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Constructive Renormalization Group

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CONSTRUCTIVE RENORMALIZATION GROUP

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The present lecture describes the attemps by A. Kupiainen and myself, paralleled by other approaches, to bring under control systems of equilibrium classical statistical mechanics exhibiting strong correlations among infinite number of degrees of freedom.

As a typical example, we shall consider a lattice unbounded spin system (ϕ_x) , $x \in \mathbb{Z}^d$ (random field) described by the Gibbs state given formally (or rigorously in finite volume) by

$$(1) \qquad \frac{1}{Z}\, e^{-V(\phi)} d\mu_{(-\Delta)^{-1}}(\phi)$$

where $d\mu_{(-\Delta)^{-1}}(\phi) = \frac{1}{Z'}\, e^{-\frac{1}{2}\sum_{\langle xy\rangle}(\phi_x-\phi_y)^2} \prod_x d\phi_x$ is the gaussian measure with covariance $(-\Delta)^{-1}$ and $V(\phi)$ is an interaction e.g. $V(\phi) = \sum_x v(\phi_x)$ or $V(\phi) = \sum_{\langle xy\rangle} v(\phi_x-\phi_y)$. The strong correlations of ϕ_x's are signaled by the power-law decay of the correlation function

$$(2) \qquad \langle\phi_x;\phi_y\rangle^c = \langle\phi_x\phi_y\rangle - \langle\phi_x\rangle\langle\phi_y\rangle \underset{|x-y|\to\infty}{\sim} \frac{1}{|x-y|^\alpha}$$

(critical regime) as opposed to the case of massive regime where

(3) $$\langle\phi_x;\phi_y\rangle^c \underset{|x-y|\to\infty}{\sim} e^{-|x-y|/\xi}$$

(ξ is called the correlation length).

For the critical systems not much has been known on the rigorous bases. It has to be contrasted with the case of systems with small ξ which by various expansion techniques are under good rigorous control. There exists however a powerful heuristic tool just designed for the study of systems with no length scale of the type of ξ : this is the renormalization group of Wilson, Kadanoff and others. The idea is to introduce by hand a sequence of length scales L^n reducing the system to an iteration of problems with correlations decaying exponentially on distance L . These problems are subsequently studied by perturbation expansion or numerically.

Our approach may be rephrased as an attempt at a non-perturbative control of the renormalization group.

Upon the rescaling the lattice spacing to 1, the problem of the existence of continuum limits of lattice approximations to the euclidean quantum field theory models becomes the problem of control of the critical regime and of the approach to it for a spin system on $\mathbb{Z}^d$. From this point of view our ultimate goal is the construction of renormalizable asymptotically free models of quantum fields, possibly also non-asymptotically free ones and the understanding of their relation to the perturbation expansion.

Finally from the probabilistic point of view, we aim at limit theorems for sums of strongly correlated random variables ϕ_x .

The idea of the renormalization group is as follows : starting with (1), we notice that

(4) $$(-\Delta)^{-1}_{xy} \sim \frac{1}{|x-y|^{d-2}} \quad \text{for} \quad |x-y| \to \infty$$

so that we can decompose

(5) $$(-\Delta)^{-1}_{xy} = L^{d-2}(-\Delta)^{-1}_{x/L\ y/L} + \Gamma_{xy}$$

where Γ has better decay properties. Changing slightly $(-\Delta)^{-1}$ on the right to some $(-\Delta)^{-1}_1$ we may attain Γ exponentially decaying on distance L and positive definite. Then with (5) we may associate the <u>decomposition of the field</u>

(6) $$\phi_x = L^{\frac{d-2}{2}}\ \psi^1_{x/L} + Z_x$$

factorizing $d\mu_{(-\Delta)^{-1}}(\phi)$:

(7) $$d\mu_{(-\Delta)^{-1}}(\phi) = d\mu_{(-\Delta)^{-1}_1}(\psi^1) \times d\mu_\Gamma(Z) .$$

The decomposition (6) is into <u>long distance component</u> ψ^1 and <u>short distance fluctuation</u> Z . Now the idea is to integrate out the short distance fluctuations and study the effective theory for the long distance component which is

(8) $$\frac{1}{Z_1}\ e^{-V_1(\psi^1)}\ d\mu_{(-\Delta)^{-1}_1}(\psi^1)$$

with the effective interaction V_1 given by

$$(9) \qquad e^{-V^1(\psi^1)} = \text{const} \int e^{-V(L^{\frac{2-d}{2}} \psi^1_{1/L} + Z)} d\mu_\Gamma(Z)$$

(coarse graining). If Z is taken to have zero averages over $L\times\ldots\times L$ blocks in $\mathbb{Z}^d$, then the ψ^1 theory describes the distribution of the averages of ϕ over these blocks (block spins), whence the ultimate relation to the limit theorems for sums of ϕ_x's .

Now, we may iterate the transformation $V \to V_1$, after n steps arriving at

$$(10) \qquad \frac{1}{Z_n} e^{-V_n(\psi^n)} d\mu_{(-\Delta)^{-1}}(\psi^n)$$

with

$$(11) \qquad e^{-V_n(\psi^n)} = \text{const} \int e^{-V_{n-1}(L^{\frac{2-d}{2}} \psi^n_{\cdot/L} + Z^{n-1})} d\mu_{\Gamma_{n-1}}(Z^{n-1}) \quad .$$

So the study of the state (1) has been reduced to the study of the general step (11). The hope is that the degrees of freedom Z^{n-1} involved in the functional integration of (11) have fast decay of correlations and the standard expansion methods with due modifications taking into account the presence of the "external field" ψ^n apply.

The heuristic renormalization group studies (9) or (10) by perturbation expansion : e.g. for $V(\phi) = \lambda \sum_{\langle xy \rangle} (\phi_x - \phi_y)^4 \equiv \lambda \sum_{\mu,x} (\nabla_\mu \phi_x)^4$,

$$(12) \qquad V_1(\psi^1) = \int V(L^{\frac{2-d}{d}} \psi^1_{\cdot/L} + Z) d\mu_\Gamma(Z) + O(V^2)$$

$$= \lambda \underbrace{L^{\frac{2-d}{2} \cdot 4 - 4 + d}}_{L^{-d}} \sum_\mu \int dx (\nabla_\mu \psi^1_x)^4 + \text{lower order terms in } \nabla\psi^1 + O(V^2) \ ,$$

so that the net effect is that λ is driven down and we may expect that $V_n(\nabla\psi) \underset{n\to\infty}{\to} O(\lambda)\sum_\mu \int dx(\nabla_\mu\psi_x)^2$ that is the convergence to a gaussian fixed point.

For $V(\phi) = \lambda \sum_x \phi_x^4 + \frac{1}{2}\mu^2 \sum_x \phi_x^2$, $\lambda \to \lambda L^{4-d}$ and we may also expect convergence to a gaussian fixed point for $d > 4$ (at the critical value of μ^2) . Second order computation in λ suggests that this still holds for $d = 4$, while for $d < 4$ the convergence to a non-gaussian fixed point should take place (the second order stops the initial growth of small λ coming from the linear term). It is not difficult to translate the information about the behaviour of V_n's into the properties of the correlation functions : in the case of the convergence to the gaussian fixed point, the correlations on long distances approach those of the corresponding gaussian (free) random field (infrared asymptotic freedom). For a non-gaussian fixed point the asymptotics of the correlations is not free.

The basic question is :

Can we trust the perturbative argument ?

In an attempt to answer this question, let us concentrate for simplicity on the easiest case of $V(\phi) = \lambda \sum_{\mu,x} (\nabla_\mu\phi_x)^4$. It is clear that in (9) the expansion in powers of V makes sense if V is small. This is the case if the argument of V is small, e.g.

(13) $$|\nabla\phi| < O(|\log \lambda|)$$

and λ is small. This condition may be violated if $\nabla\psi^1$ or Z is large. Take first $\nabla\psi^1$ small :

(14) $$|\nabla\psi^1| < O(|\log \lambda|) .$$

The perturbative expansion gives formally

(15) $$V^1(\nabla\psi^1) = \sum_n \lambda^n c_n(\nabla\psi^1) ,$$

but, as one can easily convince oneself, the right hand side diverges as $c_n \sim O(n!)$ for large n : the typical behaviour of large orders of $\lambda\phi^4$ theory. The reason is that even with $\nabla\psi^1 = 0$, $\lambda \sum_{\mu,x} (\nabla_\mu Z_x)^4$ can still be large in ∇Z . So let us introduce into (9) a characteristic function $\chi(|\nabla Z| < O(|\log \lambda|))$. Then we obtain

(16) $$V^1_\chi(\nabla\psi^1) = \sum_n \lambda^n c_{n\chi}(\nabla\psi^1)$$

where the right hand side converges for small λ . The following are the crutial observations :

1.

(17) $$|V^1(\nabla\psi^1) - V^1_\chi(\nabla\psi^1)| \le e^{-O(|\log \lambda|^2)}$$

due to small $d\mu_\Gamma(Z)$ probability of large ∇Z . This is a very small non-perturbative correction.

2.

(18) $$V^1_\chi(\nabla\psi^1) = \sum_m \sum_{\substack{x_1 \ldots x_m \\ \mu_1 \ldots \mu_m}} d_{m\chi}^{\mu_1 x_1 \ldots \mu_m x_m}(\lambda) \nabla_{\mu_1}\psi^1_{x_1} \ldots \nabla_{\mu_m}\psi^1_{x_m}$$

(which is a reshuffling of (16)) converges so that $V^1_\chi(\nabla\psi^1)$ is analytic in small $\nabla\psi^1$ region as well as the whole $V^1(\nabla\psi^1)$ so that also the right hand side of

$$V^1(\nabla\psi^1)-V^1_\chi(\nabla\psi^1) = \sum_m \sum_{\substack{x_1\ldots x_m \\ \mu_1\ldots\mu_m}} d^{\mu_1 x_1\ldots\mu_m x_m}_{m\chi^\perp}(\lambda)\nabla_{\mu_1}\psi^1_{x_1}\ldots\nabla_{\mu_m}\psi^1_{x_m}$$

converges with

$$(20) \qquad |d_{m\chi^\perp}(\lambda)| < e^{-O(|\log\lambda|^2)}O(|\log\lambda|)^{-m}$$

by (17) and the Cauchy estimate.

This way we expect $V^1(\nabla\psi^1)$ to be an analytic functional of $\nabla\psi^1$ for $|\nabla\psi^1| < O(|\log\lambda|)$ (but not of λ at $\lambda = 0$) with the Taylor coefficients given by the perturbation theory with small non-perturbative corrections (carrying the whole non-analycity in λ). In particular $V^1(\nabla\psi^n)$ is essentially $\lambda_1 \sum_\mu \int dx(\nabla_\mu\psi^1_x)^4+O(\lambda)\sum_\mu \int dx(\nabla_\mu\psi^1_x)^2$ with λ_1 (the effective quartic coupling) $\approx \lambda L^{-d}$

In order to iterate this analysis, to show the analyticity of $V_n(\nabla\psi^n)$ in the small field region $|\nabla\psi^n| < O(|\log\lambda_n|)$, we would need e.g. the analyticity of $V_{n-1}(\nabla\psi^{n-1})$ in the polystrip

$$(21) \qquad |\mathrm{Im}\ \nabla\psi^{n-1}| < O(|\log\lambda_{n-1}|)$$

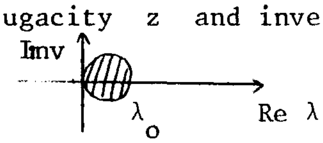
and to exhibit the small probability of ∇Z^{n-1}, the stability bound

$$(22) \qquad |e^{-V^{n-1}(\nabla\psi^{n-1})}| \le e^{\kappa\sum_\mu \int dx|\nabla_\mu\psi^{n-1}|^2}$$

with small κ. Both properties will carry over to the next step.

All this analysis has to be done locally in regions of space (depending on which fields are small-large in a given region) and combined with a cluster expansion exhibiting the approximately local character of the functional integral of (11) due to the exponential decay of Γ_{n-1} as well as of the Taylor coefficients $d_{n-1}^{\mu_1 x_1 \ldots \mu_m x_m}$ of V^{n-1} .

The result is the convergence of the $V^n(\nabla\psi)$ in the small field region $|\nabla\psi| < O(|\log \lambda_u|)$ to $O(\lambda) \sum_{\mu} \int dx(\nabla_\mu \psi_x)^2$ as well as a geometric convergence of the effective quartic couplings λ_n to zero together with the expansion of the small field region to infinity. Using this analysis one can prove :

Theorem. Let $V(\phi) = \lambda \sum_{\mu,x} (\nabla_\mu \phi_x)^4$ or $V(\phi) = - \sum_{\mu,x} 2z \cos \beta^{1/2} \nabla_\mu \phi_x$ (the latter case corresponds to the dipole gas with fugacity z and inverse temperature β in the sine-Gordon picture). Let $\lambda \in$ 

, $|z| < z_o$ with λ_o, z_o small, $d \geq 3$. Then

1. The effective potentials converge to $\frac{1}{2}(\varepsilon_\infty - 1) \sum_{\mu} \int dx(\nabla_\mu \psi_x)^2$ (uniformly in the volume).

2. The thermodynamical limit of the correlations exists.

3. $\lim_{|x-y|\to\infty} \dfrac{\langle \phi_x \phi_y \rangle}{(-\Delta)^{-1}_{xy}} = \varepsilon_\infty^{-1}$ exists (ε_∞ is the dielectric constant for the dipole gas).

4. The scaling limit $\lim_{\theta\to\infty} \theta^{\frac{d-2}{2} r} \varepsilon_\infty^{r/2} \langle \phi_{\theta x_1} \cdots \phi_{\theta x_r} \rangle$ for $x_i \neq x_j$ exists and is the free continuum massless field.

5. The perturbation theory in λ is Borel summable (with Tirozzi in preparation) and in z (the Mayer expansion) is convergent.

The long distance behaviour of $\lambda(\nabla\phi)^4$ model was also controlled by Magnen and Sénéor with the use of the phase space cell expansion, a method whose idea partly overlaps with the renormalization group approach and, for certain correlations, by Fontaine by use of correlation inequalities.

Work is presently going in the following directions :

a) φ^4_d , $d \geq 4$ - Logarithmic corrections to scaling in $d = 4$,

b) $\varphi^4_{2,3}$, using ε and $\frac{1}{N}$ perturbative arguments to get perturbative information about the non-gaussian fixed point,

c) XY model in $d = 2$ and U(1) lattice gauge theory in the Kosterlitz-Thouless phase.