline{\Delta}$. If $\Psi^2 \neq {\rm id}$ then $\underline{\Delta}^{\rm op}$ can not be given simply by $\Psi\circ\underline{\Delta}$ or $\Psi^{-1}\circ\underline{\Delta}$, and needs to be specified. It is defined as a second coproduct structure on $\underline{H}$, characterized by the condition shown in the figure with respect to a class of modules (V,α)

of $\underline{H}$ in the category[11]. If $\underline{\Delta}^{\rm op} = \underline{\Delta}$ then we say that we have a braided group rather than a quantum braided group. We note that algebraic structures in other symmetric (not braided) monoidal categories have been studied by many authors, notably [18][19]. The novel aspect of our work is to go further to the study of algebraic structures in truly braided categories.

2 Anyonic Vector Spaces

In this section we study quasitensor or "braided monoidal" categories associated to $\boldsymbol{Z}_n$, the finite group of order n. Let g be the generator of $\boldsymbol{Z}_n$ with $g^n = 1$. As a category of objects and morphisms we take the category Rep$(\boldsymbol{Z}_n)$ of finite-dimensional representations of $\boldsymbol{Z}_n$. Given an object V of Rep$(\boldsymbol{Z}_n)$ we can decompose it under the action of $\boldsymbol{Z}_n$ as

$$V = \oplus_{a=0}^{n-1} V_a, \qquad a = 0, 1, \cdots, n-1.$$

Here a runs over the set of irreducible representations ρ_a of $\boldsymbol{Z}_n$ and V_a is the subspace of V where g acts as copies of ρ_a. Explicitly,

$$g \triangleright v = e^{\frac{2\pi \imath a}{n}} v, \qquad \forall v \in V_a$$

where the action of $\boldsymbol{Z}_n$ is simply denoted $\triangleright$. If $v \in V_a$, we say that v is homogeneous of degree $|v| = a$.

On this category Rep$(\boldsymbol{Z}_n)$ we can now define the non-standard braiding

$$\Psi_{V,W}(v \otimes w) = e^{\frac{2\pi\imath |v||w|}{n}} w \otimes v \tag{4}$$

on homogeneous elements of degree $|v|$, $|w|$. This is well known to physicists in the context of anyons, e.g.[13][14]. The quantities $e^{\frac{2\pi\imath |v||w|}{n}}$ can be called fractional or anyonic statistics. We denote by $\mathcal{A}_n$ the category Rep$(\boldsymbol{Z}_n)$ equipped with this anyonic braiding. The associativity Φ is the usual vector space one.

To my knowledge the structure of this quasitensor category $\mathcal{A}_n$ has not been systematically studied before. Our main result of this section is to identify it as the category of representations of a quantum group. Of course, it is well known that quantum groups (in the strict sense, with quasitriangular structures) have quasitensor or braided monoidal categories of representations. However, given such a category it may not come from a quantum group. Our result is,

Proposition 1 *Let $\mathbb{C}\boldsymbol{Z}_n$ denote the group algebra of $\boldsymbol{Z}_n$. This is just the algebra over $\mathbb{C}$ generated by $1, g$ and the relation $g^n = 1$. It is a Hopf algebra with $\Delta g = g \otimes g$, $\epsilon g = 1$, $Sg = g^{-1}$. Then $\mathbb{C}\boldsymbol{Z}_n$ has a non-trivial quasitriangular structure*

$$\mathcal{R} = \frac{1}{n}\sum_{a,b=0}^{n-1} e^{-\frac{2\pi\imath ab}{n}} g^a \otimes g^b. \tag{5}$$

We denote the Hopf algebra $\mathbb{C}\boldsymbol{Z}_n$ equipped with this non-standard quasitriangular structure by $\mathbb{C}\boldsymbol{Z}'_n$. Moreover,

$$\mathcal{A}_n = \mathrm{Rep}\,(\mathbb{C}\boldsymbol{Z}'_n).$$

If $n \geq 3$ then $\mathbb{C}\boldsymbol{Z}'_n$ is strictly quasitriangular and $\mathcal{A}_n$ is strictly braided.

Proof It is easy to verify that the $\mathcal{R}$ shown obeys all the axioms of a quantum group as recalled in the preliminaries. The prime in $\mathbb{C}\boldsymbol{Z}'_n$ is to distinguish it from the usual group algebra $\mathbb{C}\boldsymbol{Z}_n$ with $\mathcal{R} = 1 \otimes 1$. We compute the braiding in the category of representations of $\mathbb{C}\boldsymbol{Z}'_n$. Recall, e.g.[5, Sec. 7], that for any quantum group H, the category $\mathrm{Rep}\,(H)$ of representations becomes a quasitensor category as follows: The tensor product of representations V, W is $h{\triangleright}(v \otimes w) = \sum h_{(1)}{\triangleright}v \otimes h_{(2)}{\triangleright}w$ for $v \otimes w \in V \otimes W$ and the braiding is

$$\Psi_{V,W}(v \otimes w) = \sum \mathcal{R}^{(2)}{\triangleright}w \otimes \mathcal{R}^{(1)}{\triangleright}v$$

where $\mathcal{R} \equiv \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$. The various actions are denoted $\triangleright$. In our case then, the $\otimes$ is as for $\boldsymbol{Z}_n$-representations, while $\Psi_{V,W}(v \otimes w) = \frac{1}{n}\sum_{a,b} e^{\frac{2\pi\imath b|w|}{n}} w \otimes e^{\frac{2\pi\imath a|v|}{n}} v e^{-\frac{2\pi\imath ab}{n}}$ $= \sum_b e^{\frac{2\pi\imath b|w|}{n}} w \otimes v\delta_{|v|,b} = e^{\frac{2\pi\imath|v||w|}{n}} w \otimes v$ on homogeneous elements. We use here and below the orthogonality of $\boldsymbol{Z}_n$ representations in the form

$$\frac{1}{n}\sum_{a=0}^{n-1} e^{\frac{2\pi\imath a(b-c)}{n}} = \delta_{b,c}, \qquad b, c = 0, 1, \cdots, n-1 \tag{6}$$

where $\delta_{b,c}$ is 1 if $b = c$ and 0 otherwise $\square$.

Now in any quasitensor category with duals (as there are here) there is an intrinsic notion of $\mathrm{rank}\,(V)$ for any object V, and of $\underline{\mathrm{Tr}}\, f$ for any endomorphism f. The $\underline{\mathrm{Tr}}\, f$ is defined as the morphism

$$\underline{1} \to V \otimes V^* \xrightarrow{f \otimes \mathrm{id}} V \otimes V^* \xrightarrow{\Psi_{V,V^*}} V^* \otimes V \to \underline{1}$$

and $\mathrm{rank}\,(V) = \underline{\mathrm{Tr}}\,\mathrm{id}_V$, cf[20] in the tensor case. Here $\underline{1}$ is the identity object in the category (in our case, the trivial representation $\mathbb{C}$). The definition of trace $\underline{\mathrm{Tr}}$ extends further as a morphism $\underline{\mathrm{Hom}}(V,V) = V \otimes V^* \xrightarrow{\Psi_{V,V^*}} V^* \otimes V \to \underline{1}$, where $\underline{\mathrm{Hom}}$ is the internal hom in the category. For the quasitensor categories $\mathrm{Rep}\,(H)$ where H is a quantum group, the rank was studied in [5][21]. For $H = U_q(sl(2))$ it comes out as a variant of the familiar q-dimension. In general it comes out as[5], $\mathrm{rank}\,(V) = \mathrm{Tr}\,\rho_V(\underline{u})$ where $\underline{u} \in H$ is $\underline{u} = \sum (S\mathcal{R}^{(2)})\mathcal{R}^{(1)}$ and $\rho_V(\underline{u})$ is the matrix of $\underline{u}$ acting on V. Likewise if $f : V \to V$ is an endomorphism or indeed any linear map (viewed as for vector spaces in $\underline{\mathrm{Hom}}(V,V) = V \otimes V^*$) we have

$$\underline{\mathrm{Tr}}\,f = \mathrm{Tr}\,\rho_V(\underline{u})f. \tag{7}$$

Because of Proposition 1 we can apply this general theory to the quasitensor categories $\mathcal{A}_n$. It is also evident from this formula that $\underline{\mathrm{Tr}}\,f \circ g = \underline{\mathrm{Tr}}\,g \circ f$ for endomorphisms f, g. Note that this is not necessarily true if f, g are not intertwiners for H but merely linear maps. For example, one can show that $\underline{\mathrm{Tr}}\,\rho_V(h) \circ \rho_V(g) = \underline{\mathrm{Tr}}\,\rho_V(S^{-2}g) \circ \rho_V(h)$ for $h, g \in H$. Because of Proposition 1 we can apply this general theory to $\mathcal{A}_n$. We have,

Proposition 2 *The intrinsic category-theoretic rank or "anyonic dimension" of an anyonic vector space V (an object in $\mathcal{A}_n$) is*

$$\mathrm{rank}\,(V) = \sum_{a=0}^{n-1} e^{-\frac{2\pi \imath a^2}{n}} \dim V_a \tag{8}$$

where V_a is the subspace of homogeneous degree a. Likewise if $f : V \to V$ is degree-preserving then the category-theoretic or "anyonic" trace is

$$\underline{\mathrm{Tr}}\,f = \sum_{a=0}^{n-1} e^{-\frac{2\pi \imath a^2}{n}} \mathrm{Tr}\, f|_{V_a} \tag{9}$$

where $f|_{V_a} : V_a \to V_a$ is the restriction of f to degree a. If f is not degree-preserving we project it back down to each V_a. If $n = 2$ we recover the usual super-dimension and super-trace.

Proof We compute

$$\underline{u} = \frac{1}{n} \sum_{a,b} g^{-b} g^a e^{-\frac{2\pi \imath ab}{n}} = \frac{1}{n} \sum_a g^a \theta_n(a) \tag{10}$$

where $\theta_n(a) = \sum_b e^{-\frac{2\pi\imath(a+b)b}{n}}$ is the $\boldsymbol{Z}_n$ theta-function. It is the $\boldsymbol{Z}_n$-fourier transform of a Gaussian. To compute $\underline{\mathrm{Tr}}\, f$, let $\{e_{a,\gamma_a}\}$ be a basis of V and $\{f^{a,\gamma_a}\}$ a dual basis, where the e_{a,γ_a} are homogeneous of degree a and $\gamma_a = 1, \cdots \dim V_a$. By cyclicity of the ordinary trace, we can apply $\underline{u}$ first. Then we find $\underline{\mathrm{Tr}}(f) = \frac{1}{n}\sum_a \theta_n(a) \sum_b \sum_{\gamma_b} f^{b,\gamma_b}(f(g^a{\triangleright} e_{b,\gamma_b})) = \frac{1}{n}\sum_{a,b}\theta_n(a) e^{\frac{2\pi\imath ab}{n}}(\sum_{\gamma_b} f^{b,\gamma_b}(f(e_{b,\gamma_b})))$ giving the result on using (2.6) □.

3 Anyonic Quantum Groups

In this section we show how to obtain quantum groups in the category of anyonic vector spaces by means of the general transmutation theorem in [11]. This theory applies to quasitensor categories which are generated as the representations of some quantum group H_1. Proposition 1 says that $\mathcal{A}_n$ are of this type with generating quantum group $H_1 = \mathbb{C}\boldsymbol{Z}'_n$. The general transmutation theory says that if H is any ordinary Hopf algebra into which H_1 maps by a Hopf algebra map $f : H_1 \to H$, then H acquires the additional structure of a Hopf algebra $\underline{H}$ in the quasitensor category Rep(H). It consists of the same vector space and algebra as H, but with a modified coproduct. The vector space of H becomes an object $\underline{H}$ in Rep(H) by means of the adjoint action via f. $\underline{H}$ also has a certain opposite coproduct and if H is a quantum group (with $\mathcal{R}$) then $\underline{H}$ has a quasitriangular structure $\underline{\mathcal{R}}$ in Rep(H). In our case we obtain as a generalization of [11, Cor. 2.5],

Proposition 3 *If $(H, \mathcal{R})$ is an ordinary quantum group containing a group-like element g of order n, then it has the additional structure of an anyonic quantum group $\underline{H}$ in the category $\mathcal{A}_n$. The product coincides with that of H. The quantum group structure of $\underline{H}$ is*

$$\underline{\Delta} b = \sum b_{(1)} g^{-|b_{(2)}|} \otimes b_{(2)}, \quad \underline{\epsilon} b = \epsilon b, \qquad \underline{S} b = g^{|b|} S b$$

$$\underline{\Delta}^{\mathrm{op}} b = \sum b_{(2)} g^{-2|b_{(1)}|} \otimes g^{-|b_{(2)}|} b_{(1)}, \qquad \underline{\mathcal{R}} = \mathcal{R}_g^{-1} \sum \mathcal{R}^{(1)} g^{-|\mathcal{R}^{(2)}|} \otimes \mathcal{R}^{(2)}.$$

Here g acts on H in the adjoint representation $g{\triangleright} b = gbg^{-1}$ for $b \in \underline{H}$. The above formulae are defined on homogeneous elements of eigenvalue $g{\triangleright} b = e^{\frac{2\pi\imath|b|}{n}} b$. This defines the $\boldsymbol{Z}_n$-grading of $\underline{H}$. The quantity $\mathcal{R}_g$ is the quasitriangular structure on $\mathbb{C}\boldsymbol{Z}'_n$ as given in Proposition 1.

Proof These formulae follow directly from the general formulae in [11]. In the notation there we are computing $\underline{H} = B(\mathbb{C}\mathbf{Z}'_n, H)$ where $\mathbb{C}\mathbf{Z}'_n$ is the Hopf subalgebra generated by g, equipped with the non-standard quasitriangular structure given in Section 2. In the result shown it is assumed that all tensor product decompositions are into homogeneous elements. The second coproduct $\underline{\Delta}^{\rm op}$ specified in [11] is not simply $\Psi^{-1}\circ\underline{\Delta}$ but has something of the character of this. It comes out as

$$\underline{\Delta}^{\rm op} b = \sum e^{-\frac{2\pi i |b_{(1)}||b_{(2)}|}{n}} b_{\underline{(2)}} g^{-2|b_{(1)}|} \otimes b_{\underline{(1)}} \tag{11}$$

where $\underline{\Delta} b = \sum b_{\underline{(1)}} \otimes b_{\underline{(2)}}$. This then computes to the form stated. Note also that g itself appears in $\underline{H}$ with degree 0 □.

The transmutation formulae in [11] hold in fact slightly more generally where there is a Hopf algebra map $\mathbb{C}\mathbf{Z}'_n \to H$ that need not be an inclusion. We limit ourselves here to giving two examples of the transmutation procedure. In both of these the map is an inclusion.

Our first example is with H the group algebra of a finite non-Abelian group containing an element g of order n. To be concrete we take for our example the group S_3, the permutation group on three elements, regarded as the symmetries of an equilateral triangle with fixed vertices 0,1,2, numbered clockwise. Let g denote a clockwise rotation of the triangle by $\frac{2\pi}{3}$ and let R_a denote reflections about the bisector through the fixed vertex a. Let $\mathbb{C}S_3$ denote the group Hopf algebra of S_3. It has basis $\{1, g, g^2, R_0, R_1, R_2\}$. Of course, there are many ways to work with S_3: we present it in a way that makes the generalization to higher n quite straightforward.

Example 4 *Let $H = \mathbb{C}S_3$ where S_3 is the symmetry group of an equilateral triangle as described. Its transmutation $\underline{S_3}$ by Proposition 3 is the following anyonic quantum group in $\mathcal{A}_3$. Firstly, the quantities*

$$r_a = \frac{1}{3}\sum_{b=0}^{b=2} e^{-\frac{2\pi i ab}{3}} R_b$$

are homogeneous of degree $|r_a| = a$. Together with $1, g, g^2$ of degree zero they form a basis of $\underline{S_3}$ as an anyonic vector space. Its ordinary dimension is 6 and its anyonic dimension *is* $2e^{-\frac{\pi\imath}{3}}$. *The algebra structure is that of* $\mathbb{C}S_3$. *The anyonic quantum group structure is*

$$\underline{\Delta} r_a = \sum_{c=0}^{c=2} e^{-\frac{2\pi\imath c(a-c)}{3}} r_c \otimes r_{a-c}, \qquad \underline{\epsilon} r_a = \delta_{a,0}, \qquad \underline{S} r_a = e^{-\frac{2\pi\imath a^2}{3}} r_a$$

$$\underline{\Delta}^{\rm op} r_a = \sum_{c=0}^{c=2} e^{-\frac{2\pi\imath c(a-c)}{3}} r_{a-c} \otimes r_c, \qquad \underline{\mathcal{R}} = \mathcal{R}_g^{-1}$$

Proof The reflections have the property that $gR_ag^{-1} = R_{a+1}$ (mod 3). Hence their inverse Fourier transforms r_a as shown are homogeneous of degree as stated. The Hopf algebra structure on g (of degree zero) is unmodified. The usual coproduct in the remainder of $\mathbb{C}S_3$ is $\Delta R_a = R_a \otimes R_a$, hence $\Delta r_a = \sum_c r_c \otimes r_{a-c}$. This then becomes modified as $\underline{\Delta} r_a = \sum_c r_c g^{c-a} \otimes r_{a-c}$. Now note that in S_3, $R_a g = gR_ag^{-1} = g\triangleright R_a$ for all a. Hence $r_a g = g\triangleright r_a = e^{\frac{2\pi\imath a}{3}} r_a$ giving the result shown. Likewise, the original antipode on the R_a is $SR_a = R_a^{-1} = R_a$. Hence $Sr_a = r_a$ also. From this and $g^{-1}R_a = g\triangleright R_a$ for all a (so that $g^{-1} r_a = e^{\frac{2\pi\imath}{3}} r_a$) we obtain $\underline{S}$ as shown. The computation for $\underline{\Delta}^{\rm op}$ is similar to that for $\underline{\Delta}$. The unmodified $\mathcal{R}$ of $\mathbb{C}S_3$ is $\mathcal{R} = 1 \otimes 1$, so that $\underline{\mathcal{R}} = \mathcal{R}_g^{-1}$ $\square$.

For the second example we consider the quantum groups $H = U_q(sl(2))'$ defined at q a root of unity as in [7]. Here the prime in [7] refers to the finite-dimensional versions. They are generated by K, X, Y with relations

$$KXK^{-1} = q^{\frac{1}{2}}X,\ KYK^{-1} = q^{-\frac{1}{2}}Y,\ [X,Y] = \frac{K^2 - K^{-2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},\ X^r = Y^r = 0,\ K^{4r} = 1$$

where $q = e^{\frac{2\pi\imath}{r}}$. There is a coproduct $\Delta X = X \otimes K + K^{-1} \otimes X$ etc, antipode $SX = -q^{\frac{1}{2}}X$, $SY = -q^{-\frac{1}{2}}Y$ and a quasitriangular structure[7]. We work with this quantum group in an equivalent form with new generators and the $[X, Y]$ relation taking the form

$$g = K, \qquad E = XK^3, \qquad F = YK^{-1}, \qquad qEF - FE = \frac{g^4 - 1}{q - 1}.$$

Here $g^{4r} = 1$ so we can apply Proposition 3.

Example 5 *Let* $n = 4r$, *so* $q = e^{\frac{8\pi\imath}{n}}$ *and* $H = U_q(sl(2))'$ *as described. Its transmutation* $\underline{U_q(sl(2))'}$ *by Proposition 3 is the following anyonic quantum group in* $\mathcal{A}_n$. *As an anyonic algebra it has generators* g *of degree* 0 *and* E, F *of degrees* $|E| = 2$, $|F|=-2$. *The algebra relations are those of* $U_q(sl(2))'$. *The anyonic quantum group structure is*

$$\underline{\Delta} E = E \otimes g^4 + 1 \otimes E, \qquad \underline{\Delta} F = F \otimes 1 + 1 \otimes F$$

$$\underline{\epsilon} E = \underline{\epsilon} F = 0, \qquad \underline{S} E = -Eg^{-4}, \quad \underline{S} F = -F$$

$$\underline{\Delta}^{\rm op} E = E \otimes 1 + 1 \otimes E, \qquad \underline{\Delta}^{\rm op} F = F \otimes 1 + g^4 \otimes F$$

$$\underline{\mathcal{R}} = \sum_{m=0}^{r-1} E^m \otimes F^m \frac{(q-1)^{2m}}{(q^m - 1)\cdots(q-1)}$$

where the $m = 0$ term is defined to be $1 \otimes 1$. The coproduct etc on g are unchanged.

Proof This follows from direct computation using the form of the generators shown. The degree of E, F is computed by $gEg^{-1} = e^{\frac{2\pi i |E|}{n}} E$ and similarly for $|F|$. Since g has degree 0 its structure is of course unchanged. The formula for $\underline{\mathcal{R}}$ was in fact obtained by direct computation from the axioms for an anyonic quasitriangular structure in $\underline{U_q(sl(2))'}$. Proposition 3 can then be pushed backwards to obtain an expression for $\mathcal{R}$ in $U_q(sl(2))'$, namely

$$\begin{aligned} \mathcal{R} &= \mathcal{R}_g \sum_{m=0}^{r-1} E^m g^{-2m} \otimes F^m \frac{(q-1)^{2m}}{(q^m-1)\cdots(q-1)} \\ &= \mathcal{R}_K \sum_{m=0}^{r-1} (KX)^m \otimes (K^{-1}Y)^m \frac{(1-q^{-1})^{2m}}{(1-q^{-m})\cdots(1-q^{-1})}. \end{aligned}$$

Its matrices in the standard representations coincide with those in [7]. $\mathcal{R}_g = \mathcal{R}_K$ comes from Proposition 1 □.

Note in this example the general phenomenon of transmutation: it can trade a non-cocommutative object $U_q(sl(2))'$ in an ordinary bosonic category into a more cocommutative object (see $\underline{\Delta} F, \underline{\Delta}^{\rm op} E$) in a more non-commutative (in this case anyonic) category. There are plenty of other examples along the lines of the two examples above.

4 General Construction Based on Self-Dual Hopf Algebras

In this section we briefly describe a general construction from which the results of Section 2 were obtained. For these purposes we work over an arbitrary field k of characteristic not 2. Let H be an arbitrary self-dual Hopf algebra. This means a Hopf algebra equipped with a bilinear form $<\ ,\ >: H \otimes H \to k$ such that

$$< \Delta g, a \otimes b > = < g, ab >, \quad < h \otimes g, \Delta a > = < hg, a > \tag{12}$$

$$< 1, a > = \epsilon(a), \quad < h, 1 > = \epsilon(h), \quad < Sh, a > = < h, Sa > \tag{13}$$

for all $h,g,a,b \in H$. Here $< h\otimes g, a\otimes b >=< h,a >< g,b >$. Now for any finite-dimensional Hopf algebra H there is a quasitriangular one, $D(H)$, introduced by [17] (the Drinfeld quantum double of H). It is built on $H\otimes H^*$ with certain relations. In [22] we showed how to generalize this construction to the situation of dually paired Hopf algebras, not necessarily finite-dimensional. We use this now in the case when H is dually paired with itself. The construction in [22] means that $D(H)$ is built now on $H\otimes H$ with the product

$$(h\otimes a)(g\otimes b)=\sum < Sh_{(1)}, b_{(1)} > (h_{(2)}g\otimes b_{(2)}a) < h_{(3)}, b_{(3)} > \tag{14}$$

and the tensor product coalgebra structure. The antipode is $S(h\otimes a)=(Sh\otimes 1)(1\otimes S^{-1}a)$.

Proposition 6 *Let H be an involutory self-dual Hopf algebra and $D(H)$ its quantum double as described on $H\otimes H$. Then*

$$H'=\frac{D(H)}{(h\otimes 1-1\otimes h\ :\ h\in H)}$$

is a commutative Hopf algebra. In the finite-dimensional case it is quasitriangular (a quantum group in the strict sense) with

$$\mathcal{R}=\sum_a f^a\otimes e_a\in H'\otimes H'.$$

Here $\{e_a\}$ is a basis of H and $\{f^a\}$ another, dual, basis of H and we use for $\mathcal{R}$ their projections to H'. If S does not act as the identity in H' then $\mathcal{R}$ is strictly quasitriangular (so that Rep (H) *is strictly braided).*

Proof Firstly, the coproduct on $D(H)$ is $\Delta(h\otimes a)=\sum h_{(1)}\otimes a_{(1)}\otimes h_{(2)}\otimes a_{(2)}$. Then we have

$$\begin{aligned}\Delta(h\otimes 1-1\otimes h) &= \frac{1}{2}\sum(h_{(1)}\otimes 1-1\otimes h_{(1)})\otimes(1\otimes h_{(2)}+h_{(2)}\otimes 1)\\ &\quad +(h_{(1)}\otimes 1+1\otimes h_{(1)})\otimes(h_{(2)}\otimes 1-1\otimes h_{(2)})\end{aligned}$$

where we use the same decomposition $\Delta h=\sum h_{(1)}\otimes h_{(2)}$ for the two h's. Hence the ideal generated by the relations $(h\otimes 1)=(1\otimes h)$ for all $h\in H$ is a biideal in the sense of [16, p. 87]. It is also respected by S if $S^2=\mathrm{id}$ (the condition that H is involutory). Hence the quotient is a Hopf algebra. The $\mathcal{R}$ shown is just the projection of the one on $D(H)$ found by

[17]. We used the conventions of [23]. That H' is commutative follows from the formula for the product (in our conventions, $D(H)$ includes H on the right with the opposite product). Note that in any quasitriangular Hopf algebra one has $(S\otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1}$ which leads to $\mathcal{R}^{-1} = \sum Sf^a \otimes e_a$. Note that because H' is commutative, its antipode has square 1 □.

We used this construction applied to $H = \mathbb{C}\boldsymbol{Z}_n$ to obtain the structure of $\mathbb{C}\boldsymbol{Z}'_n$ described above. To do this we take for H a basis $e_a = g^a$ for $a = 0, 1, \cdots, n-1$. The dual basis can be written in terms of $\hat{\boldsymbol{Z}}_n$ (the character group of $\boldsymbol{Z}_n$), which we identify with $\boldsymbol{Z}_n$ itself to obtain the self-duality pairing.

Of course, the input Hopf algebra H need not itself be commutative or cocommutative. Indeed some, self-dual non-commutative and non-cocommutative involutory Hopf algebras were constructed in [24]. For example, let T_n be the group of upper triangular matrices in $M_n(k)$ with 1 on the diagonal. Then in [24] we constructed a bicrossproduct $kT_n{}^{\beta}\bowtie_{\alpha}(kT_n)^*$ by means a certain action α and coaction β. These were obtained by an action of each T_n factor group on the other by a modification of the left-regular action. The left T_n factor here plays the role of momentum group, the other of position space. The Hopf algebra itself is then the quantum algebra of observables in an algebraic approach to quantum gravity[25]. Physically, in this setting Hopf algebra duality corresponds to a reversal of the roles of observables and states in the quantum system, and in this class of bicrossproduct models the dual Hopf algebra is of the same type with reversal of the roles of position and momentum. Of course, for physical models one must work with Lie groups and Hopf-von Neumann algebras rather than in an algebraic setting. This was done in [26]. The application of the above construction in this context is one direction for further work.

Finally we mention a variant of Proposition 6 which avoids some of the restrictions there. It applies also to H infinite-dimensional provided the antipode is invertible and that $\mathcal{R}$ makes sense.

Proposition 7 *Let H be a finite-dimensional antiself dual Hopf algebra. Then $H' = D(H)/(h\otimes 1 - 1\otimes h:\ h\in H)$ is a quasitriangular Hopf algebra (not necessarily commutative) with $\mathcal{R}$ as in Proposition 6.*

Proof This variant differs in that we now suppose that there is a pairing $<\ ,\ >: H\otimes H \to k$ that obeys $<\Delta h, a\otimes b> = <h, ba>$ and $<Sh, a> = <h, S^{-1}a>$ for all $h, a, b \in H$ (the rest as in (4.12)-(4.13)). In the finite-dimensional case this says $H{\cong}H^{*op}$ where the latter is H^* with the opposite product. The formulae for $D(H)$ are now similar but with $ab_{(2)}$ rather than $b_{(2)}a$ in (4.14) and $S(h\otimes a) = (Sh\otimes 1)(1\otimes Sa)$. This means that both H factors in $H\otimes H$ are sub-Hopf algebras. In this case H' is always a Hopf algebra and need not be commutative □.

On the other hand, so far, few antiself-dual Hopf algebras are known. One example of H that is antiself-dual (as well as self-dual) is the Hopf algebra generated by g, x and relations $g^2 = 1, x^2 = 0, gx = -xg$ and Hopf algebra structure $\Delta g = g\otimes g$, $\Delta x = x\otimes 1 + g\otimes x$, $\epsilon g = 1$, $\epsilon x = 0$, $Sg = g$, $Sx = xg$. The Hopf algebra is due to Sweedler but see [11] where we show that it is self-dual and isomorphic to its own opposite. In this case $H' = H$ as Hopf algebras. The proposition means that H itself is quasitriangular, as first found in [27] by other means. The search for further antiself-dual Hopf algebras is a second direction for further work.

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