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LEVEL ONE KAC-MOODY CHARACTERS AND MODULAR INVARIANCE

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1. In this work we continue our investigations on modular transformation properties of Kac-Moody characters and associated conformal theories. The A-D-E classification[1] obtained in the case of SU(2) might lead one to suspect that interesting phenomena happen when dealing with other groups. Here I shall limit myself to preliminary remarks pertaining to the group SU(N), specializing even to level one representations of its (untwisted) Kac-Moody or affine algebra. For reasons which are not totally clear, there is a striking parallel between level one representations for SU(N) (Lie algebra A_{N-1}) and level k representations of SU(2) for N = 2(k+2). A slight surprise is that modular invariance of the partition function on a torus allows a plethora of possibilities in the case of SU(N) indexed by divisors of N if N is odd or N/2 if N is even.

I have attempted to give an elementary exposition for the benefit of those reader who - like me - have a very hard time to decipher the literature. To a large extent the following presentation is therefore a paraphrase of existing work - mainly Kac's book[2] and the article by Gepner and Witten [3]. I shall also rely on considerations presented in a review on elementary integrable systems[4]. The benefit of dealing with a special case is to be able to exhibit explicit expressions and to provide examples.

We will accept the formula for characters and analyze it in some detail showing in particular that for SU(N) at level one the modular transformation properties simplify considerably. I do not know whether this simplification is well known and/or has already appeared in print.

A specific property of SU(N), $N \geqslant 3$, is the existence of an automorphism of the Lie algebra which extends to a correspondence between representations and their complexe conjugates (in physical terms charge conjugation). This has the consequence that affine restricted characters attached to conjugate ground states (or highest weight states) are undistinguishable. Therefore to characterize the content of a conformal invariant model it is not sufficient to exhibit a modular invariant partition function on a torus. A way out will be discussed below.

2. Let τ denote the ratio of two fundamental periods on a torus, $\tau = \omega_2/\omega_1$, $\mathrm{Im}\,\tau > 0$. We set $q = \exp 2i\pi\tau$, use c for the central charge, L_0 for the grading element of the Virasoro algebra generating dilatations in the plane or translations on the torus. The specialized characters give an expression for the trace

$$\chi(\tau) = \mathrm{Tr}\, q^{L_0 - c/24} \qquad (2\text{-}1)$$

in an irreducible representation of the affine Lie algebra, here $A_{N-1}^{(1)}$. The level k describes the central extension of the corresponding loop algebra, or the Schwinger term in the current algebra, im physicist's terminology. For SU(N) the quantities c, k, and N are related through

$$C = \frac{(N^2-1)k}{N+k} \qquad (2\text{-}2)$$

which reduces to $c = N-1$, the rank of A_{N-1}, when $k = 1$, while c tends to N^2-1, the dimension of the algebra, when k tends to infinity. The integer N is therefore the Coxeter number of the simply laced Lie algebra A_{N-1}.

The character $\chi(\tau)$ pertaining to an irreducible representation of the affine algebra $A_{N-1}^{(1)}$ is a generating function for the number of linearly independent states corresponding to a given eigenvalue of L_0. These states can be organized into irreducible multiplet's of A_{N-1}. In particular to the lowest eigenvalue of L_0, the conformal weight h, corresponds a unique irreducible representation of A_{N-1}, itself described by a Young tableau, or a highest weight. The characters of level k are those for which these representations correspond to Young tableaux with at most k columns (and N-1 rows). Their number is

Number of $A_{N-1}^{(1)}$ characters

$$\text{at level } k = \frac{(N+k-1)!}{k!\,(N-1)!} \qquad (2\text{-}3)$$

When $k = 0$ there exists a unique trivial character equal to unity, while for $k = 1$ we have N characters. Those are attached to representations of A_{N-1} acting on anti-symmetric tensors with λ indices, where λ runs from zero (trivial representation of A_{N-1}) to N-1. Identifying λ and $\lambda+N$, we shall later on label these characters by the integer λ mod N. As was already indicated, distinct affine representations corresponding to λ and $N-\lambda$ (distinct except when N is even and $\lambda = N/2$) have identical number of states for each eigenvalue of L_0 and therefore equal character.

In general, instead of Young tableaux specific to SU(N) we can use the language of roots and weights. Let V be a vector space of dimension equal to the rank r (here $r = N-1$) equiped with the Cartan Killing metric. Let $\underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(r)}$, be the fundamental linearly independent weight vectors. A positive weight is an integral linear combination of these vectors with non negative coefficients. Therefore any strictly positive weight is of the form $\underline{p} = \underline{\rho} + \tilde{\underline{p}}$ with $\tilde{\underline{p}}$ positive and

$$\underline{\rho} = \underline{\alpha}^{(1)} + \ldots + \underline{\alpha}^{(r)} \qquad (2\text{-}4)$$

For a simply laced algebra, such as A_{N-1}, the fundamental weigths $\underline{\alpha}^{(i)}$ are dual to the fundamental roots $\underline{\alpha}_{(i)}$

$$\underline{\alpha}^{(i)} \cdot \underline{\alpha}_{(j)} = \delta^i{}_j \qquad 1 \leqslant i,j \leqslant r \qquad (2\text{-}5)$$

and the symmetric Cartan matrix is

$$g_{ij} = \underline{\alpha}_{(i)} \cdot \underline{\alpha}_{(j)} \qquad (2\text{-}6)$$

while g^{ij} stands for the inverse matrix

$$g^{ij} = \underline{\alpha}^{(i)} \cdot \underline{\alpha}^{(j)} \qquad g^{ik} g_{kj} = \delta^i{}_j \quad (2\text{-}7)$$

The normalization is such that the diagonal terms are $g_{ii} = \underline{\alpha}^2{}_i = 2$ and the Cartan matrix has off-diagonal entries equal to zero or negative integers (here -1).

Irreducible representations of the Lie algebra are indexed by strictly positive weights $\underline{p}$ and the corresponding quadratic Casimir invariant is $\underline{p}^2 - \underline{\rho}^2$ in an appropriate normalization. Thus level k irreducible representations of $A_{N-1}^{(1)}$ will be labelled by $\underline{p}$, the strictly positive weight characterizing the lowest irreducible representation of A_{N-1}, lowest in the sense that L_0 assumes its lowest eigenvalue

$$h_{\underline{p}} = \frac{\underline{p}^2 - \underline{\rho}^2}{2(N+k)} \qquad (2\text{-}8)$$

Higher eigenvalues of L_0 differ from h_ρ by a positive integer. The factor 2 occuring in the denominator is related to the convention the fundamental roots have square length equal to 2.

It follows therefore that when τ is increased by unity

$$\chi_\rho(\tau+1) = \exp 2i\pi\left(h_\rho - \frac{c}{24}\right) \chi_\rho(\tau) \qquad (2\text{-}9)$$

Let us describe in more detail the roots and weights of SU(N). Consider in an N-dimensional space a set of N orthonormal vectors $\underline{e}_{(\mu)}$

$$\underline{e}_{(\mu)} \cdot \underline{e}_{(\nu)} = \delta_{\mu\nu} \qquad 1 \leqslant \mu, \nu \leqslant N \qquad (2\text{-}10)$$

Let $\underline{e} = \sum_{\mu=1}^{N} \underline{e}_{(\mu)}$ and V be the (N-1) dimensional (metric) subspace orthogonal to $\underline{e}$. The fundamental roots

$$\underline{\alpha}_{(1)} = \underline{e}_{(1)} - \underline{e}_{(2)}, \ldots, \underline{\alpha}_{(N-1)} = \underline{e}_{(N-1)} - \underline{e}_{(N)} \qquad (2\text{-}11)$$

span V. The set of $\frac{N(N-1)}{2}$ positive roots are the vectors $\underline{e}_{(\mu)} - \underline{e}_{(\nu)}$, $1 \leqslant \mu < \nu \leqslant N$, easily expressed as linear combinations of the $\underline{\alpha}_{(i)}$'s with non negative coefficients. The Cartan matrix is therefore

$$g_{ij} = \begin{cases} 2 & \text{if} \quad |i-j| = 0 \\ -1 & \text{if} \quad |i-j| = 1 \\ 0 & \text{if} \quad |i-j| > 1 \end{cases} \qquad 1 \leqslant i, j \leqslant N-1 \qquad (2\text{-}12)$$

Let P denote the orthogonal projection on V along $\underline{e}$

$$P\,\underline{x} = \underline{x} - \frac{1}{N}\,\underline{e}(\underline{e}.\underline{x}) \qquad (2\text{-}13)$$

The fundamental weights are then

$$\begin{aligned} \underline{\alpha}^{(1)} &= P\,\underline{e}_{(1)} \quad \underline{\alpha}^{(2)} = P(\underline{e}_{(1)} + \underline{e}_{(2)}) \ldots \\ \underline{\alpha}^{(N-1)} &= P(\underline{e}_{(1)} + \ldots + \underline{e}_{(N-1)}) \end{aligned} \qquad (2\text{-}14)$$

and

$$g^{ij} = \mathrm{Inf}\{i,j\} - \frac{ij}{N} \qquad 1 \leqslant i, j \leqslant N-1 \qquad (2\text{-}15)$$

The fundamental roots generate over the integers $\mathbf{Z}$ the root lattice M and the fundamental weights its dual, the weight lattice M^*. Every vector in the root lattice M has an even square length, and the root lattice is a sublattice of the weight lattice of index N

$$M^*/M = \mathbf{Z}/N\mathbf{Z} \qquad (2\text{-}16)$$

The sum of fundamental weigths, equal to half the sum over positive roots, reads

$$\begin{aligned} \underline{\rho} &= \sum_{i=1}^{N-1} \underline{\alpha}^{(i)} = \frac{1}{2} \sum_{1 \leqslant \mu < \nu \leqslant N} (\underline{e}_{(\mu)} - \underline{e}_{(\nu)}) \\ &= \sum_{\mu=1}^{N} \left(\frac{N+1}{2} - \mu\right) \underline{e}_{(\mu)} \end{aligned} \qquad (2\text{-}17)$$

We have also to indroduce the Weyl group W, a finite group of orthogonal transformations generated by r reflections. Seen in N-dimensional space this is the permutation group S_N acting on the N vectors $\underline{e}_{(\mu)}$, generated by the r transpositions $\underline{e}_{(\mu)} \leftrightarrow \underline{e}_{(N+1)}$, $\mu=1, \ldots, N-1$, and therefore in the subspace V by the N-1 reflections, for i=1,... N-1

$$\underline{x} \in V \quad \underline{x} \to \underline{x}' = \underline{x} - \underline{\alpha}_{(i)}(\underline{\alpha}_{(i)} \cdot \underline{x}) \qquad (2\text{-}18)$$

If a vector $\underline{x} \in V$ is expanded on the weight basis as

$$\underline{x} = \sum_{i=1}^{N-1} x_i\, \underline{\alpha}^{(i)} \qquad (2\text{-}19)$$

the closure of a fundamental domain for W (a Weyl chamber) is characterized by

$$(\underline{x}_{(i)} \cdot \underline{x}) = x_i \geqslant 0 \qquad 1 \leqslant i \leqslant N-1 \qquad (2\text{-}20)$$

The semi-direct product of W by the translations of the root lattice M is a Coxeter group generated by N reflections. Those include the (N-1) reflections in the hyperplanes orthogonal to the roots through the origin (equation (2-18)) as well as the reflection in the hyperplane orthogonal to the vector

$$\underline{\alpha}_{(N)} = - \sum_{i=1}^{(N-1)} \underline{\alpha}_{(i)} = \underline{e}_{(N)} - \underline{e}_{(1)} \qquad (2\text{-}21)$$

the equation of the hyperplane being

$$\underline{\alpha}_{(N)} \cdot \underline{x} = 1 \qquad (2\text{-}22)$$

A characterisation of the vector $-\underline{\alpha}_{(N)}$ is that it is the largest positive root, in the sense that the sum of its components in the basis of fundamental roots is the largest possible.

The closure of a fundamental domain B for the Coxeter group (also called inhomogeneous Weyl group) is a simplex defined by the inequalities

$$x_i \geqslant 0 \qquad \sum_{i=1}^{N-1} x_i \leqslant 1 \qquad (2\text{-}23)$$

The quotient of V by the translations of the root lattice M is a torus isomorphic to the Cartan torus of SU(N). By quotienting further by W, we split this torus into N! (the order of W) pieces equivalent to B, each one having a unique intersection with a generic equivalence class in SU(N), generic in the sense that each matrix in this class has distinct eigenvalues. Non generic equivalence classes, with some subset of equal eigenvalues, intersect the boundary of B.

With this information we return to the formula for conformal weights (2-8), and apply it to $A_{N-1}^{(1)}$ at level one. We write for short

$$h_\lambda = \frac{\left(\rho + \underline{\alpha}^{(\lambda)}\right)^2 - \rho^2}{2(N+1)} \qquad (2\text{-}24)$$

with λ running from 0 to N-1 and $\underline{\alpha}^{(0)} \equiv 0$ by convention (therefore $h_0 = 0$). For $1 \leqslant \lambda \leqslant N-1$ we find

$$\left(\underline{\alpha}^{(\lambda)}\right)^2 = g^{\lambda\lambda} = \frac{\lambda(N-\lambda)}{N}$$

$$2\,\underline{\alpha}^{(\lambda)} \cdot \rho = (\underline{e}_{(1)} + \ldots + \underline{e}_{(\lambda)}) \cdot \sum_{1 \leqslant \mu < \nu \leqslant N} (\underline{e}_{(\mu)} - \underline{e}_{(\nu)})$$

$$= \sum_{\mu=1}^{\lambda} \sum_{\nu=\lambda+1}^{N} 1 = \lambda(N-\lambda)$$

and

$$h_\lambda = \frac{\lambda(N-\lambda)}{2N} \qquad 0 \leqslant \lambda \leqslant N-1 \qquad (2\text{-}25)$$

In the transformation law for characters under the translation $\tau \longrightarrow \tau+1$ (equation (2-9)) there appears the exponential $\exp 2i\pi \dfrac{\lambda(N-\lambda)}{2N}$ which is invariant when λ is increased by a multiple of N as well as under the symmetry

$$\lambda \longleftrightarrow N-\lambda \qquad (2\text{-}26)$$

The latter is a consequence of the following automorphism (with square equal to unity). In N-dimensional space consider the map

$$\underline{e}_{(\mu)} \longleftrightarrow -\underline{e}_{(N-\mu+1)} \qquad (2\text{-}27)$$

Under this transformation, a combination of a symmetry and a permutation of the Weyl group, we have for the roots and weigths

$$\begin{aligned} \underline{\alpha}_{(i)} &\longleftrightarrow \underline{\alpha}_{(N-i)} \\ \underline{\alpha}^{(i)} &\longleftrightarrow \underline{\alpha}^{(N-i)} \end{aligned} \qquad (2\text{-}28)$$

This automorphism will leave (restricted) affine characters invariant and is reflected in an automorphism of the Cartan matrix (as well as its representative Dynkin diagramm) and of the root and weight lattices. We note here that the signature of the permutation $\mu \longleftrightarrow N-\mu+1$ is $(-1)^{\frac{N(N-1)}{2}}$.

3. A consistent Wess-Zumino-Witten model has a modular invariant partition function on a torus $Z(\tau,\bar{\tau})$ of the form

$$Z(\tau,\bar{\tau}) = \sum_{\underline{p},\underline{p}'} \mathcal{N}_{\underline{p},\underline{p}'} \chi^{*}_{\underline{p}}(\tau)\, \chi_{\underline{p}'}(\tau) \qquad (3\text{-}1)$$

The sum runs over strictly positive weigths $\underline{p}$ and $\underline{p}'$ pertaining to level k, $\mathcal{N}_{\underline{p},\underline{p}'}$ are non negative integers, and for normalization $\mathcal{N}_{\underline{\rho},\underline{\rho}} = 1$. To investigate such invariant sesquilinear forms we need to know the behaviour of the characters under the generators

$$\begin{array}{ll} T & \tau \longrightarrow \tau + 1 \\ S & \tau \longrightarrow -\tau^{-1} \end{array} \qquad (3\text{-}2)$$

of the modular group. We know already this behaviour under T according to equations (2-8)-(2-9) and (2-25) . We also require the transformation under S which will follow from the explicit Kac-Weyl formula to be described below. It is interesting to guess first the result by invoking plausibility arguments. These are based on suggestions by Verlinde[5], which, as we shall check later are happily fulfilled in the present context. As before, we restrict ourselves to level one characters $\chi_\lambda(\tau)$ considered as even periodic functions of the index λ mod N, as suggested by the transformation under T. Assume that we have a linear transformation formula of the type

$$\chi_\lambda(-\tau^{-1}) = \sum_{\lambda' \bmod N} S_{\lambda\lambda'}\chi_{\lambda'}(\tau) \qquad (3\text{-}3)$$

Because of the symmetry $\lambda \leftrightarrow N-\lambda$ this would not define the matrix $S_{\lambda\lambda'}$ (assuming it exists). However characters can be extended to involve an argument $\underline{x}$ (a set of angles characterizing an equivalence class in SU(N)) in which case under the change of periods $\underline{\omega}_1 \longrightarrow -\underline{\omega}_1$, $\underline{\omega}_2 \longrightarrow -\underline{\omega}_2$, which does not affect τ, and corresponds to the square of the above operation, $\underline{x}$ transformed as $-\underline{x}$ and $\chi_\lambda(\tau,\underline{x})$ into $\chi_\lambda(\tau,-\underline{x}) = \chi_{N-\lambda}(\tau,\underline{x})$. Thus extending appropriately (3-3) when $\underline{x}$ is different from zero we should have

$$\begin{aligned} (S^2)_{\lambda,\lambda'} &= \delta_{\lambda+\lambda',0 \bmod N} \\ (S^4)_{\lambda,\lambda'} &= \delta_{\lambda,\lambda' \bmod N} \end{aligned} \qquad (3\text{-}4)$$

This is of course reminiscent of the finite Fourier transform as will be confirmed below.

Verlinde's ideas apply to chiral conformal theories, with primary operator $\varphi_\lambda(z)$ and corresponding characters $\chi_\lambda(\tau)$. It is of course understood in a situation like the one we are dealing with, that $\varphi_\lambda(z)$ carries internal indices such as tensor components for the SU(N) representation. For the time being we assume that λ runs over a finite set Λ. There is a distinguished value denoted $\lambda = 0$, corresponding to the weight $h_0 = 0$, representing the identity operator. One then introduces linear operators on characters denoted Φ_λ with the properties

$$\text{(i)} \qquad \Phi_\lambda\chi_0(\tau) = \chi_\lambda(\tau) \qquad (3\text{-}5)$$

(ii) The operators Φ_λ generate an associative, commutative algebra such that the generators satisfy

$$\Phi_\lambda\Phi_{\lambda'} = \sum_{\lambda''\in\Lambda} N_{\lambda\lambda'}{}^{\lambda''}\Phi_{\lambda''} \qquad (3\text{-}6)$$

with non negative integral "structure constants" $N_{\lambda\lambda'}{}^{\lambda''}$. Moreover Φ_0 is the identity in this algebra. It then follows by action on $\chi_0(\tau)$ that

$$\Phi_\lambda\chi_{\lambda'}(\tau) = \sum_{\lambda''\in\Lambda} N_{\lambda\lambda'}{}^{\lambda''}\chi_{\lambda''}(\tau) \qquad (3\text{-}7)$$

so that Φ_λ is faithfully represented on the set of characters by the matrices N_λ with matrix elements $(N_\lambda)_{\lambda'}{}^{\lambda''} \equiv N_{\lambda\lambda'}{}^{\lambda''}$ as is the multiplication by Φ_λ on the algebra.

(iii) Finally it is assumed that if under $\tau \longrightarrow -\tau^{-1}$ the characters transform linearly as in (3-3) with λ and λ' runing over the set Λ, the same matrix S (which is now understood to have a square equal to unity) diagonalizes simultaneously the commuting matrices N_λ.

For the present application the set Λ is the set of integers mod N and by extending the previous properties it will be more convenient to relax the condition of symmetry under $\lambda \longrightarrow N-\lambda$ so that S^2 instead of being the identity implements the conjugation automorphism.

As shown by Verlinde, knowing the matrix S we can reconstruct the algebra and vice-versa. Here I will content myself with the observation that when dealing with level one characters of $A_{N-1}^{(1)}$ it is easy to guess that the only sensible algebra is expected to be

$$\Phi_\lambda \Phi_{\lambda'} = \Phi_{\lambda+\lambda' \bmod N} \qquad (3\text{-}8)$$

Of course this will be verified later, but given that the index refers to antisymmetric tensors with λ indices, which other associative and commutative algebra could we think of in that case which does not involve arithmetic properties of N? Assuming this to be the case the corresponding matrix S is the (complex conjugate) matrix of finite Fourier transform up to a sign, acting here on even characters

$$S_{\lambda\lambda'} \equiv F^{*}_{\lambda\lambda'} = \frac{1}{\sqrt{N}} \exp\left\{-2i\pi \frac{\lambda\lambda'}{N}\right\} \qquad (3\text{-}9)$$

To summarize, the modular transformation properties of level one $A_{N-1}^{(1)}$ characters are expected to be

$$\begin{aligned} T \quad & \chi_\lambda(\tau+1) = \exp 2i\pi\left(\frac{\lambda(N-\lambda)}{2N} - \frac{N-1}{24}\right) \chi_\lambda(\tau) \\ S \quad & \chi_\lambda(-\tau^{-1}) = \frac{1}{\sqrt{N}} \sum_{\lambda' \bmod N} \exp -2i\pi \frac{\lambda\lambda'}{N} \chi_{\lambda'}(\tau) \\ & \chi_{N-\lambda}(\tau) = \chi_\lambda(\tau) \end{aligned}$$

(3-10)

We will need some classical properties of the matrix of finite Fourier transform F which we will collect here without proof. Since $F^4 = I$, its eigenvalues are ± 1 or $\pm i$. The precise set of eigenvalues is then obtained as follows. Consider the sequence starting with $a_1 = 1$ and continued indefinitely with period 4 by setting $a_2 = -1$, $a_3 = i$, $a_4 = 1$, $a_5 = -i$, so that $a_6 = -1$, etc... The first N terms in this sequence are the N eigenvalues of $F^{[N]}$, where the superscripts recalls that we are dealing with integers mod N. In particular the determinant of $F^{[N]}$ is given by

$$\begin{aligned} \det F^{[2\ell+1]} &= (-i)^\ell \\ \det F^{[2\ell+2]} &= -i^\ell \end{aligned} \qquad \ell \geqslant 0 \qquad (3\text{-}11)$$

One observes a strong similarity between the transformation formulas (3-10) for $A_{N-1}^{(1)}$ and those pertaining to $A_1^{(1)}$ characters[1] at level k, such that $N = 2(k+2)$, of course N is then even. The latter characters are also indexed by an integer λ mod N but are odd under $\lambda \leftrightarrow N-\lambda$. To distinguish them from the preceding ones we write $\tilde{\chi}_\lambda(\tau)$ for the $A_1^{(1)}$ characters which obey

$$\begin{aligned} T \quad & \tilde{\chi}_\lambda(\tau+1) = \exp 2i\pi\left(\frac{\lambda^2}{2N} - \frac{1}{8}\right) \tilde{\chi}_\lambda(\tau) \\ S \quad & \tilde{\chi}_\lambda(-\tau^{-1}) = \frac{-i}{\sqrt{N}} \sum_{\lambda' \bmod N} \exp 2i\pi \frac{\lambda\lambda'}{N} \tilde{\chi}_{\lambda'}(\tau) \\ & \tilde{\chi}_{N-\lambda}(\tau) = \tilde{\chi}_\lambda(\tau) \end{aligned}$$

(3-12)

Here the central is given by $-\frac{c}{24} = \frac{1}{2N} - \frac{1}{8}$, i.e. $c = \frac{3k}{k+2}$ in agreement with equation (2-2) for SU(2) level k. The two sets of formula (3-10) and (3-12) should agree when we set $N = 2$ in (3-10) and $N = 6$ in (3-12) corresponding both to $A_1^{(1)}$ level 1, with $\chi_0(\tau) \equiv \tilde{\chi}_1(\tau)$ and $\chi_1(\tau) = \tilde{\chi}_2(\tau)$. Both transformation laws read

$$\begin{aligned} T \quad & \begin{pmatrix} \chi_0(\tau+1) \\ \chi_1(\tau+1) \end{pmatrix} = \begin{pmatrix} \exp -\frac{2i\pi}{24} & 0 \\ 0 & \exp 2i\pi \frac{5}{24} \end{pmatrix} \begin{pmatrix} \chi_0(\tau) \\ \chi_1(\tau) \end{pmatrix} \\ S \quad & \begin{pmatrix} \chi_0(-\tau^{-1}) \\ \chi_1(-\tau^{-1}) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \chi_0(\tau) \\ \chi_1(\tau) \end{pmatrix} \end{aligned}$$

(3-13)

The corresponding algebra has of course two elements $\Phi_0 = I$ the identity and $\Phi_1 = \Phi$ such that $\Phi^2 = I$.

4. It is not a priori obvious that acting on even character the formulas (3-10) generate a representation of the modular group. Before giving a proof of their validity we shall indeed check that $(ST)^3$ acts as the identity as required. From

$$(ST)_{\lambda\lambda'} = \frac{1}{\sqrt{N}} \exp 2i\pi\left(\frac{\lambda'(N-\lambda'-2\lambda)}{2N} - \frac{N-1}{24}\right)$$

so that understanding sums over integers mod N we find

$$(ST)^3_{\lambda\lambda'} = \frac{1}{N^{3/2}} \exp -2i\pi \frac{N-1}{8} \times$$

$$\sum_{\lambda_1,\lambda_2} \exp\frac{2i\pi}{2N}[\lambda_2(N-\lambda_2-2\lambda)+\lambda_1(N-\lambda_1-2\lambda_2)+\lambda'(N-\lambda'-2\lambda_1)] \tag{4-1}$$

The intermediate sum over λ_1 reads

$$\frac{1}{\sqrt{N}}\sum_{\lambda_1} \exp \frac{2i\pi}{2N}\lambda_1(N-\lambda_1-2\lambda_2-2\lambda')$$
$$= \frac{1}{\sqrt{N}} \exp \frac{2i\pi}{2N}\left(\lambda_2+\lambda'-\frac{N}{2}\right)^2\sigma_N$$

where

$$\sigma_N = \frac{1}{\sqrt{N}}\sum_{\lambda_1} \exp -\frac{2i\pi}{2N}\left(\lambda_1-\frac{\epsilon}{2}\right)^2 \tag{4-2}$$

with $\epsilon = 0$ if N is even and $\epsilon = 1$ if N is odd. We compute σ_N successively in these two cases.

(i) N even, $\epsilon = 0$. We extend the sum (4-2) from λ_1 mod N to λ_1 mod 2N at the price of a factor 1/2, and obtain a Gauss sum, the trace of $F^{[2N]^*}$. Thus

$$\sigma_N = \frac{1}{2\sqrt{N}} \sum_{\lambda \bmod 2N} \exp -2i\pi\frac{\lambda^2}{2N}$$
$$= \frac{1}{\sqrt{2}}\operatorname{Tr} F^{[2N]^*} = \frac{1-i}{\sqrt{2}} = \exp -\frac{i\pi}{4}$$

from the sequence of eigenvalues given above.

(ii) N odd, $\epsilon = 1$. In this case

$$\sigma_N = \frac{1}{\sqrt{N}} \sum_{\lambda \bmod p} \exp -2i\pi\frac{(2\lambda-1)^2}{4p}$$

Again if λ is increased by p the exponent is invariant. Thus

$$\sigma_N = \frac{1}{\sqrt{8p}} \sum_{\substack{x \bmod 4p \\ x\equiv 1 \bmod 2}} \exp -\frac{2i\pi}{4p}x^2$$

On the other hand, from the properties of $F^{[4P]^*}$

$$\sqrt{2}\exp -\frac{i\pi}{4} = \frac{1}{\sqrt{4p}}\sum_{x \bmod 4p} \exp -\frac{2i\pi}{4p}x^2$$
$$= \sqrt{2}\,\sigma_N + \frac{1}{\sqrt{4p}}\sum_{\substack{x \bmod 4p \\ x\equiv 0 \bmod 2}} \exp -2i\pi\frac{x^2}{4p}$$

The second sum on the r.h.s. is

$$\frac{1}{2\sqrt{p}}\sum_{y \bmod 2p} \exp -2i\pi\frac{y^2}{p} = \frac{1}{\sqrt{p}}\sum_{y \bmod p}\exp -2i\pi\frac{y^2}{p} = 0$$

The last statement comes from $p = 2N \equiv 2 \bmod 4$ and the properties of the Fourier transform. In conclusion for any of N even or odd

$$\sigma_N = \exp -\frac{i\pi}{4} \tag{4-3}$$

Returning to the computation of $(ST)^3$ we have thus

$$(ST)^3_{\lambda\lambda'} = \exp\left[-\frac{2i\pi}{8}\right]$$
$$\frac{1}{N}\sum_{\lambda_2}\exp\frac{2i\pi}{N}\left[\lambda_2(N-\lambda_2-2\lambda)+\lambda'(N-\lambda')+\left(\lambda_2+\lambda'-\frac{N}{2}\right)^2\right]$$
$$= \frac{1}{N}\sum_{\lambda_2}\exp\frac{2i\pi}{N}\lambda_2(\lambda-\lambda') = \delta_{\lambda\lambda' \bmod N} \tag{4-4}$$

We conclude therefore that on even characters (to ensure $S^2 \sim I$) the transformation laws (3-10) define a representation of the modular group (a unitary one) and will allow a discussion at possible invariant partition functions. Before doing that, let us now make sure that (3-10) is in agreement with the explicit expression for characters.

5. The character formula is related to an elementary problem in classical and quantum mechanics, for a free particle bouncing elastically in a simplicical box B described by the inequalities (2-23), the box B lying in the subspace V. Since the reflections in the walls of B generate a tiling of V, we find for the free motion in B an integrable system both classically and quantum mechanically. Closed orbits are obtained by folding in B straight segments joining points equivalent under a translation of the root lattice M. In a appropriate scale, the square of weight vectors are the energy levels. The characters are given by ratios of the heat kernel relative to B and the latter is expressed in terms of Θ-functions.

Explicitly consider the heat kernel on the torus V/M. With Ret > 0 and $\underline{x} \in V$ it is defined through

$$\sum_{\underline{\gamma} \in M} \exp -\pi t(\underline{\gamma}+\underline{x})^2 = \frac{1}{t^{\frac{N-1}{2}} N^{1/2}} \times \sum_{\underline{q} \in M^*} \exp -\frac{\pi}{t} q^2 + 2i\pi\, \underline{q}.\underline{x} \tag{5-1}$$

exhibiting the equivalence between a sum over closed orbits (hence the invariance under $\underline{x} \longrightarrow \underline{x} + M$) and a sum over energy levels, as a result of Poisson's formula. This motivates the definition of the Θ-function pertaining to the Lie algebra A_{N-1} through the substitutions

$$t \longrightarrow \kappa \frac{\tau}{i} \qquad \kappa \text{ positive integer,} \qquad \mathrm{Im}\,\tau > 0$$
$$\underline{x} \longrightarrow \frac{\underline{x}}{\tau} + \frac{\underline{p}}{\kappa} \qquad \underline{p} \in M^* \tag{5-2}$$

and an overall multiplication by a factor $\exp -i\pi\kappa \frac{\underline{x}^2}{\tau}$

$$\Theta_{\kappa,\underline{p}}(\underline{x},\tau) = \sum_{\underline{\gamma} \in M} \exp\left[i\pi\,\tau\kappa\left(\underline{\gamma}+\frac{\underline{p}}{\kappa}\right)^2 + 2i\pi\,\kappa\left(\underline{\gamma}+\frac{\underline{p}}{\kappa}\right).\underline{x}\right]$$
$$= \left(\frac{i}{\tau\kappa}\right)^{\frac{N-1}{2}} \frac{1}{N^{1/2}} \exp -i\pi\kappa \frac{\underline{x}^2}{\tau} \times \sum_{\underline{q} \in M^*} \exp\left[-i\pi\frac{\kappa}{\tau}\left(\frac{\underline{q}}{\kappa}\right)^2 + 2i\pi\,\underline{q}.\left(\frac{\underline{p}}{\kappa}+\frac{\underline{x}}{\tau}\right)\right] \tag{5-3}$$

If $\underline{\gamma}$ belongs to the lattice M, we have

$$\Theta_{\kappa,\underline{p}}(\underline{x}+\underline{\gamma},\tau) = \Theta_{\kappa,\underline{p}}(\underline{x},\tau)$$
$$\Theta_{\kappa,\underline{p}}(\underline{x}+\tau\underline{\gamma},\tau) = \exp\left[-i\pi\,\kappa\tau\underline{\gamma}^2 - 2i\pi\,\kappa\underline{\gamma}.\underline{x}\right]\Theta_{\kappa,\underline{p}}(\underline{x},\tau) \tag{5-4}$$

where the prefactor in the second equality is independent of $\underline{p}$.

Returning to the definition (5-3) we may perform the second sum in two steps, by writing $\underline{q} = \underline{p}' + \kappa\,\underline{\gamma}$, where γ runs over M and $\underline{p}'$ over the finite abelian group $M^*/\kappa M$, with the result that

$$\Theta_{\kappa,\underline{p}}(\underline{x},\tau) = \exp\left[-i\pi\kappa\frac{\underline{x}^2}{\tau}\right]\left(\frac{i}{\tau\kappa}\right)^{\frac{N-1}{2}} \frac{1}{N^{1/2}} \times \sum_{\underline{p}' \in M^*/\kappa M} \exp\left[2i\pi\frac{\underline{p}.\underline{p}'}{\kappa}\right] \Theta_{\kappa,\underline{p}'}\left(\frac{\underline{x}}{\tau}, -\frac{1}{\tau}\right) \tag{5-5}$$

By quotienting further by the Weyl group W, we are led to the expression for the heat kernel for the Dirichlet problem in B, through the definition of the antisymmetrized Θ-function

$$\Theta^{\wedge}_{\kappa,\underline{p}}(\underline{x},\tau) = \sum_{w\in W} \epsilon(w)\ \Theta_{\kappa,\underline{p}}\left({}^{w}\ \underline{x},\tau\right) \qquad (5\text{-}6)$$

where $\epsilon(w) = \pm 1$ is the signature of the permutation $w \in W$. Since both M and M^{*} are invariant under W, it follows that

$$\Theta^{\wedge}_{\kappa,\underline{p}}\left({}^{w}\ \underline{x},\tau\right) = \Theta^{\wedge}_{\kappa,{}^{w^{-1}}\underline{p}}(\underline{x},\tau) = \epsilon(w)\ \Theta^{\wedge}_{\kappa,p}(\underline{x},\tau) \quad (5\text{-}7)$$

This property implies that when dealing with $\Theta^{\wedge}$ the weight $\underline{p}$ can be restricted to the set of strictly positive weights up to a translation in κM. Said otherwise, we can restrict ourselves to weights $\underline{p}$ such that $\underline{p}/\kappa$ belongs to the interior of the box B. Write $\underline{p}$ in the basis of fundamental weights

$$\underline{p} = \sum_{i=1}^{N-1} p_i\ \underline{\alpha}^{(i)} \qquad (5\text{-}8)$$

with integral components p_i. The above conditions mean $p_i > 0$ for any i as well as $\sum_{i=1}^{N-1} p_i < \kappa$. This requires the integer κ to be larger or equal to N. For $\kappa = N$ the only solution is $\underline{p} = \underline{\rho} = \sum_{i=1}^{N-1} \underline{\alpha}^{(i)}$ (and $\frac{\underline{p}}{N}$ is then the "center" of the box). In general we set

$$\kappa = N + k \qquad k \text{ non negative integer} \quad (5\text{-}9)$$

where k is the level. Correspondingly

$$p_i = 1 + p_i' \quad , \ p_i' \geqslant 0.$$

The number of possible weights at level k is therefore the number of distinct solutions of $\sum_{i=1}^{N-1} p_i' \leqslant k$ with non negative integral p_i'. Define $p_N' \geqslant 0$ such that $\sum_{i=1}^{N} p_i' = k$. The required number is then the coefficient of z^k in the Taylor expansion of $(1-z)^{-N}$ around the origin, namely $\binom{N-1+k}{k}$ as claimed in equation (2-3). Clearly this is also the number of Young tableaux with at most k columns and N-1 lines.

The "unspecialized" character pertaining to the representation of the affine algebra $A^{(1)}_{N-1}$ of level k and highest weight $\underline{p}$ is then given the ratio

$$\chi_{k,\underline{p}}(\underline{x},\tau) = \frac{\Theta^{\wedge}_{N+k,\underline{p}}(\underline{x},\tau)}{\Theta^{\wedge}_{N,\underline{\rho}}(\underline{x},\tau)} \qquad (5\text{-}10)$$

The meaning of this character is that, in the corresponding representation, it is the trace

$$\chi_{k,\underline{p}}(\underline{x},\tau) = \mathrm{Tr}_{k,\underline{p}} \exp 2i\pi\left[\tau\left(L_0 - \frac{c}{24}\right) + \underline{H}.\underline{x}\right] \quad (5\text{-}11)$$

where the operators $\underline{H}$ form a basis of generators for a maximal commuting Cartan subalgebra in A_{N-1} appropriately normalized, i.e. such that $2i\pi\ \underline{H}.\underline{\gamma}$ is unity for any $\underline{\gamma} \in M$. Combining the definition (5-6) with the transformation law (5-5), we find

$$\Theta^{\wedge}_{N+k,\underline{p}}(\underline{x},\tau) = \exp\left[-i\pi(k+N)\ \frac{\underline{x}^2}{\tau}\right]\left(\frac{i}{\tau}\right)^{\frac{N-1}{2}} \frac{1}{(k+N)^{\frac{N-1}{2}}}$$

$$\times\ \frac{1}{N^{1/2}} \sum_{\underline{p}'}^{(k)} \psi\left(\underline{p},\frac{\underline{p}'}{k+N}\right)\ \Theta^{\wedge}_{N+k,\underline{p}'}\left(\frac{\underline{x}}{\tau},-\frac{1}{\tau}\right) \qquad (5\text{-}12)$$

The function $\psi(\underline{p},\underline{y})$ is the antisymmetric sum

$$\psi(\underline{p},\underline{y}) = \sum_{w\in W} \epsilon(w)\ \exp 2i\pi\ \underline{p}\ .{}^{w}\underline{y} \quad (5\text{-}13)$$

and is interpreted as the un-normalized wave function corresponding to the weight (momentum) $\underline{p}$ for the particle in B. In equation (5-12) the symbol $\sum_{\underline{p}'}^{(k)}$ means that $\underline{p}'$ runs over weights of level k, meaning that $\underline{p}'/_{(k+N)}$ lies in B.

In the denominator of the character formula (5-10) we have the value of the [illegible] $k = 0$ and $p = \rho$. In this case the sum in the transformation formula (5-12) reduces to a single term with $p = p' = \rho$ and a coefficient

$$\psi\left(\rho, \frac{\rho}{N}\right) = \sum_{w\in W} \epsilon(w) \exp 2i\pi \frac{\rho \cdot {}^{w}\rho}{N} = N^{1/2} i^{\frac{N(N-1)}{2}} \tag{5-14}$$

To prove this identity we write $\psi\left(\rho, \frac{\rho}{N}\right)$ as an $N \times N$ determinant using cartesian components with ρ given by equation (2-17)

$$\psi\left(\rho, \frac{\rho}{N}\right) = \det\left\{\exp\frac{2i\pi}{N}\left(\frac{N+1}{2}-\mu\right)\left(\frac{N+1}{2}-\nu\right)\right\} \quad 1\leq\mu,\nu\leq N \tag{5-15}$$

When N is odd, $\frac{N+1}{2} - \mu$ runs over a complete set of residues mod N when μ runs from 1 to N. Thus the above determinant is $N^{\frac{N}{2}}$ times the determinant of the finite Fourier transform $F^{[N]}$ given by (3-11) as $(-i)^{\ell} = i^{\frac{N(N-1)}{2}}$ if $N = 2\ell+1$. When N is even $\frac{N+1}{2} - \mu$ runs over the set of residues mod N + 1/2 when μ runs from 1 to N. The above determinant is $\frac{N^{1/2}}{i}$ times the determinant of $F^{[N]}$. If $N = 2\ell+2$ the latter is $-i^{\ell}$. Putting these factors together produces again (5-14). As a consequence we have

$$\Theta^{A}_{N,\rho}(\underline{x},\tau) = \exp\left[-i\pi \frac{N\underline{x}^2}{\tau}\right] i^{\frac{N(N-1)}{2}} \left(\frac{i}{\tau}\right)^{\frac{N-1}{2}} \Theta^{A}_{N,\rho}\left(\frac{\underline{x}}{\tau}, -\frac{1}{\tau}\right) \tag{5-16}$$

and for the characters at level k

$$\chi_{k,p}(\underline{x},\tau) = \exp\left[-i\pi k \frac{\underline{x}^2}{\tau}\right] (-i)^{\frac{N(N-1)}{2}} \frac{1}{(N+k)^{\frac{N-1}{2}}}$$

$$\times \frac{1}{N^{1/2}} \sum_{p'}^{(k)} \psi\left(\rho, \frac{p'}{k+N}\right) \chi_{k,p'}\left(\frac{\underline{x}}{\tau}, -\frac{1}{\tau}\right) \tag{5-17}$$

The matrix which implements the transformation $(\underline{x},\tau) \longrightarrow \left(\frac{\underline{x}}{\tau}, -\frac{1}{\tau}\right)$ must have its fourth power equal to unity (its square transforms $(\underline{x},\tau)$ into $(-\underline{x},\tau)$).

In the limit $\underline{x} \longrightarrow 0$, both numerator and denominator tend to zero in equation (5-10). However dividing first both quantities by $\psi(\rho,\underline{x})$ one finds finite limits. This defines the restricted character as

$$\chi_{k,p}(\tau) \equiv \chi_{k,p}(0,\tau) = \frac{\lim_{\underline{x}\to 0}\Theta^{A}_{N+k,p}(\underline{x},\tau)/\psi(\rho,\underline{x})}{\lim_{\underline{x}\to 0}\Theta^{A}_{N,\rho}(\underline{x},\tau)/\psi(\rho,\underline{x})} \tag{5-18}$$

For any weight p

$$\dim[p] = \lim_{\underline{x}\to 0} \frac{\psi(p,\underline{x})}{\psi(\rho,\underline{x})} = \prod_{\text{positive roots } \underline{\alpha}} \frac{\underline{\alpha}\cdot p}{\underline{\alpha}\cdot\rho} \tag{5-19}$$

is the dimension of the representation labelled by p (for p strictly positive) and extended as an antisymmetric function over the weight lattice, a celebrated result due to Weyl. Consequently

$$\chi_{k,p}(\tau) = \frac{\sum_{\underline{\gamma}\in M} \exp\left[i\pi(k+N)\tau\left(\underline{\gamma}+\frac{p}{k+N}\right)^2\right]\dim[p+(k+N)\underline{\gamma}]}{\sum_{\underline{\gamma}\in M} \exp\left[i\pi N\tau\left(\underline{\gamma}+\frac{\rho}{N}\right)^2\right]\dim[\rho+N\underline{\gamma}]} \tag{5-20}$$

Parenthetically we note the factorization formula of Dyson and Macdonald, relating the denominator in (5-20) to the Dedekind function

$$\eta(\tau) = \exp\left[i\pi \frac{\tau}{12}\right] \prod_{n=1}^{\infty} (1-\exp 2i\pi\, n\, \tau) \quad (5\text{-}21)$$

in the form

$$[\eta(\tau)]^{N^2-1} = \sum_{\underline{\gamma} \in M} \exp\left[i\pi\, N\, \tau\left(\underline{\gamma}+\frac{\underline{\rho}}{N}\right)^2\right] \dim[\underline{\rho}+N\,\underline{\gamma}] \quad (5\text{-}22)$$

The power of η is of course the dimension of the Lie algebra A_{N-1}. It is nice to check that the standard transformation law

$$\eta(\tau) = \left(\frac{i}{\tau}\right)^{1/2} \eta\left(-\frac{1}{\tau}\right) \quad (5\text{-}23)$$

is in agreement with (5-22) and (5-16) since

$$\lim_{\underline{x}\to 0} \frac{\Theta^{\wedge}_{N,\underline{\rho}}(\underline{x},\tau)}{\psi(\underline{\rho},\underline{x})} = \left[\lim_{\underline{x}\to 0} \frac{\psi\left(\underline{\rho},\frac{\underline{x}}{\tau}\right)}{\psi(\underline{\rho},\underline{x})}\right] i^{\frac{N(N-1)}{2}} \left(\frac{i}{\tau}\right)^{\frac{N-1}{2}} \times \lim_{\underline{x}\to 0} \frac{\Theta^{\wedge}_{N,\rho}\left(\frac{\underline{x}}{\tau},-\frac{1}{\tau}\right)}{\psi\left(\underline{\rho},\frac{\underline{x}}{\tau}\right)}$$

The first factor on the r.h.s. is $\tau^{-\frac{N(N-1)}{2}}$ where the exponent is (minus) the number of positive roots. Altogether the prefactor involves $(i/\tau)^{1/2}$ to a power $N(N-1) + N-1 = N^2-1$ the dimension of the algebra. It is also readily seen that under $\tau \to \tau+1$ both sides of equation (5-22) are multiplied by a factor $\exp 2i\pi\frac{(N^2-1)}{24}$. As a consequence the ratio of the right to the left hand side of this equation is a modular invariant analytic function of τ in the half plane $\mathrm{Im}\,\tau > 0$ (since $\eta(\tau)$ never vanishes for $\mathrm{Im}\,\tau > 0$). Since it is bounded when $\tau \to i\infty$ (in fact tends to unity) it is a constant equal to one, as one readily checks. This yields a simple analytic proof of this identity.

Taking (5-22) into account we can rewrite the character formula in the alternative form

$$\chi_{k,\underline{p}}(\tau) = \frac{\sum_{\underline{\gamma} \in M} \exp\left[i\pi \frac{\tau}{k+N}(\underline{p}+(k+N)\underline{\gamma})^2\right] \dim[\underline{p}+(k+N)\underline{\gamma}]}{\eta(\tau)^{N^2-1}}. \quad (5\text{-}24)$$

And under $\tau \to -\tau^{-1}$ we have

$$\chi_{k,\underline{p}}(\tau) = \frac{(-i)^{N(N-1)/2}}{(N+k)^{N-1/2}N^{1/2}} \sum_{\underline{p}'}^{(k)} \psi\left(\underline{\rho},\frac{\underline{p}'}{k+N}\right) \chi_{k,\underline{p}'}\left(-\frac{1}{\tau}\right) \quad (5\text{-}25)$$

This transformation must of course agree with the invariance of χ under the automorphism which replaces $\underline{p}$ by its conjugate (and leaves $\underline{\rho}$ invariant).

6. Every root has an even square length, and if $\underline{\gamma} \in M$ and $\underline{p} \in M^*$ the scalar product $\underline{\gamma}.\underline{p}$ is an integer. As a consequence, for any integer κ

$$\frac{\kappa}{2}\left(\underline{\gamma} + \frac{\underline{p}}{\kappa}\right)^2 \equiv \frac{\underline{p}^2}{2\kappa} \bmod \mathbf{Z} \qquad \underline{\gamma} \in M \qquad \underline{p} \in M^* \quad (6\text{-}1)$$

From equations (5-3) and (5-6) it follows therefore that

$$\Theta^{\wedge}_{\kappa,\underline{p}}(\underline{x},\tau+1) = \exp\left[i\pi \frac{\underline{p}^2}{\kappa}\right] \Theta^{\wedge}_{\kappa,\underline{p}}(\underline{x},\tau) \quad (6\text{-}2)$$

Thus the behaviour of characters under a shift $\tau \to \tau+1$ is readily found to be

$$\chi_{k,\underline{p}}(\tau+1) = \exp\left[2i\pi\left(\frac{\underline{p}^2}{2(k+N)} - \frac{\underline{\rho}^2}{2N}\right)\right] \chi_{k,\underline{p}}(\tau) \quad (6\text{-}3)$$

The exponent can be rearranged in the form $h_{\underline{p}} - \frac{c}{24}$ with

$$h_{\underline{p}} = \frac{1}{2}\,\frac{\underline{p}^2-\rho^2}{(k+N)} \qquad c = 12\rho^2\,\frac{k}{N(k+N)} = (N^2-1)\,\frac{k}{k+N} \tag{6-4}$$

where we have used the fact that

$$\rho^2 = \sum_{\mu=1}^{N}\left(\frac{N+1}{2}-\mu\right)^2 = \frac{N(N^2-1)}{12} \tag{6-5}$$

Equation (6-4) agrees with the statements made in section 2. When k=1 we find as claimed c=N-1 and $h_\lambda = \frac{\lambda(N-\lambda)}{2N}$.

It is worth to check the action of the automorphism (2-28) on characters. Let $\underline{x} \longrightarrow \tilde{\underline{x}}$ be this automorphism. Clearly $\tilde{\rho} = \rho$ and $\Theta^{\Lambda}_{\kappa,\tilde{\underline{p}}}(\underline{x},\tau) = \Theta^{\Lambda}_{\kappa,\underline{p}}(\tilde{\underline{x}},\tau)$. We can write

$$\tilde{\underline{x}} = -{}^{\omega_0}\,\underline{x} \tag{6-6}$$

where $\omega_0 \in W$ is the element in the Weyl group corresponding to the permutation $\underline{e}_{(\mu)} \leftrightarrow \underline{e}_{(N-\mu+1)}$ of signature $(-1)^{N(N-1)/2}$, with square equal to unity. In other words

$$\begin{aligned}\Theta^{\Lambda}_{\kappa,\tilde{\underline{p}}}(\underline{x},\tau) &= (-1)^{N(N-1)/2}\,\Theta^{\Lambda}_{\kappa,\underline{p}}(-\underline{x},\tau)\\ \chi_{k,\tilde{\underline{p}}}(\underline{x},\tau) &= \chi_{k,\underline{p}}(-\underline{x},\tau)\end{aligned} \tag{6-7}$$

The last relation is the expected one and shows the invariance of the character when $\underline{x}=0$.

7. In order to fully justify formula (3-10) for the action of the modular group on characters at level one, it remains to restrict the general inversion property (5-25) to the special case where $\underline{p}$ and $\underline{p}'$ belong to the set of weights of level one of the form

$$\underline{p}_\lambda = \rho + \underline{\alpha}^{(\lambda)} \tag{7-1}$$

with $\underline{\alpha}^{(0)} \equiv 0$ by convention. This requires the computation of

$$\psi_{\lambda,\lambda'} \equiv \psi\left(\underline{p}_\lambda, \frac{\underline{p}_{\lambda'}}{N+1}\right) = \sum_{\omega\in W} \epsilon(\omega)\exp 2i\pi\frac{(\rho+\underline{\alpha}^{(\lambda)})\cdot{}^{\omega}(\rho+\underline{\alpha}^{(\lambda')})}{(N+1)} \tag{7-2}$$

Let us look first at $\psi_{0,0}$ which we cast in the form of an N×N determinant similar to the one discussed in investigating the denominator of the character formula

$$\psi_{0,0} = \det\left\{\exp 2i\pi\,\frac{(\frac{N+1}{2}-\mu)(\frac{N+1}{2}-\nu)}{N+1}\right\} \qquad 1 \leqslant \mu,\ \nu \leqslant N \tag{7-3}$$

As compared to (5-15) the difference lies in the denominator (N+1) instead of N.

$$\psi_{0,0} = (-1)^{\frac{N(N+1)}{2}}(N+1)^{N/2}\det\left\{\frac{1}{(N+1)^{1/2}}\exp 2i\pi\,\frac{\mu\nu}{N+1}\right\}\qquad 1 \leqslant \mu,\ \nu \leqslant N$$

The last determinant is the (0,0) minor of the finite Fourier transform $F^{[N+1]}$

$$F^{[N+1]}_{a,b} = \frac{1}{(N+1)^{1/2}}\exp 2i\pi\,\frac{ab}{N+1} \qquad 0 \leqslant a,\ b \leqslant N$$

Its (0,0) minor is equal to its determinant (namely $i\,[-i]^{(N+1)(N+2)/2}$) times the (0,0) element of its inverse (namely $(N+1)^{-1/2}$). Thus

$$\begin{aligned}\psi_{0,0} &= (-1)^{\frac{N(N+1)}{2}}(N+1)^{\frac{N-1}{2}}\,i(-i)^{\frac{(N+1)(N+2)}{2}}\\ &= (N+1)^{\frac{N-1}{2}}\,i^{\frac{N(N-1)}{2}}\end{aligned} \tag{7-4}$$

To compute $\psi_{\lambda,0} = \psi_{0,\lambda}$ for $\lambda=1,\ldots,N$ we consider the ratio

$$\frac{\psi_{\lambda,0}}{\psi_{0,0}} = \frac{\sum\limits_{\omega \in W} \epsilon(\omega)\exp 2i\pi\ (\rho + \underline{\alpha}^{(\lambda)}) . \frac{{}^{\omega}\rho}{N+1}}{\sum\limits_{\omega \in W} \epsilon(\omega)\exp 2i\pi\ \rho . \frac{{}^{\omega}\rho}{N+1}}$$

This is interpreted as the character of the representation of SU(N) on antisymmetric tensors with λ indices, evaluated on the diagonal matrix with entries

$$\exp\left[2i\pi \frac{\rho^{\mu}}{N+1}\right] = -\exp\left[-\frac{2i\pi\ \mu}{N+1}\right] \qquad 1 \leqslant \mu \leqslant N$$

In turn this is equal to $(-1)^{\lambda}\ \Sigma_{\lambda}$ where Σ_{λ} is the λ-th elementary symmetric function of the N quantities $\exp\left[-2i\pi \frac{\mu}{N+1}\right]$ given by

$$z^{N}-\Sigma_{1} z^{N-1}+\ldots+(-1)^{\lambda}\ \Sigma_{\lambda}\ z^{N-\lambda}+\ldots+(-1)^{N}\Sigma_{N}=$$
$$\prod_{\mu=1}^{N} \left(z-\exp\left[-2i\pi \frac{\mu}{N+1}\right]\right)= z^{N}+ z^{N-1}+\ldots+1$$

showing that $(-1)^{\lambda}\Sigma_{\lambda}=1$ and

$$\psi_{\lambda,0}= \psi_{0,\lambda} = \psi_{0,0} \qquad (7\text{-}5)$$

Finally the quantity

$$\frac{\psi_{\lambda,\lambda'}}{\psi_{0,0}} = \frac{\psi_{\lambda,\lambda'}}{\psi_{0,0}}$$

is similarly the same SU(N) character evaluated now on the diagonal matrix with entries

$$\exp\left[2i\pi \frac{\rho^{\mu}+\alpha^{(\lambda')\mu}}{N+1}\right] = \begin{cases} -\exp\left[-\frac{2i\pi\mu}{N+1} + \frac{2i\pi}{N+1}\left(1-\frac{\lambda'}{N}\right)\right] & 1 \leqslant \mu \leqslant \lambda' \\ -\exp\left[-\frac{2i\pi\mu}{N+1} - \frac{2i\pi\lambda'}{(N+1)N}\right] & \lambda' < \mu \leqslant N \end{cases}$$

$$= -\exp\left[-2i\pi\frac{\lambda'}{N}\right] \begin{cases} \exp\left[\frac{2i\pi}{N+1}\ (\lambda'+1-\mu)\right] & 1 \leqslant \mu \leqslant \lambda' \\ \exp\left[\frac{2i\pi}{N+1}\ (\lambda'-\mu)\right] & \lambda' < \mu \leqslant N \end{cases}$$

Up to the prefactor $\left\{-\exp\left[-2i\pi\frac{\lambda'}{N}\right]\right\}$ these are again the (N+1)-th roots of unity except 1. Therefore

$$\frac{\psi_{\lambda,\lambda'}}{\psi_{0,0}} = \frac{\psi_{\lambda',\lambda}}{\psi_{0,0}} = \exp -2i\pi \frac{\lambda\lambda'}{N} \qquad 1 \leqslant \lambda,\ \lambda' \leqslant N-1 \qquad (7\text{-}6)$$

Combining these expressions we get

$$\psi_{\lambda,\lambda'}=(N+1)^{\frac{N-1}{2}}\ i^{\frac{N(N-1)}{2}}\ \exp\left[-2i\pi\frac{\lambda\lambda'}{N}\right] \quad 0 \leqslant \lambda,\lambda' \leqslant N-1 \qquad (7\text{-}7)$$

Inserting this result in (5-25) and recalling all properties proved before, we reach the conclusion that we have fully justified formula (3-10) for the modular transformations of $A^{(1)}_{N-1}$ characters at level k=1. From this follows the Verlinde algebra (3-8) in this case.

8. We turn to the classification of modular invariant partition functions of the form (3-1) for a W-Z-W model based on SU(N) at level one. With λ standing for an integer mod N such a partition function assumes the form

$$Z(\tau,\bar{\tau}) = \sum_{\lambda,\lambda' \bmod N} \mathcal{N}_{\lambda,\lambda'}\ \chi^{*}_{\lambda}(\tau)\ \chi_{\lambda}(\tau) \qquad (8\text{-}1)$$

where the normalization is such that

$$\mathcal{N}_{0,0} = 1 \qquad (8\text{-}2)$$

and the coefficients $\mathcal{N}_{\lambda,\lambda'}$ are non negative integers. Eventhough $\chi_{\lambda}(\tau) = \chi_{N-\lambda}(\tau)$, we extend the summation in λ,λ' over all integers mod N. This means that we do distinguish the sectors λ and $N-\lambda$ with equal restricted characters but corresponding to inequivalent Kac-Moody representations.

It is immediate from (3-10) that the two combinations

$$Z_{N,0}(\tau,\bar{\tau}) = \sum_{\lambda \bmod N} \chi^{*}_{\lambda}(\tau)\chi_{\lambda}(\tau) \qquad (8\text{-}3)$$

$$Z_{N,1}(\tau,\bar{\tau}) = \sum_{\lambda \bmod N} \chi^{*}_{\lambda}(\tau)\chi_{N-\lambda}(\tau) \qquad (8\text{-}4)$$

fulfil all critera. They are of course numerically equal as partition functions on the torus but their operator content is distinct, as explained above. The notation for the indices will be justified in the sequel. We shall show that for N an odd prime they are the only possibilities satisfying the requirements of conformal invariance. We study now the constraints on the matrix $\mathcal{N}$. For a matrix element $\mathcal{N}_{\lambda,\lambda'}$ to be different from zero it follows from invariance under T in (3-10) that the indices must satisfy

$$\lambda(N-\lambda) \equiv \lambda'(N-\lambda') \bmod 2N \tag{8-5}$$

If N is even this implies $\lambda \equiv \lambda' \bmod 2$ and is therefore equivalent to

$$\lambda^2 \equiv \lambda'^2 \bmod 2N \qquad N \text{ even} \tag{8-6a}$$

If N is odd, since $\lambda(N-\lambda) \equiv \lambda'(N-\lambda') \equiv 0 \bmod 2$, condition (8-5) is equivalent to

$$\lambda^2 \equiv \lambda'^2 \bmod N \qquad N \text{ odd} \tag{8-6b}$$

We put $n = N$ if N is odd and $n = \frac{N}{2}$ if N is even. For each divisor of n define the $N \times N$ matrix Ω_δ as follows. Let α be the greatest common divisor of δ and n/δ, denoted $\alpha = (\delta, n/\delta)$, so that α^2 divides n and therefore N. Note that when N is even so is N/α^2. There exist integers ρ and σ such that

$$\rho \frac{n}{\alpha\delta} - \sigma \frac{\delta}{\alpha} = 1 \tag{8-7}$$

Set

$$\omega = \rho \frac{n}{\alpha\delta} + \sigma \frac{\delta}{\alpha} \qquad \bmod N/\alpha^2 \tag{8-8}$$

From $\omega^2 - 1 = 4\rho\sigma \frac{n}{\alpha^2}$ we have

$$\omega^2 \equiv 1 \quad \bmod 2N/\alpha^2 \quad \text{if N is even} \tag{8-9a}$$
$$\omega^2 \equiv 1 \quad \bmod N/\alpha^2 \quad \text{if N is odd} \tag{8-9b}$$

The matrix elements $(\Omega_\delta)_{\lambda,\lambda'}$ vanish unless λ and λ' are both multiples of α in which case

$$(\Omega_\delta)_{\lambda,\lambda'} = \sum_{\xi \bmod \alpha} \delta_{\lambda',\omega\lambda+\xi N/\alpha} \qquad \alpha|\lambda,\ \alpha|\lambda' \tag{8-10}$$

Note that Ω_δ is symmetric. In particular if $\delta = n$, $\alpha = 1$ a possible choice of ω is $2n+1 \equiv 1 \bmod N$ and Ω_n is the unit matrix

$$(\Omega_n)_{\lambda,\lambda'} = \delta_{\lambda,\lambda'} \tag{8-11}$$

Exchanging the roles of δ and n/δ changes $\omega(\delta)$ into $\omega(n/\delta) = -\omega(\delta) \bmod N/\alpha^2$. In particular

$$(\Omega_1)_{\lambda,\lambda'} = \delta_{\lambda,-\lambda'} \tag{8-12}$$

These two matrices lead to the two invariants (8-3) and (8-4). In general so does Ω_δ as we show now.

The multiples of α can be written in the form

$$\alpha|\lambda \qquad \lambda = r\frac{n}{\delta} + s\delta \tag{8-13}$$

Since $\omega - 1 = 2\sigma\frac{\delta}{\alpha}$ and $\omega + 1 = 2\rho\frac{n}{\alpha\delta}$ we find

$$\omega = (\omega-1+1)r\frac{n}{\delta} + (\omega+1-1)s\delta = r\frac{n}{\delta} - s\delta + 2(\sigma r + \rho s)\frac{n}{\alpha}$$
$$\equiv r\frac{n}{\delta} - s\delta \bmod N/\alpha \tag{8-14}$$

which shows that among the multiples of α, δ is the smallest which changes sign under multiplication by ω, mod N/α (of course $-n \equiv n \bmod N$). Furthermore

$$(\omega\lambda+\xi N/\alpha)^2 - \lambda^2 = (\omega^2-1)\lambda^2 + 2\xi\omega\frac{\lambda}{\alpha}N + \xi^2\frac{N}{\alpha^2}N \tag{8-15}$$

In view of the fact that α^2 divides λ^2 and using equations (8-9a) and (8-9b) this proves that $(\omega\lambda+\xi N/\alpha)^2$ is equal to λ^2 mod 2N if N is even and mod N if N is odd. Thus Ω_δ commutes with T. As for commutation with S we find readily that

$$(S\Omega_\delta)_{\lambda,\lambda'} = (\Omega_\delta S)_{\lambda,\lambda'} = \begin{cases} \dfrac{1}{(N/\alpha)^{1/2}} \exp\left[-\dfrac{2i\pi\omega\lambda\lambda'}{N}\right] & \alpha|\lambda\ \alpha|\lambda' \\ 0 & \text{otherwise} \end{cases} \tag{8.16}$$

A slight generalization to N odd of a proof given in reference[6] and repeated in [1] shows that conversely any element of the commutant of S and T is a linear combination of the matrices Ω_δ. Since the matrix elements of Ω_δ are non negative integers and $(\Omega_\delta)_{00}=1$ it follows that any partition function of the form

$$Z_{N,\delta}(\tau,\bar\tau) = \sum_{\lambda,\lambda' \bmod N} \chi^*(\tau)_\lambda (\Omega_\delta)_{\lambda\lambda'} \chi_{\lambda'}(\tau)$$

$$= \sum_{\substack{\lambda,\lambda' \bmod N \\ \alpha|\lambda,\alpha|\lambda'}} \sum_{\xi \bmod \alpha} \chi_\lambda^*(\tau)\chi_{\omega\lambda+\xi N/\alpha}(\tau) \qquad (8\text{-}17)$$

fulfils all positivity, integrality and invariance criteria. It is correctly normalized. Thus we have at least $\varphi(n)$ WZW-SU(N) models at level one, where $\varphi(n)$ is Euler's function counting the number of divisors of n.

It would seem that a priori one could envisage more general superpositions of the form

$$Z(\tau,\bar\tau) = \sum_{\delta|n} c_\delta \, Z_{N,\delta}(\tau,\bar\tau) \qquad (8\text{-}18)$$

If one can prove that all coefficients c_δ have to be non negative integers, by looking at the coefficient of $\chi_0^*(\tau)\chi_0(\tau)$ which reads $\sum_\delta c_\delta$ and which has to equal, 1 it would follow that all c_δ have to vanish except one, showing that (8-17) is the most general solution. This can be readily ascertained for the first few values of N and I conjecture that it is generally true. In particular it is true of N prime where the only solutions are (8-3) and (8-4) (when N=2 they coïncide). A general proof is still lacking but should not be too difficult.

As a first non-trivial example consider the case of SU(9) at level 1 with central charge c = 8. Beyond the two invariants (8-3) and (8-4) corresponding to $\delta = 9$ and $\delta = 1$ respectively we have an extra possibility with $\delta = 3$ such that

$$Z_{9,3}(\tau,\bar\tau) = |\chi_0(\tau)+\chi_3(\tau)+\chi_6(\tau)|^2 \qquad (8\text{-}19)$$

Numerically $\chi_3(\tau)=\chi_6(\tau)$. Let us set

$$\psi(\tau)=\chi_0(\tau)+\chi_3(\tau)+\chi_6(\tau)=\chi_0(\tau)+2\chi_3(\tau) \qquad (8\text{-}20)$$

It is easy to check from the general formulas (3-10) that

$$\psi(\tau+1) = \exp\left[-\frac{2i\pi}{3}\right]\psi(\tau) \qquad \psi(-\tau^{-1}) = \psi(\tau) \qquad (8\text{-}21)$$

It follows that $\psi^3(\tau)$ is a modular invariant and therefore a rational function of the classical modular invariant $j(\tau)$. From equation (5-24) $\psi(\tau)$ holomorphic in the upper half plane $\mathrm{Im}\tau > 0$ behaves for $\tau \to +i\infty$ as $q^{h_0-c/24} = q^{-1/3}$ and therefore ψ^3-j is bounded at infinity. One readily verifies that ψ^3-j is of order q for $q \to 0$ and therefore

$$\psi^3(\tau) = j(\tau) \qquad (8\text{-}22)$$

If we explicit the formula (5-24) in the present case we find

$$\begin{aligned}\chi_0(\tau) &= q^{-1/3}[1+80q+1376q^2+\ldots] \\ \chi_3(\tau) &= \chi_6(\tau)=q^{-1/3}[84q+1374q^2+\ldots]\end{aligned} \qquad (8\text{-}23)$$

Hence

$$\psi(\tau)=q^{-1/3}[1+248q + 4124q^2+\ldots] \qquad (8\text{-}24)$$

and

$$\psi^3(\tau)=j(\tau)=q^{-1}[1+744q + 196.884q^2+\ldots]$$

It is interesting to find an interpretation of $j(\tau)^{1/3}$ in terms of SU(9) while it is usually related to the theory of the exceptional group E_8. One could of course multiply the examples.

9. Our analysis of level one SU(N) models although very elementary is not even complete. It would remain to prove the conjecture stated in the previous paragraph, to show that the corresponding invariants defined on the torus lead to consistent short distance expansions and to provide an interpretation in terms of various "twists" explaining the particular structure occurring in

the expressions (8-17). On the other hand the analysis at higher levels is by no means trivial. We are currently investigating models based on rank two Lie algebras, A_2, B_2 and G_2. One purpose of this work is to get a better understanding of the implications of conformal and modular invariance, an other is to see if this could illuminate the A-D-E classification found in the case A_1.

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