R. STORA

Algebraic Structure of Chiral Anomalies

Les rencontres physiciens-mathématiciens de Strasbourg - RCP25, 1986, tome 36
« Conférences de : C. Bardos, L. Baulieu, H.J. Borchers, M. Dubois-Violette, P. Pansu et R. Stora », , exp. n° 3, p. 27-60
<http://www.numdam.org/item?id=RCP25_1986__36__27_0>

LAPP-TH-143
September 1985

ALGEBRAIC STRUCTURE OF CHIRAL ANOMALIES

by

R. STORA
LAPP, Annecy-le-Vieux, France

Lectures given at the GIFT Seminar,
1-8 June 1985, JACA, Spain

L.A.P.P. CHEMIN DE BELLEVUE BOITE POSTALE 909 74019 ANNECY-LE-VIEUX CÉDEX TÉLÉPHONE 50.23.32.45. · TÉLEX 385 180 F

I. INTRODUCTION

Chiral anomalies are objects discovered as such about fifteen years ago[1). Preceding the discovery of dilatation anomalies, they constitute one of the major conquests of the period of the seventies through which the analysis of perturbative expansions essentially came to a final acceptable stage.

They were recognized as important, not only in connection with our views on symmetry breaking, but also, somewhat later, as major obstructions to the consistency of the perturbative treatment of fully quantized gauge theories.

Over the last five years or so their mathematical structure has been substantially better understood, but there are indications that more work may be needed in view, for instance, of the idea, vigorously defended by L. Faddeev[2), among others, that an anomalous gauge theory may very well turn out to possess a consistent interpretation, at the non perturbative level, of course. The example of the chiral Schwinger model analyzed by R. Jackiw and R. Rajaraman[3) does indeed force us to keep in mind that, so far, anomalies have only been shown to spoil the perturbative regime.

To come back to the present, developments have taken place in several complementary directions: in view of the often severe computational difficulties met in direct Feynman graph perturbative computations of anomalies[4) several lines of attack have been devised to analyze their structure.

At the most primitive level, the algebraic analysis of the Wess Zumino consistency conditions allows to reduce the computations to those of a finite number of numerical coefficients. This method often requires sharpening the algebraic formulation of the symmetry affected by anomalies. On the other hand, neither does it explain the role of chirality, nor does it explain the arithmetic regularity of the various coefficients involved, which has proved crucial in the analysis of anomaly cancellation mechanisms.

These more detailed, very important structural aspects can be reached by application of sophisticated versions of index theory which have however so far failed to incorporate ab initio the crucial concept of locality[5)].

It is to be hoped that, in the long range, the whole subject will find itself fully contained within a "local" index theory, but, at the moment, the algebraic theory and the index theory appeal to different cohomology theories and actually both approaches have provided results which the other one is not able to produce[6)].

Since the coverage of the subject has fortunately been shared between Paul Ginsparg[7)] and myself, I will mostly describe here the algebraic aspects of chiral anomalies, exercising however due care about the topological delicacies involved here and there. I will most of the time illustrate the structure and methods in the context of gauge anomalies and will eventually make contact with results obtained from index theory. I will then go into two sorts of generalizations: on the one hand, generalizing the algebraic set up yields e.g. gravitational and mixed gauge anomalies, supersymmetric gauge anomalies, anomalies in supergravity theories (σ model anomalies will be treated in P. Ginsparg's lectures); on the other hand most constructions applied to the first - and eventually second - cohomologies which characterize anomalies easily extend to higher cohomologies. Although the latter have not so far received firmly founded physical interpretations, they have appeared in the topological analysis and definitely belong to the theoretical framework.

Section II is devoted to a description of the general set up as it applies to gauge anomalies. It owes much to an article by J. Mañes, B. Zumino[8)] and myself, whose writing has now come to an end (referred as MSZ in the body of these notes).

Section III deals with a number of algebraic set ups which characterize more general types of anomalies: gravitational and mixed gauge anomalies, supersymmetric gauge anomalies, anomalies in supergravity theories. It also includes brief remarks on σ models and a reminder on the full BRST algebra of quantized gauge theories.

A mathematical appendix is devoted to a description of the general cohomological constructions which underly the whole analysis.

II. GAUGE ANOMALIES

(Perturbative current algebra anomalies[9]):

Let $S(\phi)$ be a classical action involving "matter" fields ϕ, transforming linearly under an internal compact Lie symmetry group G, of the renormalizable type, $\Gamma(\phi)$ the corresponding vertex functional

$$\Gamma(\phi) = S(\phi) + \sum_{n=1}^{\infty} \hbar^n \Gamma^{(n)}(\phi) \tag{1}$$

If S is invariant under the action of G it is in particular invariant under the action of Lie G. This is expressed by a Ward identity

$$W_{cl.}(\omega)\, S(\phi) = 0 \qquad \omega \in \text{Lie } G \tag{2}$$

where $W_{cl.}(\omega)$ is a functional differential operator linear in ω. This set up covers the situation where the internal symmetry G is softly broken. One can then show that the renormalized perturbative series representing $\Gamma(\phi)$ can be defined in such a way that

$$W(\omega)\, \Gamma(\phi) = 0 \tag{3}$$

where $W(\omega)$ fulfills the commutation relations

$$[W(\omega), W(\omega')] = W([\omega,\omega']) \qquad \omega,\omega' \in \text{Lie } G \tag{4}$$

Let $\mathcal{G}$ be the gauge group associated with G (maps from space time to G in the simplest case). Then it is easy to extend $S(\phi)$ into $S(\phi,a)$, where

a is an external classical gauge field transforming under $\mathcal{G}$ in the well known way, and $W_{cl.}(\omega)$ into $\mathcal{W}_{cl.}(\underline{\omega})$, $\underline{\omega} \in \mathrm{Lie}\,\mathcal{G}$, in such a way that

$$\mathcal{W}_{cl.}(\underline{\omega})\; S(\phi,a) \;=\; 0 \tag{5}$$

This is easily carried out by "minimal coupling".

The question is then whether $\Gamma(\phi)$ can be extended into $\Gamma(\phi,a)$ and $\mathcal{W}_{cl.}(\underline{\omega})$ into $\mathcal{W}(\underline{\omega})$ in such a way that

$$\begin{aligned} &\mathcal{W}(\underline{\omega})\;\Gamma(\phi,a) \overset{?}{=} 0 \\ &[\mathcal{W}(\underline{\omega}), \mathcal{W}(\underline{\omega}')] = \mathcal{W}([\underline{\omega},\underline{\omega}']) \end{aligned} \qquad \underline{\omega},\underline{\omega}' \in \mathrm{Lie}\,\mathcal{G} \tag{6}$$

It is found that this is in general not the case, precisely when the matter fields contain chiral spinors. Rather, one has an anomalous Ward identity

$$\mathcal{W}(\underline{\omega})\;\Gamma(\phi,a) \;=\; \int_M \mathfrak{A}(\underline{\omega},a) \tag{7}$$

where $\mathfrak{A}(\underline{\omega},a)$ is a differential form linear in $\underline{\omega}$, a <u>local</u> polynomial in a and its derivatives, and $\int_M$ denotes space time integration. This describes the situation for $d = 4$-dimensional perturbatively renormalizable theories. The locality property of $\mathfrak{A}$ is due to the locality properties of perturbative expansions; the fact it has canonical dimension 4 comes from power counting. $\mathfrak{A}$ is partly characterized by the Wess Zumino consistency conditions which follow from the commutation relations Eq.(6):

$$\mathcal{W}(\underline{\omega}) \int \mathfrak{A}(\underline{\omega}',a) - \mathcal{W}(\underline{\omega}') \int \mathfrak{A}(\underline{\omega},a) - \int \mathfrak{A}([\underline{\omega},\underline{\omega}']) \;=\; 0 \;. \tag{8}$$

By the algebraic Poincaré lemma[10)] applied to the local functional $\mathfrak{A}(\underline{\omega},a)$, one has

$$\mathcal{W}(\underline{\omega})\; \mathfrak{A}(\underline{\omega}',a) - \mathcal{W}(\underline{\omega}')\; \mathfrak{A}(\underline{\omega},a) - \mathfrak{A}([\underline{\omega},\underline{\omega}']) \;=\; 0 \mod d \tag{9}$$

where "mod d" means "modulo" the exterior differential of a form local in $\underline{\omega},\underline{\omega}'$,a, and their derivatives.

The Wess Zumino consistency condition characterizes $\int \alpha(\underline{\omega},a)$ as an element of $H^1(\mathrm{Lie}\,\mathcal{G}, \Gamma^{loc}(a))$ (cf. Appendix) where $\Gamma^{loc}(a)$ denotes the space of local functionals of a, because a change of renormalization prescriptions, which does not introduce a dependence on ϕ and alters $\Gamma(\phi,a)$ by a local counterterm $\Gamma^{loc}(a)$, alters $\alpha(\underline{\omega},a)$ by an amount $\mathcal{W}(\underline{\omega})\ \Gamma^{loc}(a)$. In the case of d = 4 renormalizable theories the elimination of the linearly transforming "matter" fields requires a bit more than the Wess Zumino consistency conditions when G contains U(1) factors; once $\Gamma^{loc}(a,\phi)$ is reduced to $\Gamma^{loc}(a)$ there only remains to compute $H^1(\mathrm{Lie}\,\mathcal{G}, \Gamma^{loc}(a))$. The result, which will be described later turns out to be completely expressible in terms of the differential form $\underline{a}$, and its exterior derivative, not separately on its coefficients and their derivatives. Besides the complete results for dimension d = 4 summarized here, there is now one complete result known, for arbitrary d, namely the general computation of $H^*(\mathrm{Lie}\,\mathcal{G}, \Gamma^{loc}(\underline{a}))$, recently carried out by M. Dubois Violette, M. Talon, C.M. Viallet[11], to which one may apply the various general constructions described in the appendix. For instance, given an n cocycle mod d, $Q^n_p(\omega,\underline{a})^{\dagger}$ of Lie $\mathcal{G}$ with values in $\Gamma^{loc}(\underline{a})$, we may construct the corresponding Wess Zumino cocycle††

$$\int_{g_t \in T_n(g_o \ldots g_n) \times \mathcal{E}_p} Q^n_p\,(g_t^{-1} d_t g_t, {}^{g_t}\underline{a}) \tag{10}$$

where ${}^{g_t}\underline{a}$ is the gauge transform of a :

$${}^{g}\underline{a} = g^{-1}\underline{a}g + g^{-1}dg \tag{11}$$

---

†from now on ω denotes the generator of $H^*(\mathrm{Lie}\,\mathcal{G})$, cf. Appendix, eq.(A9).

††cf. Appendix eq.(A13).

and $\mathcal{C}_p$ is a cycle in the base manifold, of dimension p, the degree of Q^n_p as a differential form.

In particular, for n=1, p=d, the dimension of the base manifold we get the Wess Zumino action

$$\Gamma_{WZ}(g,\underline{a}) = \int_{T_1(e,g)\times M_d} \alpha(g_t^{-1}d_t g_t, {}^{g_t}\underline{a}) \tag{12}$$

which fulfills the Ward identity

$$\delta\, \Gamma_{WZ}(g,\underline{a}) = \int_{M_d} \alpha(\omega,a) \tag{13}$$

where δ is defined by

$$\begin{aligned} \delta\underline{a} &= -\,d\omega - [\underline{a},\omega] \\ \delta g &= \omega g \end{aligned} \tag{14}$$

(cf. Eqs (A27),(A10)).

Note that a non trivial cocycle with values in $\Gamma^{loc}(\underline{a})$ becomes trivial - up to non uniformity - in $\Gamma^{loc}(g,\underline{a})$ where a new variable $g \in \mathcal{G}$, behaving non linearly under $\mathcal{G}$, has been introduced. This is one classical procedure to "kill cohomology". The other known procedure, the so called Green Schwartz procedure requires the introduction of a 2-form with suitable transformation properties and will be mentioned in the next section.

The non uniformity of Γ_{WZ}[12] is clearly parametrized by $\pi_1(\mathcal{G})$, the first homotopy group of the gauge group and, upon proper normalization, can be gotten rid of by exponentiation in most cases of interest[13] (e.g. compact space-time, see MSZ for further details).

We shall now describe in some details the results of M. Dubois Violette, M. Talon, C.M. Viallet[11] (referred to as DTV), namely exhibit the construction of canonical representatives $Q^n_p(\omega,\underline{a})$ for $H^*(\text{Lie}\,\mathcal{G}, \Gamma^{loc}(\underline{a}))$.

Some technical preparation is in order: Let

$$\underline{F} = d\underline{a} + \frac{1}{2}[\underline{a},\underline{a}] \qquad (15)$$

be the curvature of $\underline{a}$.

Let J_n be a symmetric polynomial on Lie G of degree n, invariant under the adjoint action of G.

One has the following transgression formulae inherited from Chern, Weil, Cartan:

$$J_n(F(\underline{a})\dots F(\underline{a})) - J_n(F(\underline{a}_o)\dots F(\underline{a}_o))$$

$$= n\,(d+\delta)\int_0^1 J_n(d_t A_t,\ F(\underline{A}_t)\dots F(\underline{A}_t))$$

$$= (d+\delta)\,Q_n \qquad (16)$$

where $\underline{a}_o$ is a fixed background connection ($\neq 0$ if connections live on a non trivial bundle, cf. MSZ), and

$$\underline{A}_t = t\,\underline{a}_o + (1-t)\,(\underline{a}+\omega) \qquad (17)$$

is a family interpolating between $\underline{a}_o$ and $\underline{a}+\omega$. Q_n is an element of total degree 2n+1 bigraded by the form degree p and the degree in ω,g (p+g = 2n-1):

$$Q_n = \sum_{g=0}^{g=2n-1} Q^g_{2n-1-g} \qquad (18)$$

Expanding Eq.(16) in powers of ω we get the hierarchy of identities

$$\begin{aligned}
J_n(F(\underline{a})) - J_n(F(\underline{a}_o)) &= d\,Q^o_{2n-1}\\
0 &= \delta\,Q^o_{2n-1} + d\,Q^1_{2n-2}\\
\vdots\quad &= \delta\,Q^g_{2n-1-g} + d\,Q^{g+1}_{2n-1-g-1}\\
\vdots\quad&\\
0 &= \delta\,Q^{2n-1}_o
\end{aligned} \qquad (19)$$

If J_n is irreducible, namely is not the product of several invariant polynomials, Q_o^{2n+1} is an irreducible antisymmetric invariant polynomial in ω. Conversely, one can show that these invariant antisymmetric polynomials generate $H^*(\text{Lie } G)$ when J_n spans a system of generators of invariant symmetric polynomials. The Q_n's appear as building blocks for $H^*(\text{Lie } \mathcal{G}, \Gamma^{loc}(\underline{a}))$:

<u>Th</u> (D.T.V.): A system of representative cocycles for $H^*(\text{Lie } \mathcal{G}, \Gamma^{loc}(\underline{a}))$ is given by expanding in powers of ω all expressions of the type:

$$\prod_{\substack{n_i \\ n_1 \leq n_2 \leq \cdots \leq n_k}} \sum_{g_i=0}^{g_i=\min(2n_i+1)} Q^{g_i}_{2n_i-1-g_i} \prod_{m_i \geq n_1} (J_{m_i}(F(\underline{a})) - J_{m_i}(F(\underline{a}_o))), \tag{20}$$

integrating them over cycles of the base manifold, considering as distinct those for which the expressions

$$\prod_{\substack{n_i \\ n_1 \leq n_2 \leq \cdots \leq n_k}} \sum_{g_i=0}^{g_i=\min 2n_i+2} Q^{g_i}_{2n_i-1-g_i} \prod_{m_i \geq n_1} (J_{m_i}(F(\underline{a})) - J_{m_i}(F(\underline{a}_o))) \tag{21}$$

are independent, with the notation

$$Q_{-1}^{2n_i+2} = J_{n_i+1}(F(\underline{a})) - J_{n_i+1}(F(\underline{a}_o)) . \tag{22}$$

The system of generators of $H^*(\text{Lie } \mathcal{G}, \Gamma^{loc}(\underline{a}))$ has first been written down by J. Thierry-Mieg[14]. That these expressions are δ cocycles mod d is a consequence of eq.(19). This was partly conjectured by J. Dixon, as reported in ref. 10). On the other hand, the independence statement proved by D.T.V. requires heavy use of homological algebra which we cannot report on in any detail here.

For practical purpose this analysis shows that anomalies are all of the form Q^1_{2n-2} (eq.19), and in one to one correspondence with non necessarily irreducible invariants J_n, for even dimensional space time, but it also provides amusing examples in odd dimensional space time when G contains at least two U(1) factors[11),14)].

III. MORE GENERAL DIFFERENTIAL ALGEBRAS[15)]

It must be clear from the preceding section what anomalies are in other similar situations which are of essentially two types:

- the anomaly is still associated with the cohomology of a Lie algebra;
- the anomaly is associated with a differential algebra which is not necessarily - known to be or to contain - the cohomology algebra of a Lie algebra.

Of course the first type is a particular case of the second. It covers the following situations of interest:

- gravitational and mixed gauge anomalies[4),8),16)]; this is both useful and educative because the problem is often set up in such a way that it is not apparent whether one is dealing with the cohomology of a Lie algebra, a situation which still prevails in a number of cases touching supergravity theories;
- supersymmetric gauge anomalies[17)]: the interesting feature here is that the non polynomial structure of the anomaly is rather obscure from the point of view of differential geometry. The reduction of the anomaly to the Wess Zumino gauge, on the other hand, poses a problem of the second type[18),32)];
- anomalies in σ models (see in particular P. Ginsparg's lectures[7),19)]). In this case there remains to compute the relevant cohomology with values local functionals of the σ model fields.

In the second class one finds for instance:

- the full BRST cohomology relevant to the perturbative renormalization of gauge theories; $H^1(\text{BRST}, \Gamma^{loc})$ was computed by BRS under restrictions provided by power counting. H. Kluberg Stern, J.B. Zuber, B.W. Lee, S.D. Joglekar, John Dixon[20] performed a number of additional steps but this problem may have to be taken up again in order to investigate the structure of the Ward identity characterizing a fully quantized anomalous gauge theory[2],[21];
- supergravity theories: in terms of various sets of component fields restricted by gauge choices, these theories are characterized by differential algebras[22] which are not obvious extensions of Lie algebra cohomology algebras. In most cases there remains to check that, extending the gravitational case, local supersymmetry does not modify the classification of anomalies; e.g. the Green Schwartz[23] cancellation mechanism can be performed in a way compatible with supersymmetry[24].

This list is certainly not exhaustive; a number of other differential algebras are in sight e.g. the algebra associated with differential forms[25], the algebras associated with constrained dynamical systems[26].

In the following, we shall gather some results pertaining to a few examples of recent interest we have listed above.

III.1 Gravitational and mixed anomalies[4],[8b],[16]

An external gravitational field on space time M may be represented by a vielbein form

$$e^a = e^a_\mu \, dx^\mu \qquad \begin{array}{l} a = 1 \ldots d = \dim M \\ \mu = 1 \ldots d = \dim M \end{array} \tag{23}$$

and a spin connection

$$\theta^a_b = \theta^a_{b\mu} \, dx^\mu \tag{24}$$

which is a Lie(SOd) valued one form. At the classical level, the action describing the coupling of matter with external gravity is invariant under SO(d) gauge transformations and diffeomorphisms of M. At the quantum level the question is whether this is true for the vacuum functional $\Gamma(e,\theta)$. The action of SO(d) gauge transformations is clear but that of the diffeomorphisms of M is not uniquely defined: a sensible definition requires the introduction of a fixed background spin connection $\overset{o}{\theta}$.

The structure equations of the corresponding differential algebra are[16]:

$$\begin{aligned}
\delta e &= -\Omega e - \overset{o}{\mathcal{L}}_\xi e \\
\delta\theta &= -D(\theta)\Omega - i_\xi R(\theta) + D(\theta) i_\xi(\theta - \overset{o}{\theta}) \\
\delta\Omega &= -\frac{1}{2}[\Omega,\Omega] - \overset{o}{\mathcal{L}}_\xi \Omega - \frac{1}{2} i_\xi i_\xi R(\overset{o}{\theta}) \\
\delta\xi &= -\frac{1}{2}[\xi,\xi]
\end{aligned} \qquad (25)$$

where Ω and ξ are the cohomology generators corresponding to SO(d) gauge transformations and diffeomorphisms of M, $D(\theta) = d.+ [\theta,.]$, $R(\theta) = d\theta + \frac{1}{2}[\theta,\theta]$, i_ξ denotes saturation of a differential form by the vector field ξ and

$$\overset{o}{\mathcal{L}}_\xi = i_\xi D(\overset{o}{\theta}) - D(\overset{o}{\theta}) i_\xi \qquad (26)$$

is the covariant Lie derivative operator acting on forms. These are the structure equations for the cohomology algebra of a Lie algebra $\varepsilon(\overset{o}{\theta})$ with values in $\Gamma^{loc}(e,\theta)$. The commutation relations defining $\varepsilon(\overset{o}{\theta})$ are:

$$\begin{aligned}
[W(\Omega), W(\Omega')] &= W([\Omega,\Omega']) \\
[W(\xi), W(\Omega)] &= W(\overset{o}{\mathcal{L}}_\xi \Omega) = W(i_\xi \overset{o}{D}\Omega) \\
[W(\xi), W(\xi')] &= W([\xi,\xi']) - W(i_\xi i_{\xi'}, R(\overset{o}{\theta}))
\end{aligned} \qquad (27)$$

If M is parallelizable one may of course choose $\overset{o}{\theta} = 0$.

Given two background connections $\overset{o}{\theta}$, $\overset{1}{\theta}$, one has an isomorphism

$$\varepsilon(\overset{o}{\theta}) \simeq \varepsilon(\overset{1}{\theta}) \simeq \varepsilon \tag{28}$$

given by

$$\begin{aligned} \overset{o}{W}(\Omega) &= \overset{1}{W}(\Omega) \overset{\text{def}}{=} W(\Omega) \\ \overset{o}{W}(\xi) &= \overset{1}{W}(\xi) + W(i_\xi(\overset{o}{\theta}-\overset{1}{\theta})) \ . \end{aligned} \tag{29}$$

A change of generators in the differential algebra (Eq.25):

$$\tilde{\Omega} = \Omega - i_\xi(\theta-\overset{o}{\theta}) \tag{30}$$

allows to cast the structure equations into the form

$$\begin{aligned} \delta e &= -\tilde{\Omega}e - (i_\xi D(\theta) - D(\theta)i_\xi)\, e \\ \delta\theta &= -D(\theta)\tilde{\Omega} - i_\xi R(\theta) \\ \delta\tilde{\Omega} &= -\frac{1}{2}[\tilde{\Omega},\tilde{\Omega}] + \frac{1}{2} i_\xi i_\xi R(\theta) \\ \delta\xi &= -\frac{1}{2}[\xi,\xi] \end{aligned} \tag{31}$$

from which $\overset{o}{\theta}$ has dropped out but which is no more obviously the set of structure equations of a Lie algebra; as one often says, it is related to a "field dependent" Lie algebra in which $W(\xi)$ is interpreted as generating "parallel transport". This may be physically appealing but is mathematically obscure. The cohomology $H^*(\varepsilon, \Gamma^{loc}(\underline{e},\underline{\theta}))$ has not yet been computed, but it is a relatively easy exercise[16] to construct some representatives of $H^*(\varepsilon, \Gamma^{loc}(\underline{\theta}))$: it turns out that

$$\begin{aligned} \mathcal{R}(\theta+\tilde{\Omega}) &= (d+\delta)(\theta+\tilde{\Omega}) + \frac{1}{2}[\theta+\tilde{\Omega}, \theta+\tilde{\Omega}] \\ &= e^{-i_\xi} R(\theta) \end{aligned} \tag{32}$$

It follows that

$$J_n(\mathcal{R}^n) - J_n(R(\overset{o}{\theta})^n) = (d+\delta)\, Q_{2n-1} \tag{33}$$

where

$$Q_{2n-1} = n \int_0^1 J_n(\theta-\overset{o}{\theta}+\tilde{\Omega}, \mathcal{R}_t^{n-1}) \tag{34}$$

with

$$\mathcal{R}_t = \mathcal{R}(\overset{o}{\theta} + t\, e^{-i\xi}(\theta-\overset{o}{\theta}+\Omega)) \tag{35}$$

So,

$$e^{-i\xi} J_n(R(\theta)^n) - J_n(R(\overset{o}{\theta})^n)$$
$$= (d+\delta)\, n \int_0^1 J_n\, (e^{-i\xi}(\theta-\overset{o}{\theta}+\Omega),\, \mathcal{R}(\overset{o}{\theta} + t\, e^{-i\xi}(\theta-\overset{o}{\theta}+\Omega))^{n-1} \tag{36}$$

Now, in dimension $\leq 2n-2$

$$(i_\xi)^k J_n(R(\theta)^n) - J_n(R(\overset{o}{\theta})^n) = 0 \tag{37}$$

$\forall\ k \geq 1$. The terms Q^g_{2n-1-g}, $g \geq 1$, of the expansion of Q_{2n-1} thus yields representatives for $H^*(\varepsilon, \Gamma^{loc}(\theta))$ upon integration over cycles of the correct dimension. One may thus conjecture[16] that $H^*(\varepsilon, \Gamma^{loc}(\underline{\theta}))$ $\simeq H^*(\mathrm{Lie}\,\mathcal{G}, \Gamma^{loc}(\underline{\theta}))$ where $\mathrm{Lie}\,\mathcal{G} \subset \varepsilon$ is the relevant gauge group (corresponding to $SO(d) \times G$ in the mixed gravitational gauge case).

In the language of ref. 8a) the derivation i_ξ is a homotopy which intertwines $d+\delta$. and $\delta+\delta_{\tilde{\Omega}}$, where $\delta_{\tilde{\Omega}} = \delta|_{\xi=0}$.

The first cohomology has been more extensively studied, since it is of immediate physical interest. In particular other particular representatives have been found by W. Bardeen and B. Zumino[27] for $H^1(\varepsilon, \Gamma^{loc}(e,\theta))$ from which the Ω component has been eliminated via a Wess Zumino counterterm.

We now turn to:

III.2 Supersymmetric gauge anomalies in flat d=4 dimensional superspace[17)28)]

The previous examples have lead to results rather compactly expressed in terms of differential forms, most of which can be obtained by topological arguments, via the index theorems[39], through the isomorphisms $H^*(\text{Lie } \mathcal{G}, .) \simeq H^*_{DR,inv.}(\mathcal{G}, .)$ as explained in the appendix. Once the role of locality gets incorporated into the index type analysis, the latter offers of course more accurate results in each specific case since it provides an identification of the relevant invariant polynomials.

The supersymmetric case to be described now is one in which the geometry is still in a complete state of obscurity and in which the algebraic set up yields an answer through an equally obscure route.

Whereas the connection form $\underline{a}$ naturally appears in the classical case, the corresponding gauge superfield V appears by solving constraints to be imposed upon the super-connection forms $\varphi, \bar{\varphi}$ which respectively transform under the chiral and antichiral gauge groups. The search for a manifestly supersymmetric solution of the consistency conditions is ascertained to be a difficult task in view of the theorem that, if the structure group is semi-simple, the anomaly cannot be polynomial in the components of $\varphi, \bar{\varphi}$[29]. It is known to be parametrized in terms of invariant polynomials of degree 3, J_3, by a theorem of O. Piguet and K. Sibold[30] (the P.S. theorem).

On the other hand, remarkably enough, it has been shown by L. Bonora, P. Pasti, M. Tonin (B.P.T.)[17] that a representative, polynomial in the components of $\varphi, \bar{\varphi}$, can be found for the first cohomology of the graded Lie algebra generated by gauge transformations, space time translations and rigid supersymmetry transformations: the algebra used to find gravitational anomalies applies, with ξ restricted to vector fields generating space time translations and rigid supersymmetry transformations. In the present case however the vanishing of $e^{-i_\xi} J_n(R(\theta)^n)$ (cf. eq.(36)), in the useful dimensions, does not apply in rigid superspace. However, remarkably enough, the use of the constraints on $\varphi, \bar{\varphi}$, allow to throw the l.h.s. of Eq.(33) into the r.h.s. under the $d+\delta$ symbol! There follow polynomial expressions involving an explicitly non vanishing supersymmetry anomaly. By a theorem of O. Piguet, M. Schweda, K. Sibold[31], tha latter is trivial. However it turns out to be the coboundary of a non polynomial Wess Zumino type local functional and its elimination provides the manifestly supersymmetric, non polynomial form of the anomaly precedingly announced, which can also be directly obtained by superspace analysis.[17] We refer the reader to the original articles, since the final formulae are neither appetizing nor methodologically illuminating.

Clearly, more work has to be done in order to reach some decent understanding of this structure, which should reasonably emerge from a clear description of the geometry.

Another interesting exercise is to find the anomaly directly in the Wess Zumino gauge. This has been done by H. Itoyama, V.P. Nair, H.C. Ren[18] who find both a gauge and a supersymmetry component, with vanishing translation component. The interest of the exercise lies in the fact that the corresponding differential algebra (cf. ref. 32, eqs 6.2, 6.3) is the prototype of one for which it is not known whether it is an extension of the cohomology algebra of some - in this case graded - Lie algebra.

III.3 σ models[7],[19]

Anomalies in σ models are treated in detail in P. Ginsparg lectures[7]. They have been investigated by topological methods first by G. Moore and P. Nelson, then, together with A. Manohar. This work has now been somewhat deepened and generalized by L. Alvarez Gaumé and P. Ginsparg.

The Lie algebra defining the anomalies is generated by the isometries of the target space and the gauge transformations of the corresponding bundle of frames. It is a relatively simple matter to find representative 1-cocycles by substituting into the usual formulae the expressions of the gauge field in terms of the σ field. But there is at the moment no complete result concerning the non triviality of these expressions (there is no proof either that these are the only possible anomalies). At the moment, anomalies associated with topological obstructions are known to be non trivial in the local sense. In the particular case where the target space is a homogeneous space G/H sufficient triviality conditions on the known candidates have been found by L. Alvarez Gaumé and P. Ginsparg, namely that the H representation of the chiral fermion fields originates from the reduction to H of a representation of G.

More complete results would be welcome in particular because of the role played by σ models in supergravity theories[19).

III.4 B.R.S.T. cohomology[9),33)

At the classical level, the action for a fully quantized renormalizable gauge theory reads

$$\Gamma^{cl.} = \int (\mathcal{L}_{inv}(a,\phi) + b.\mathcal{G}^{(a,\phi)} + \bar{\omega}\, s\, \mathcal{G} + Q(b)) \tag{38}$$

where a is the gauge field, ϕ the matter field, b the Stueckelberg field, ω the Faddeev-Popov $(\Phi\pi)$ cohomology generator, $\bar{\omega}$ the $\Phi\pi$ multiplier, $\mathcal{G}$ a gauge function. On has

$$s\Gamma^{cl.} = 0 \tag{39}$$

with

$$\begin{aligned} sa &= -(d\omega + [a,\omega]) \qquad & s\omega &= -\frac{1}{2}[\omega,\omega] \\ s\phi &= t(\omega)\phi & & \\ s\bar{\omega} &= -b & sb &= 0 \end{aligned} \tag{40}$$

where $t(.)$ is the representation of Lie $\mathcal{G}$ on the matter field.

In order to allow the invariance property (eq.39) to go through the construction of the perturbative series describing quantum corrections, one introduces sources A, Φ, Ω linearly coupled to sa, $s\phi$, $s\omega$

$$\Gamma^{source} = \int A\ sa + \Phi\ s\phi + \Omega\ s\omega \tag{41}$$

So that

$$\Gamma^{o} = \Gamma^{d} + \Gamma^{source} \tag{42}$$

fulfills

$$\int \left(\frac{\delta\Gamma}{\delta a}\frac{\delta\Gamma}{\delta A} + \frac{\delta\Gamma}{\delta\phi}\frac{\delta\Gamma}{\delta\Phi} + \frac{\delta\Gamma}{\delta\omega}\frac{\delta\Gamma}{\delta\Omega} - \frac{\delta\Gamma}{\delta\bar{\omega}}\ b \right) = 0 \tag{43}$$

This is the Legendre transform of the linear Ward identity which Z^{c}, the connected Green's functional fulfills.

Assuming that

$$\Gamma = \Gamma^{o} + \Sigma\ \hbar^{n}\ \Gamma^{(n)} \tag{44}$$

has been proved to fulfill eq.(43) up to and including order $n-1$, the question is whether eq.(43) can be fulfilled at order n:

$$\int \left(\frac{\delta\Gamma^{o}}{\delta a}\frac{\delta\Gamma^{n}}{\delta A} - \frac{\delta\Gamma^{o}}{\delta A}\frac{\delta\Gamma^{n}}{\delta a} + \frac{\delta\Gamma^{o}}{\delta\phi}\frac{\delta\Gamma^{n}}{\delta\Phi} - \frac{\delta\Gamma^{o}}{\delta\Phi}\frac{\delta\Gamma^{n}}{\delta\phi} + \frac{\delta\Gamma^{o}}{\delta\omega}\frac{\delta\Gamma^{n}}{\delta\Omega} - \frac{\delta\Gamma^{o}}{\delta\Omega}\frac{\delta\Gamma^{n}}{\delta\omega} - \frac{\delta\Gamma^{n}}{\delta\bar{\omega}}\ b \right)$$

$$+ \sum_{p=1}^{n-1} \frac{\delta\Gamma^{p}}{\delta a}\frac{\delta\Gamma^{n-p}}{\delta A} + \frac{\delta\Gamma^{p}}{\delta\phi}\frac{\delta\Gamma^{n-p}}{\delta\Phi} = 0\ . \tag{45}$$

This is an equation for the n^{th} term in the local effective action

$$\Gamma_{eff} = \Gamma^{o}_{eff} + \Sigma\ \hbar^{n}\Gamma^{(n)}_{eff} \overset{e.g}{=} T_{4}\ \Gamma \tag{46}$$

obtained, for instance (in the BPHZ framework), by truncating the p-space

Taylor expansions of the coefficients of Γ in such a way that only dimension 4 monomials survive, from which renormalized Feynman graphs are constructed. One may also appeal to a regularization procedure and carry out renormalization by the separation of divergent parts. Equation (45) may be rewritten as

$$\mathcal{S}^o \Gamma^n + R^n = 0 \tag{47}$$

where

$$R^n = \int \sum_{p=1}^{n-1} \frac{\delta\Gamma^p}{\delta a}\frac{\delta\Gamma^{n-p}}{\delta A} + \frac{\delta\Gamma^p}{\delta\phi}\frac{\delta\Gamma^{n-p}}{\delta\Phi} + \frac{\delta\Gamma^p}{\delta\omega}\frac{\delta\Gamma^{n-p}}{\delta\Omega} \tag{48}$$

is fully determined by lower order terms assumed to fulfill the Ward identity and

$$\mathcal{S}^o = \int \frac{\delta\Gamma^o}{\delta a}\frac{\delta}{\delta A} - \frac{\delta\Gamma^o}{\delta A}\frac{\delta}{\delta a} + \frac{\delta\Gamma^o}{\delta\phi}\frac{\delta}{\delta\Phi} - \frac{\delta\Gamma^o}{\delta\Phi}\frac{\delta}{\delta\phi} + \frac{\delta\Gamma^o}{\delta\omega}\frac{\delta}{\delta\Omega} - \frac{\delta\Gamma^o}{\delta\Omega}\frac{\delta}{\delta\omega} - b\frac{\delta}{\delta\overline{\omega}} \tag{49}$$

which fulfills identically

$$(\mathcal{S}^o)^2 = 0 \tag{50}$$

by virtue of

$$\mathcal{S}(\Gamma^o) \equiv \int \frac{\delta\Gamma^o}{\delta a}\frac{\delta\Gamma^o}{\delta A} + \frac{\delta\Gamma^o}{\delta\phi}\frac{\delta\Gamma^o}{\delta\Phi} + \frac{\delta\Gamma^o}{\delta\omega}\frac{\delta\Gamma^o}{\delta\Omega} - b\frac{\delta\Gamma^o}{\delta\overline{\omega}} = 0 \tag{51}$$

Now, it follows from the action principle that

$$\mathcal{S}^o \Gamma^n + R^n = \Delta^n \tag{52}$$

where Δ^n is a local expression.

We will see in a short while that the recursion hypothesis implies

$$\mathcal{S}^o R^n = 0 \tag{53}$$

It follows that

$$\mathcal{S}^o \Delta^n = 0 \tag{54}$$

Thus, if the first local cohomology of $\mathcal{S}^o$ is trivial[41] one has

$$\Delta^n = \mathcal{S}^o \Delta \Gamma^n_{loc} \tag{55}$$

so that, replacing Γ^n by

$$\Gamma^n - \Delta \Gamma^n_{loc} = \tilde{\Gamma}^n \tag{56}$$

which amounts to altering $T_4 \Gamma^n = \Gamma^n_{eff}$ by the local counterterm $\Delta \Gamma^n_{loc}$, one fulfills the Ward identity at order n :

$$\mathcal{S}^o \tilde{\Gamma}^n + R^n = 0 . \tag{57}$$

The first local cohomology of $\mathcal{S}^o$ was computed by B.R.S.[9),33] and shown to boil down to the Adler Bardeen obstruction. Extension of this to the definition of invariant local operators has been performed by S.D. Joglekar and B.W. Lee, H. Kluberg Stern and J.B. Zuber and a general analysis undertaken by J. Dixon[20] and pursued by J. Thierry Mieg[14], L. Baulieu and others, but at the moment there is no complete result on the local cohomology of $\mathcal{S}_o$.

We end up this section proving eq.(54) from the recursion hypothesis[33]:

$$\begin{aligned}
&\mathcal{S}(\Gamma^o) = 0 \\
&\mathcal{S}^o \Gamma^1 = 0 \\
&\mathcal{S}^o \Gamma^2 + \frac{1}{2} \mathcal{S}^1 \Gamma^1 = 0 \\
&\vdots \\
&\mathcal{S}^o \Gamma^{n-1} + \frac{1}{2} [\mathcal{S}^1 \Gamma^{n-2} + \mathcal{S}^2 \Gamma^{n-3} + \ldots + \mathcal{S}^{n-2} \Gamma^n] = 0
\end{aligned} \tag{58}$$

with the definitions

$$\begin{aligned}
\mathcal{S}^{(p)} &= \int \frac{\delta \Gamma^p}{\delta \psi} \frac{\delta}{\delta \Psi} - \frac{\delta \Gamma^p}{\delta \Psi} \frac{\delta}{\delta \psi} \\
&= \sigma(\Gamma^p) \qquad\qquad p \geq 1
\end{aligned} \tag{59}$$

where for any functional F,

$$\sigma(F) = \int \frac{\delta F}{\delta\psi}\frac{\delta}{\delta\Psi} - \frac{\delta F}{\delta\Psi}\frac{\delta}{\delta\psi} \tag{60}$$

and

$$\psi = (a,\phi,\omega)$$
$$\Psi = (A,\Phi,\Omega) \tag{61}$$

One has here a graded symplectic structure[42]:

$$B = \int \delta\psi\Lambda\delta\Psi \tag{62}$$

$\sigma(F)$ is a vector field defined by

$$B(\sigma(F)) = \delta F \tag{63}$$

For two functionals F, G, one has:

$$\sigma(F)G = (-)^{\mathrm{dgF dgG}}\,\sigma(G)F \stackrel{\mathrm{Def}}{=} [F,G]_{\text{graded Poisson bracket}} \tag{64}$$

with

$$\sigma([F,G]_{\mathrm{graded}}) = [\sigma(F),\sigma(G)]_{\mathrm{graded}} \tag{65}$$

Thus the graded Poisson bracket fulfills the graded Jacobi identity. With these notations

$$R^n = \frac{1}{2}\sum_{p-1}^{n-1}[\Gamma^p,\ \Gamma^{n-p}] = \frac{1}{2}\sum_{\substack{(pq),p+q=n\\ p\neq 0,n\ p\leq q}}[\Gamma^p,\Gamma^q] \tag{66}$$

Now $\mathcal{S}^o$ acts as a graded derivation on the graded Poisson algebra:

$$\mathcal{S}^o[F,G] = \mathcal{S}^o\,\sigma(F)G = [\mathcal{S}^o,\sigma(F)]_{\mathrm{graded}}G + (-)^{\mathrm{degF}}\,\sigma(F)\mathcal{S}^oG \tag{67}$$

$$= \sigma(\mathcal{S}^oF)G + (-)^{\mathrm{degF}}\,\sigma(F)\,\mathcal{S}^oG$$

$$= [\mathcal{S}^oF,G] + (-)^{\mathrm{degF}}\,[F,\mathcal{S}^oG], \tag{67}$$

(The main step of this derivation is to check that $[\mathscr{s}^o, \sigma(F)] = \sigma(\mathscr{s}^o F)$).

So,

$$\mathscr{s}^o R^n = \frac{1}{2} \Sigma [\mathscr{s}^o\Gamma^p,\Gamma^q]_+ + [\Gamma^p,\mathscr{s}^o\Gamma^q]$$

$$= -\frac{1}{4} \Sigma [\mathscr{s}^{p_1}\Gamma^{p_2},\Gamma^q] - \frac{1}{4} \Sigma [\Gamma^p,\mathscr{s}^{q_1}\Gamma^{q_2}]$$

$$= -\frac{1}{4} \Sigma [[\Gamma^{p_1},\Gamma^{p_2}],\Gamma^q] - \frac{1}{4} \Sigma [\Gamma^p,[\Gamma^{q_1},\Gamma^{q_2}]]$$

$$= -\frac{1}{2} \Sigma [[\Gamma^p,\Gamma^q],\Gamma^r] \tag{68}$$

where the summations range respectively over:

p,q : $p+q = n$ $p \neq 0,n$;

p_1,p_2,q : $p_1+p_2+q = n$ $q\neq 0$, $p_1\neq 0$, $p_2\neq 0$

p,q_1,q_2 : $p+q_1+q_2 = n$ $p\neq 0$, $q_1\neq 0$, $q_2\neq 0$

p, q, r : $p+q+r = n$ $p\neq 0$, $q\neq 0$, $r\neq 0$.

The final expression vanishes since each triple p,q,r satisfying the correct conditions occurs in three terms which cancel by the Jacobi identity[34]. This concludes the proof which, in the present form, I dedicate to V. Glaser.

As a last remark, let us point out that eq.(43) is also fulfilled by Γ_{eff} and provides in particular a deformation of the classical invariance defined by eq.(40).

III.5 Supergravity theories

We shall limit ourselves to N=1 supergravity, possibly coupled to Yang Mills and chiral matter. In a superspace formalism, the geometrical set up is reasonably clear, namely one has invariance under superdiffeomorphisms, Lorentz supergauge transformations, gauge transformations. The variables are superdifferential forms subject to invariant constraints[35].

However, solving constraints and imposing Wess Zumino - like gauge conditions leads to various systems of component fields and auxiliary fields. The corresponding differential algebras[22)] expressing the invariance of the action have been written down explicitly by K. Stelle and P.C. West in the case of the minimal, auxiliary field system, by L. Baulieu and M. Bellon in the case of the Sohnius-West system. The structure equations look like a fairly opaque amplification of those describing the Wess Zumino gauge[32)].

This renders the solution of the anomaly problem rather difficult to treat thoroughly. In the case of the Sohnius West system the absence of U(1) anomalies has however been carried out by L. Baulieu and M. Bellon[22)].

The question of the cancellation of anomalies in the zero slope limit of string theories has led M.B. Green and J.H. Schwarz[23)] to the discovery of a new mechanism which allows to trivialze anomalies in Γ^{loc}(gauge fields) by going to Γ^{loc}(gauge fields, antisymmetric tensor fields[*)]) similar to the Wess Zumino trick. However, in order to complete this program, it is necessary to show it can be carried out without violating local supersymmetry[24)]. Clearly this requires working within the framework of a well defined differential algebra[40)]. There are several indications that supersymmetry, be it local, is cohomologically trivial as it is in the pure gauge case[30)], 2-dimensional supergravity[36)], (see also ref. 22)). This matter deserves to be pursued.

[*)] present in supergravity theories.

IV. CONCLUSION AND OUTLOOK

In these notes we have attempted to insist on the structural definition of perturbative anomalies as they show up in a number of geometrical contexts arising from and usually extending symmetries to be implemented at the classical level. The mathematical framework has considerably cleared up, but many of the differential algebras which arise deserve further study. Some of these anomalies are connected with the topological anomalies so far investigated (see P. Ginsparg's lectures) through the canonical constructions summarized in the following appendix. Whereas the local structure of perturbative theories is essential here, its connection with the topological properties of various field spaces seem at the moment rather miraculous. The miracle is presumably related to the structure of the local formulae which express the various index theorems. Both the local cohomologies envisaged here and the topological cohomologies which are the objects considered by the index theorems are awaiting some common cohomological denominator[39].

ACKNOWLEDGEMENTS

The author wishes to express collectively his gratitude to all those who communicated their results prior to publication. He apologizes to those who have been only implicitly referred to, in the bibliography. Setting up a complete list of relevant references would have gone beyond the reasonably modest goal of these notes. On the other hand, the author wishes to thank all his collaborators and many colleagues for discussions and correspondence, among which: O. Alvarez, M. Asorey, M.F. Atiyah, M. Bellon, L.D. Faddeev, P.K. Mitter, D. Quillen, A.G. Reiman, K. Sibold, M. Talon, C.M. Viallet, J. Wess, E. Witten. Special thanks are due to L. Alvarez Gaumé, L. Baulieu, L. Bonora, M. Dubois Violette, G. Girardi, R. Grimm, D. Kastler, O. Piguet, I.M. Singer, M. Tonin, B. Zumino.

APPENDIX

COHOMOLOGY OF LIE ALGEBRAS, LIE GROUPS[37)]

Let G be a Lie group, Lie G its Lie algebra, V a representation space for G, and, consequently, Lie G.

C^n(Lie G, V), the vector space of n cochains of Lie G with values in V is the vector space of multilinear alternate n forms on Lie G, with values in V. Evaluated at $X_o,\ldots,X_{n-1} \in$ Lie G such an n-cochain $\vec{f}_n$ is an element of V : $\vec{f}_n(X_o,\ldots,X_{n-1})$.

On $C^*(\text{Lie } G, V) = \bigoplus_{n=o}^{\infty} C^n(\text{Lie } G, V)$, one defines the coboundary operator δ: $C^n(\text{Lie } G, V) \xrightarrow{\delta} C^{n+1}(\text{Lie } G, V)$, such that $\delta^2 = 0$, by

$$(\delta\vec{f}_n)(X_o\ldots X_n) = \sum_{i=o}^{n} (-)^i\, t(X_i)\, \vec{f}\,(X_o\ldots\hat{X}_i\ldots X_n)$$

$$+ \sum_{o\le i<j\le n} (-)^{i+j}\, \vec{f}\,([X_i,X_j],\, X_o\ldots\hat{X}_i\ldots\hat{X}_j\ldots X_n) \tag{A1}$$

where $t(X_i)$ denotes the representation of Lie G in V.
As is usual one then defines

$$Z^n(\text{Lie } G, V) = \{\, f_n \,/\, \delta f_n = 0 \,\}, \tag{A2}$$

$$B^n(\text{Lie } G, V) = \{\, f_n \,/\, f_n = \delta f_{n-1}, \quad f_{n-1} \in C^{n-1} \,\} \tag{A3}$$

and, since obviously $B^n(\text{Lie } G, V) \subset Z^n(\text{Lie } G, V)$,

$$H^n(\text{Lie } G, V) = Z^n(\text{Lie } G, V) \,/\, B^n(\text{Lie } G, V)\ . \tag{A4}$$

The elements of Z^n are called cocycles, those of B^n are called coboundaries, and those of H^n are called cohomology classes. Sometimes

δ is appended as a subscript or after V, within parentheses e.g. $Z^n_\delta(\text{Lie } G, V)$, or $Z^n(\text{Lie } G, V; \delta)$.

The easiest way to check[+] that $\delta^2 = 0$ is to observe that the antisymmetry of $\vec{f}$ in its n arguments allows to substitute them through the Maurer Cartan form $\omega = g^{-1}dg$ thus obtaining a left invariant differential form on G with values in V : $f_n(\omega,\ldots\omega)$. Conversely, $f_n(X_o\ldots X_n)$ is recovered by evaluating $f_n(\omega,\ldots\omega)$ at the set of left invariant vector fields $X^*_o\ldots X^*_n$ and substituting X^*_i through the corresponding element X_i of Lie G. One then finds

$$(\delta f_n)(\omega,\ldots\omega) = [d+t(\omega)]f_n(\omega,\ldots\omega) \tag{A5}$$

and

$$[d+t(\omega)]^2 = 0 \tag{A6}$$

follows from the structural equation

$$d\omega + \frac{1}{2}[\omega,\omega] = 0 \tag{A7}$$

The substitution $X_i \to \omega$ transforms $C^n(\text{Lie } G, V)$ into $\Omega^n_{inv.}(G,V)$, the space of invariant differential forms on G, with values in V.

$C^*(\text{Lie } G, V)$ is then isomorphic to $\Omega^*_{inv.}(G,V) = \oplus\,\Omega^n_{inv.}(G,V)$.

If V, besides being a vector space has a graded commutative algebra structure, $\Omega^*_{inv.}(G,V)$ becomes also a graded commutative algebra $\Omega^*_{inv.}(G) \underset{\text{graded}}{\otimes} V$, and so does $C^*(\text{Lie } G, V) \simeq C^*(\text{Lie } G) \otimes V$. $C^*(\text{Lie } G)$ is nothing else than the exterior algebra of $\widetilde{\text{Lie } G}$, the dual of Lie G (as a vector space). As an algebra, it is generated in dimension 1 - the product being the exterior product. In terms of a basis e_α of Lie G and a dual basis e^β of $\widetilde{\text{Lie } G}$ $(e^\beta(e_\alpha) = \delta^\beta_\alpha)$, the structure equations read[15)]

[+] see however the remark after equation (A12).

$$\delta e = -\frac{1}{2} [e,e] \tag{A8}$$

where e is the "Lie algebra valued generator of $\widetilde{\text{Lie G}}$"

$$e = \Sigma e^{\alpha} e_{\alpha} . \tag{A9}$$

e is named by physicists the geometrical Faddeev Popov ghost of Lie G. It is a purely algebraic concept. Note that the covariant differential $d+t(\omega)$ can be transformed into the ordinary differential d by the transformation

$$f \to \tilde{f} = g.f \tag{A10}$$

where g.f. denotes the action of g in V, so that

$$d \tilde{f} = \widetilde{(d+t(\omega))f} \tag{A11}$$

Finally, the complete structure equations of C^{*}(Lie G, V) are given by eq.(A8) together with

$$\delta u = \theta(e) u \quad \text{resp.} \quad \tilde{\delta} \tilde{f} = 0 \tag{A12}$$

where u resp. $\tilde{u}$ denotes a set of generators of V and θ the corresponding representation of Lie G.

Thus every $f \in C^{*}$(Lie G, V) can be represented as a polynomial in e and u, and δf is computed from the equations (A9), (A12).

The differential form version of cohomology is denoted

$H^{*}_{inv.}(G,V) \simeq H^{*}_{DR,inv.}(G) \otimes V$

where DR is an abbreviation for de Rham.

An isomorphic representation[37a)38)] is given by integrating n-forms over n-simplexes in G: Let T_n be a simplex in G, with vertices $g_o \ldots g_{n-1}$, and let

$$\tilde{f}_n(T_n) = \int_{T_n} f_n(\omega, \ldots \omega) \tag{A13}$$

Then, by Stokes theorem

$$(\delta\tilde{f}_n)(T_{n+1}) = \tilde{f}_n(\partial T_{n+1})$$

$$= \sum_{i=o}^{n} (-)^i f_n(\partial T^i_{n+1}) \qquad \text{(A14)}$$

where ∂T^i_{n+1} is the oriented ith n-face of T_{n+1}. In terms of the corresponding vertices

$$(\tilde{\delta}\tilde{f}_n)(T_{n+1}(g_o \ldots g_{n+1})) = \sum_{i=o}^{n} (-)^i \tilde{f}_n(T_n(g_o \ldots \hat{g}_i \ldots g_n)) \qquad \text{(A15)}$$

Note that under a smooth deformation of T_n with fixed vertices, a cocycle $\tilde{f}_n(T_n)$ changes by a coboundary

$$\tilde{f}_n(T_n) - \tilde{f}_n(T'_n) = \int_{(\tau)\times T_n(\tau)} d_\tau\, f_n(\omega_\tau \ldots \omega_\tau) \qquad \text{(A16)}$$

where $\omega_\tau = g_\tau^{-1} d\, g_\tau$ where g_τ interpolates between T_n and T'_n, $0 \leq \tau \leq 1$.

Using the cocycle condition in the form

$$(d_t + d_\tau)\tilde{f}_n(g_\tau^{-1} d_\tau g_\tau + g_t^{-1} d_t g_t, \ldots) = 0 \qquad \text{(A17)}$$

yields

$$\tilde{f}_n(T_n) - \tilde{f}_n(T'_n) = \sum_{i=o}^{n} (-)^i \tilde{\phi}_{n-1}\,(T_n(g_o, \ldots \hat{g}_i \ldots g_n)) \qquad \text{(A18)}$$

with

$$\tilde{\phi}_{n-1}(T_n(g_o, \ldots \hat{g}_i \ldots g_n)) = - \int_{T_n(\tau)(g_o \ldots \hat{g}_i \ldots g_{n-1})\times(\tau)} n\, \tilde{f}_n\,(g_\tau^{-1} d_\tau g_\tau, g_t^{-1} d_t g_t \ldots g_t^{-1} d_t g_t) \qquad \text{(A19)}$$

The corresponding version of cohomology is denoted $H^*_{top}(G,V)$, and the corresponding cocycles are usually referred to as Wess-Zumino cocycles and sometimes named (not everywhere defined, possibly non uniform) discrete group cocycles. This is because "discrete" group cohomology is defined via V valued cochains $C^n_{discr.}(G,V)$ which are functions of $n-1$ group elements, with the coboundary operation

$$(\delta f_n)(g_o \cdots g_{n+1}) = \sum_{i=o}^{n+1} (-)^i f_n(g_o \cdots \hat{g}_i \cdots g_{n+1}). \tag{A20}$$

These so-called homogeneous cochains[+] are furthermore supposed to be invariant in the sense that

$$f_n(g_o \cdots g_n) = \gamma \; f_n(\gamma^{-1} g_o, \ldots \gamma^{-1} g_n) \; . \tag{A21}$$

Invariance implies the possible non triviality of cohomology defined by eq.(A20) (which is otherwise trivial).

If it so happens that every $n+1$-tuple of points in G are the vertices of some simplex $T_n(g_o \cdots g_n)$, unique up to homotopy, then one is allowed to write

$$\tilde{f}_n(T_n(g_o \cdots g_n)) = \tilde{f}_{n\ discr.}(g_o \cdots g_n) \tag{A22}$$

and one may identify the Wess Zumino elements of $H^n_{top,inv.}(G,V)$ with those of $H^n_{discr.}(G,V)$, since the Wess Zumino cocycles do fulfill the invariance property. It is however safer to consider Wess Zumino cocycles as elements of $H^n_{top}(G,V)$.

Although no general analysis of the non uniformity of the Wess Zumino cocycles is known to the author, it can be reduced via the exponential map[13)]:

[+] We shall not go here into the description of discrete group cohomology via inhomogeneous cochains, which, also more widespread in the current literature, is less suitable for the present purpose.

$$a \in \tilde{V}, \quad f_n \in Z^n_{top,inv.}(G,V)$$

$$\rightarrow \quad \exp_a f_n \;=\; \exp i\, a(f_n) \tag{A23}$$

whenever, upon proper normalization the change in homotopy class of T_n results into the addition of an integer. Of course, then, cohomology definitions have to be written in multiplicative, rather than additive, notation.

It is finally worthwhile describing the procedure which allows to convert a discrete group cocycle into a de Rham cocycle[37a]:

Let $f_n(g_o \ldots g_n)$ be a smooth discrete group cocycle. Then construct

$$\omega_n(f_n) \;=\; \Delta^* d_{g_1} \ldots d_{g_n} f_n(g_o \ldots g_n) \tag{A24}$$

where Δ^* is the pull back of the diagonal application $G \rightarrow G \times \ldots G$: $g \rightarrow (g, \ldots g)$.

It is a simple matter to show that

$$\omega_{n+1}(\delta f_n) \;=\; d\, \omega_n(f) \tag{A25}$$

In particular, applying this construction to a Wess Zumino cocycle

$$f_n(g_o \ldots g_n) \;=\; \int_{T_n(g_o \ldots g_n)} \tilde{f}_n(\omega, \ldots \omega) \tag{A26}$$

yields

$$\Delta^*(d_{g_1} \ldots d_{g_n} \int_{T_n(g_o \ldots g_n)} \tilde{f}(\omega, \ldots \omega) \;=\; \tilde{f}(\omega, \ldots \omega) \text{ mod. exact form.} \tag{A27}$$

REFERENCES AND FOOTNOTES

1) A thorough review of and bibliography on the early period can for instance be found in:
R. Jackiw: "Topological investigations of quantized gauge theories" in: Relativity, Groups, Topology II, Les Houches lectures 1983, B.S. de Witt, R. Stora eds., North Holland Publ. Amsterdam 1984.

2) L.D. Faddeev, private communication.

3) R. Jackiw, R. Rajaraman, P.L. 154B, 305, 1985.

4) A good example of the complexity involved is given by:
L. Alvarez Gaumé, E. Witten, N.P.B. 234, 269-330, 1984.

5) See, for instance O. Alvarez, I.M. Singer, B.Zumino, C.M.P. 94, 409, 1984.
M.F. Atiyah, I.M. Singer, P.N.A.S. 81, 2597, 1984.

6) Typically, the family's index treatment is related to the topology of the field configuration space whereas the algebraic treatment based on locality is not.

7) P. Ginsparg, Lectures in this volume.

8) a) J. Mañes, R. Stora, B. Zumino, "Algebraic study of chiral anomalies", submitted to C.M.P.;
b) L. Alvarez Gaumé, P. Ginsparg, Ann. Phys. 161, 423, 1985.

9) C. Becchi, A. Rouet, R. Stora: "Renormalizable theories with symmetry breaking" in: Field theory quantization and statistical mechanics, E. Tirapegui ed., Reidel 1981.

10) R. Stora: "Continuum gauge theories" in: New developments in quantum field theory and statistical mechanics, 1976 Cargèse Lectures, M. Levy, P. Mitter eds, Plenum New York 1977.

11) M. Dubois Violette, M. Talon, C.M. Viallet, "BRS cohomologies", to appear in C.M.P.

12) E. Witten, "Global aspects of current algebra", N.P.B. 223, 422, 1983.

13) G. Falqui, C. Reina, "BRS cohomology and topological anomalies", Annales IHP to be published.

14) J. Thierry-Mieg, P.L. 147B, 430, 1984.

15) I am indebted to P. Van Nieuwenhuizen for pointing out and discussing the article by D. Sullivan, IHES Pub. n°47, p.269, 1977, whose reading led in particular to the algebraic interpretation of the Faddeev Popov ghost which can be found in the appendix.

16) L. Baulieu, J. Thierry-Mieg, P.L. 145B, 53, 1984;
F. Langouche, T. Schücker, R. Stora, P.L. 145B, 342, 1984;
L. Bonora, P. Pasti, M. Tonin, N.P.B. 1985 to appear.

17) L. Bonora, P. Pasti, M. Tonin, P.L. 155B, 341, 1985 and to appear in NPB;
G. Girardi, R. Grimm, R. Stora, P.L. 156B, 203, 1985);
J. Wess, B. Zumino, in preparation.

18) H. Itoyama, V.P. Nair, H.C. Ren, IAS Princeton preprint 1985.

19) G. Moore, P. Nelson, P.R.L. 53, 1519, 1984;
G. Moore, P. Nelson, HUTP-84/A076;
A. Manohar, G. Moore, P. Nelson, HUTP-84/A088;
L. Alvarez Gaumé, P. Ginsparg, HUTP 85/A015;
P. di Vecchia, S. Ferrara, L. Girardello, P.L. 151B, 199, 1985;
E. Cohen, C. Gomez, N.P.B. 254, 235, 1985;
J. Bagger, D. Nemechansky, S. Yankielowicz in Anomalies, Geometry, Topology, Argonne-Chicago 1985.

20) H. Kluberg Stern, J.B. Zuber, P.R.D. 12, 467, 1975; P.R.D. 12, 3159, 1975;
S.D. Joglekar, B.W. Lee, Ann. Phys. 97, 160, 1976;
J. Dixon, 1976-1978, unpublished.

21) L.D. Faddeev, S.L. Shatashvili, TMФ 60, 206, 1984.

22) K.S. Stelle, P.C. West, N.P.B. 140, 285, 1978;
L. Baulieu, M. Bellon, to be published.

23) M.B. Green, J.H. Schwarz, P.L. 149B, 117, 1984.

24) S. Cecotti, S. Ferrara, L. Girardello, CERN.TH.4253/85.

25) J. Thierry-Mieg, unpublished, 1979;

L. Baulieu, J. Thierry-Mieg, N.P.B. 228, 259, 1983;

Although both BRS and $\overline{\text{BRS}}$ are used in these papers, BRS alone is sufficient if one does not want to restrict oneself to linear gauge functions.

26) E.S. Fradkin, G.A. Vilkovisky, P.L. 55B, 224, 1975;

I.A. Batalin, G.A. Vilkovisky, P.L. 69B, 309, 1977.

27) W.A. Bardeen, B. Zumino, N.P.B. 244, 421, 1984.

28) L. Bonora, P. Pasti, M. Tonin, P.L.B. to appear.

29) S. Ferrara, L. Girardello, O. Piguet, R. Stora, P.L. 157B, 179, 1985.

30) O. Piguet, K. Sibold, N.P.B. 247, 484, 1984.

31) O. Piguet, M. Schweda, K. Sibold, N.P.B. 174, 183, 1980.

32) B. Zumino in Symposium on Anomalies, Geometry and Topology, Argonne-Chicago, 1985.

33) a) G. Bandelloni, C. Becchi, A. Blasi, R. Collina, Ann. IHP XXVIII, 225-254 and 255-285, 1978.

b) O. Piguet, A. Rouet, Physics Reports 76, 1-77, 1981.

34) cf. Eq. 6.16 of Ref. 33b).

35) I am indebted to R. Grimm for stimulating discussions on this subject.

36) Y. Tanii, TIT/HEP 85;

P. Howe, P. West, P.L. 156B, 335, 1985.

37) a) A. Guichardet, "Cohomologie des groupes topologiques et des algèbres de Lie", CEDIC-NATHAN Paris 1980.

b) S. Mac Lane, "Homology", Springer 1975.

c) H. Cartan, S. Eilenberg, "Homological algebra", Princeton University Press, 1956.

d) N. Jacobson, "Lie Algebras", Interscience, 1962.

38) A.G. Reiman, M.A. Semenov-Tjan-Shansky, L.D. Faddeev, in Journal of functional analysis and its applications, 1984-1985;

L.D. Faddeev, P.L. 145B, 81, 1984;

B. Zumino, N.P.B. 253, 477, 1985;

and references found in 8a).

39) I.M. Singer, in Colloque Elie Cartan, Lyon, June 25-29, 1984, to appear in Astérisque, and references therein.

40) G. Girardi and R. Grimm have just informed me they have constructed a consistent superspace geometry [namely, a correct system of constraints] which allows to incorporate (in a manifestly supersymmetric way) Chern Simons terms into the curvature of the super two form appearing in the Sohnius West multiplet.

41) When an anomaly is present, one can prove a modified Ward identity of the form

$$\int \frac{\delta\Gamma}{\delta a}\frac{\delta\Gamma}{\delta A} + \frac{\delta\Gamma}{\delta\varphi}\frac{\delta\Gamma}{\delta\Phi} + \frac{\delta\Gamma}{\delta\omega}\frac{\delta\Gamma}{\delta\Omega} - \frac{\delta\Gamma}{\delta\bar{\omega}} b - \frac{\partial}{\partial\alpha}\Gamma = 0$$

where α is a constant odd scalar with dimension -1, $\phi\pi$ charge -1. (C. Becchi, private communication).

42) C. Becchi, unpublished.