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ON THE STATISTICAL MECHANICS OF SURFACES§

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0. Introduction.

The purpose of these notes is to report some recent results and speculations concerning the statistical mechanics of surfaces or interfaces and to try to convey an impression of the beauty of and interest in a mathematical theory of random surfaces.

Random surfaces and their statistical mechanics appear in many different physical contexts among which one might mention :

(i) Crystal growth and the statistical mechanics of crystal surfaces in a solution.

(ii) Interfaces between different phases of a physical system; (e.g. Bloch walls, or the liquid-vapor interface in water, etc.)

(iii) Gauge theories; (the high temperature expansion expresses a lattice gauge theory as a theory of random surfaces; the low temperature expansion expresses a four-dimensional lattice gauge theory with discrete gauge group as a theory of two-dimensional vortex sheets.)

(iv) Dual resonance models; (string theory in its Euclidean formulation can be formulated as a theory of random surfaces. It may be viewed as a generalization of Brownian motion, from random paths to random surfaces.)

Needless to say that random surfaces appear in other problems of condensed

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matter physics, in geophysics (surfaces of mountains),... .

In the following, we briefly review some rigorous results concerning random surfaces and interfaces. We discuss :

1. The interface in the three-dimensional Ising- and rotator model [1].
2. The solid-on-solid model [2].
3. Self-avoiding random surfaces and string theories [3].
4. Lattice gauge theories [4].

We refer to the literature quoted here and in the following for information concerning the physical situations described by these models, detailed statements of results and proofs.

We have profitted from collaboration and/or discussions with M. Aizenman, J. Bricmont, J.-L. Lebowitz and E. Seiler.

1. The interface in the Ising- and rotator model.

We start by recalling the definition of the Ising- and the rotator (classical XY-) model on a simple, (hyper) cubic lattice $\mathbb{Z}^d$, $d \geq 3$: With each site $j \in \mathbb{Z}^d$ we associate a spin S_j, and

1) $S_j = \pm 1$ in the Ising model;

2) $S_j \in S^1$, in the rotator model, i.e. S_j can be parametrized by an angle $\theta_j \in [0,2\pi)$.

We use the convention

$$S_j = \pm \Leftrightarrow \begin{cases} S_j = \pm 1 \text{ , in the Ising model} \\ \theta_j = 0,\pi \text{ , in the rotator model.} \end{cases}$$

Let Λ be some finite sublattice of $\mathbb{Z}^d$, e.g.

$$\Lambda = \Lambda_{L,T} = \{j \in \mathbb{Z}^d : -T \leq j_1 \leq T, -L \leq j_\alpha \leq L \text{ , } \alpha = 2,\dots,d\} \quad .$$

The energy of a configuration $S_\Lambda = \{S_j\}_{j\in\Lambda}$ of spins in Λ, given a fixed configuration $S_{\Lambda^c} = \{S_j\}_{j\in\Lambda^c}$ of spins in the complement, Λ^c, of Λ, is given by the Hamilton function

$$H_\Lambda = - \sum_{(ij)\subset\Lambda} S_i \cdot S_j + W(S_\Lambda | S_{\Lambda^c}) \ , \tag{1}$$

where W is a boundary term defined by

$$W(S_\Lambda | S_{\Lambda^c}) = - \sum_{\substack{(ij) \\ i\in\Lambda, j\in\Lambda^c}} S_i \cdot S_j \ , \tag{2}$$

and (ij) indicates that i and j are nearest neighbors. The equilibrium state for a spin system in Λ with Hamilton function H_Λ given by (1), (2) and some fixed b.c. S_{Λ^c}, at inverse temperature β, is defined to be

$$d\mu_\beta(S_\Lambda | S_{\Lambda^c}) = Z_{\beta,\Lambda}(S_{\Lambda^c})^{-1} e^{-\beta H_\Lambda(S_\Lambda | S_{\Lambda^c})} \prod_{j\in\Lambda} dS_j \ , \tag{3}$$

where dS is the counting measure on $\{-1,1\}$, in the Ising model, and the Lebesgue measure on S^1, in the rotator model. Furthermore

$$Z_{\beta,\Lambda}(S_{\Lambda^c}) = \int e^{-\beta H_\Lambda(S_\Lambda | S_{\Lambda^c})} \prod_{j\in\Lambda} dS_j$$

is the partition function. We shall impose the following kinds of boundary conditions:

(+b.c.) $S_j = +$, for all $j \in \Lambda^c$

(±b.c.) $S_j = +$, for all $j \in \Lambda^c$ with $j_1 \geq 0$

$S_j = -$, for all $j \in \Lambda^c$ with $j_1 < 0$

(step b.c.) $S_j = +$ if $j_1 > 0$, or $j_1 = 0$ and $j_2 \geq 0$

$S_j = -$, otherwise.

Let $A(S)$ be some continuous function depending only on finitely many S_j. The equilibrium expection of A in the thermodynamic limit, for X b.c. ($X = +, \pm$, step), is given by

$$\langle A \rangle_{\beta,X} = \lim_{L,T\to\infty} \int A(S) d\mu_\beta(S_{\Lambda_{L,T}} | X) \ . \tag{4}$$

The limit is known to exist for + b.c., but some limit can always be obtained by passing to subsequences. We define $\langle(\cdot)\rangle_{\beta,-}$ by

$$\langle A(S)\rangle_{\beta,-} = \langle A(-S)\rangle_{\beta,+} \, .$$

The spontaneous magnetization, $M(\beta)$, is given by

$$M(\beta) = \langle S_j\rangle_{\beta,+} \quad . \tag{5}$$

It is known that for $d \geq 3$

$$M(\beta) \neq 0 \text{ , for large enough } \beta \quad .$$

In two dimensions this remains true in the Ising model, but the two-dimensional rotator model does not exhibit spontaneous magnetization, except at $\beta = \infty$, (a well-known theorem due to Mermin.) However, this model shows a Kosterlitz-Thouless transition, from a high temperature phase with exponentially decaying correlations to a low temperature phase where correlations have only power law decay. This has been rigorously established in [5]. This transition appears to be closely related to the roughening transition in the three-dimensional Ising model, (see Sect. 2). It is essentially the same phenomenon as the roughening transition in the solid-on-solid model described in the next section.

Next, we define thermodynamic functions :

(a) The <u>free energy</u>

$$f(\beta) = \lim_{L,T\to\infty} (TL^{d-1})^{-1} \log Z_{\beta,\Lambda_{L,T}}(S_{\Lambda^c}) \tag{6}$$

which is independent of the b.c. that are imposed.

(b) The <u>surface tension</u> (or surface free energy)

$$\tau(\beta) \equiv f^{(1)}(\beta) \equiv \lim_{L\to\infty} \lim_{T\to\infty} L^{1-d} \log \frac{Z_{\beta,\Lambda_{L,T}}^{(+)}}{Z_{\beta,\Lambda_{L,T}}^{(\pm)}} \tag{7}$$

(c) The <u>step free energy</u>

$$\sigma(\beta) \equiv f^{(2)}(\beta) \equiv \lim_{L\to\infty} \lim_{T\to\infty} L^{2-d} \log \frac{Z_{\beta,\Lambda_{L,T}}^{(\pm)}}{Z_{\beta,\Lambda_{L,T}}^{(\text{step})}} \tag{8}$$

Similarly, $f^{(k)}(\beta)$, $k = 3,\ldots,d-1$, can be defined.

Theorem 1.

1) [6] In the $d \geq 2$ dimensional Ising model

$$\tau(\beta) > 0 \Leftrightarrow M(\beta) > 0 \text{ , (i.e. } \beta > \beta_c \text{ .)}$$

(See also Sect. 4.)

2) [7] In the rotator model

$$\tau(\beta) = 0 \text{ , for all } \beta < \infty \text{ and arbitrary } d \text{ .}$$

3) [7] If $\tau(\beta) = 0$ then there is no interface, in the sense that

$$\langle(\cdot)\rangle_{\beta,\pm} = 1/2 \langle(\cdot)\rangle_{\beta,+} + 1/2 \langle(\cdot)\rangle_{\beta,-} \quad ,$$

(provided $\langle(\cdot)\rangle_{\beta,\pm}$ is invariant under translations in directions perpendicular to the 1-direction.)

We define the roughening temperature $T_R = \beta_R^{-1}$ as the smallest temperature for which $\langle(\cdot)\rangle_{\beta,\pm} = 1/2 \langle(\cdot)\rangle_{\beta,+} + 1/2 \langle(\cdot)\rangle_{\beta,-}$. It was first proven by Dobrushin [8] (see also [9,10] for simplifications and extentions) that β_R is finite for the Ising model in three or more dimensions, i.e. the Ising model in $d \geq 3$ dimensions has non-translation-invariant equilibrium states at sufficiently low temperatures. In two dimensions, all equilibrium states of the Ising model are convex combinations of $\langle(\cdot)\rangle_{\beta,+}$ and $\langle(\cdot)\rangle_{\beta,-}$, hence translation-invariant. This result is due to Aizenman [11].

It is conjectured that

$$\begin{aligned} \beta_R &> \beta_c \text{ , in } d = 3 \text{ ;} \\ \beta_R &= \beta_c \text{ , in } d \geq 4 \text{ .} \end{aligned} \tag{9}$$

A theoretical argument for the truth of this conjecture is described in the next section.

Next, we introduce some order parameters for the roughening transition in the Ising model. (Our discussion serves mainly as a preparation for the considerations in Sect. 4.) A convenient order parameter to locate the interface is

$$D(\beta,n) = \langle S_{(n,\vec{o})} \cdot S_{(-n-1,\vec{o})} \rangle_{\beta,\pm} \quad , \tag{10}$$

where $\vec{j} = (j_2,\ldots,j_d)$, and

$$D(\beta) = \lim_{n\to\infty} D(\beta,n) \ . \tag{11}$$

For $\beta < \beta_R$,

$$D(\beta) = M(\beta)^2 \geq 0$$

Moreover,

$$D(\beta) \to -1 \ , \quad \text{as} \quad \beta \to \infty \ ,$$

in dimension $d \geq 3$. We conjecture that for all $\beta > \beta_R$

$$D(\beta) < M(\beta)^2 \ ,$$

in fact, that $D(\beta,n)$ is negative, for n large enough. (We are not aware of any proof of this very plausible conjecture.) We define

$$\beta_R' = \inf\{\beta : D(\beta) < M(\beta)^2\} \geq \beta_R \tag{12}$$

Another convenient "order parameter" for the roughening transition might be the step free energy, $\sigma(\beta)$, defined in (c) above. For $\beta < \beta_c$,

$$\sigma(\beta) = 0 \ .$$

In dimension $d \geq 3$

$$\lim_{\beta\to\infty} \beta^{-1}\sigma(\beta) = -2 \ .$$

It is conjectured that

$$\begin{aligned} &\sigma(\beta) = 0 \ , \text{ for } \beta \leq \beta_R \\ &\sigma(\beta) > 0 \ , \text{ for } \beta > \beta_R \ . \end{aligned} \tag{13}$$

We set

$$\beta_R'' = \inf\{\beta : \sigma(\beta) > 0\} \tag{14}$$

Theorem 2.

$$\beta_c(d=3) \leq \beta_R, \beta_R', \beta_R''(d=3) \leq \beta_c(d=2) \ .$$

For β_R and β_R' this result was proven by van Bejeren [12], for β_R'' it has recently been established in [13]. The expected result would be

$$\beta_R = \beta_R' = \beta_R'' \quad , \quad \text{for all } d$$

$$\beta_c(d=3) < \beta_R(d=3) < \beta_c(d=2) \tag{15}$$

$$\beta_c = \beta_R \quad , \quad d \geq 4 \; .$$

Next, we sketch a suitable notion of an <u>interface</u> in an Ising model : We notice that if $\pm$ b.c. are imposed at the boundary of $\Lambda_{L,T}$ then there is a (Peierls) contour $\mathring{\Sigma}_L$ decomposing $\Lambda_{L,T}$ into two disjoint subsets, $\Lambda^+_{L,T}$ and $\Lambda^-_{L,T}$, such that $(T,\vec{0}) \in \Lambda^+_{L,T}$, $S_j = +$ if $j \in \Lambda^+_{L,T}$ borders $\mathring{\Sigma}_L$, $S_j = -$ if $j \in \Lambda^-_{L,T}$ borders $\mathring{\Sigma}_L$ and

$$\partial\mathring{\Sigma}_L = \partial\Lambda_{L,T} \cap \{x : x_1 = -1/2\} \; .$$

We define the interface Σ_L to consist of the union of $\mathring{\Sigma}_L$ and all closed contours which are $*$ connected with $\mathring{\Sigma}_L$.

In two dimensions, the interface, Σ_L, has finite width, uniformly in L, and has long wave length fluctuations on a scale of $\sqrt{L}$, provided the temperature is small enough, $(\beta > \beta_c)$. This result is due to Gallavotti [14].

In $d \geq 3$ dimensions the conjectured behaviour of the interface is as follows:

For $d = 3$ and $\beta > \beta_R$, or for $d \geq 4$ and all $\beta > \beta_c$, the interface Σ_L is well localized near $\{x : x_1 = -1/2\}$ and very rigid and thin, uniformly in L; $D(\beta,n)$ is negative, for n large enough – presumably for all $n \geq 1$. (For rigorous results valid at large β see [8,9,10,12].)

For $d = 3$ and $\beta < \beta_R$ (but β close to β_R) the interface Σ_L still has finite width but fluctuates on a <u>logarithmic scale</u>. (A theoretical argument supporting this claim is reviewed in the next section.) As β is decreased, some of the following phenomena may occur : Interlacing chains of $-$ spins will start to percolate into Λ^+ and, as a consequence, the interface grows many <u>handles</u>. In addition, short wave length fluctuations may cause a lot of wrinkles on the interface. Hence the interface fattens. When β approaches β_c, the interface might approach some self-similar surface, and below β_c it will become "space-filling". Unfortunately, there are no rigorous results, except for very large β. (See also [13] for some speculations.)

We now turn to the discussion of the <u>rotator model</u> : By Theorem 1, parts 2) and 3), <u>the rotator model never exhibits an interface</u>.

Does this mean that all equilibrium states are translation-invariant ? Before attempting to answer this question we quote a result that characterizes the translation-invariant equilibrium states of the Ising- and the rotator models. For the rotator model, define $\langle(\cdot)\rangle_{\beta,\theta}$ by

$$\langle A(S)\rangle_{\beta,\theta} = \langle A(R(\theta)S)\rangle_{\beta,+} \quad , \tag{16}$$

where $R(\theta)$ rotates each spin S_j through an angle θ . If $M(\beta) = 0$ (i.e. $\beta < \beta_c$) the states $\langle(\cdot)\rangle_{\beta,\theta}$ coincide with $\langle(\cdot)\rangle_{\beta,+}$, for all $\theta \in [0,2\pi)$. Indeed, for $\beta < \beta_c$, $\langle(\cdot)\rangle_{\beta,+}$ is the unique translation-invariant equilibrium state. This result is valid in the Ising- and the rotator model [15].

Theorem 3.

Let $\beta > \beta_c$ be such that the free energy is continuously differentiable at β.[1] *Then*

1) [16] *In the Ising model, every translation invariant equilibrium state is a convex combination of $\langle(\cdot)\rangle_{\beta,+}$ and $\langle(\cdot)\rangle_{\beta,-}$.*

2) [7] *In the rotator model, every translation-invariant equilibrium state has a representation*

$$\int d\rho(\theta)\ \langle(\cdot)\rangle_{\beta,\theta} \quad , \tag{17}$$

where ρ is some probability measure.

We now return to the question as to whether all equilibrium states of the rotator model are translation-invariant. The physical reason why there are no interfaces in the rotator model, as remarked, is quite obvious : One might wish to measure the profile of an interface in the rotator model in terms of

$$D_{L,T}(\beta,n) = \int S_{(n,\vec{0})}\cdot S_{(-n-1,\vec{0})}\, d\mu_\beta(S_{\Lambda_{L,T}} | \pm \text{ b.c.}) \quad .$$

But

$$\lim_{L,T\to\infty} D_{L,T}(\beta,n) = \langle S_{(n,\vec{0})}\cdot S_{(-n-1,\vec{0})}\rangle_{\beta,+} \quad ,$$

---

1) Since $f(\beta)$ is concave, this condition is satisfied for almost all values of β .

for all β and n ; (see [7] for a precise statement.) Thus, the interface becomes very wide (fat.) Since the model has a continuous symmetry, this is no surprise : In order to fulfil $\pm$ b.c., it suffices to turn the spins upside down extremely slowly as one moves from $j_+ = (T,\vec{0})$ down to $j_- = (-T,\vec{0})$. Interfaces (Bloch walls) are not among the "topologically stable" defects of this model.

The role of Bloch walls (Peierls contours) or interfaces in the Ising model is really played, in the rotator model, by another type of "topologically stable defects", the vortices. They are characterized by an integer winding number of the spin configuration, and, since the spin takes values in the unit circle, must have co-dimension 2. The easiest way of describing vortex configurations in the rotator model proceeds by applying a duality transformation, i.e. Fourier transformation in the angular variables (see [5,17] , and refs. given there.)

Let

$$r_\beta(\theta) := \exp[\beta\cos\theta] \quad (\text{or } := \sum_{n\in\mathbb{Z}} \exp[-\tfrac{\beta}{2}(\theta+2\pi n)^2] \ ,$$

the Villain approximation.) Let $\hat{r}_\beta(n)$ denote the n^{th} Fourier coefficient of r_β. The equilibrium state of the rotator is given by the measure

$$d\mu_\beta(\theta) = Z^{-1} \prod_{(ij)} r_\beta(\theta_i-\theta_j) \prod_j d\theta_j \quad . \tag{18}$$

The Fourier coefficients of μ_β are thus given by

$$\hat{\mu}_\beta(n) = Z^{-1} \prod_{(ij)} \hat{r}_\beta(n_{ij}) \prod_j \delta_{(\delta n)_j,0} \ , \tag{19}$$

where ij is the oriented bond pointing from i to j ,

$$(\delta n)_j \equiv \sum_{b\ni j} n_b \ , \quad \text{and} \quad n_b \equiv -n_{-b} \quad .$$

The factor $\prod_j \delta_{(\delta n)_j,0}$ arises by integrating the factors $\exp[i\theta_j(\delta n)_j]$ over θ_j , for all j . It imposes the constraint

$$\delta n = 0$$

which is solved (Poincaré's lemma) by

$$n = \delta m \ ,$$

where $m : p \to m_p \in \mathbb{Z}$ is defined on oriented unit squares (plaquettes), $p \in \mathbb{Z}^d$. We may write

$$m_p \equiv \alpha_c \in \mathbb{Z} \quad ,$$

where c is the oriented $(d-2)$-cell in $(\mathbb{Z}^d)^*$ dual to p .

In the Villain approximation,

$$\hat{r}_\beta(n) = \exp[-\frac{1}{2\beta} n^2] \quad .$$

Applying now the Poisson summation formula, we conclude that the Villain approximation to the rotator model is isomorphic to a model whose equilibrium state is given by

$$d\mu_\beta(\alpha) = \hat{Z}^{-1} \prod_{c \in (\mathbb{Z}^d)^*} (\sum_{\varphi_c \in \mathbb{Z}} e^{i\varphi_c \alpha_c}) d\mu_G(\alpha) \ ,$$

where $d\mu_G$ is a Gaussian measure on the space of "orbits" $[\alpha]$, where $[\alpha]$ is the equivalence class $\{\alpha_c + \sum_{c' \in \partial c} \chi_{c'}\}$, and $\chi : c' \to \chi_{c'} \in \mathbb{R}$ is a function defined on $(d-3)$-cells, c' , of $(\mathbb{Z}^d)^*$. The inverse covariance of $d\mu_G$ is $\beta^{-1}\delta d$. It follows from the (gauge invariance) properties of $d\mu_G$ that all configurations $\varphi = \{\varphi_c \in \mathbb{Z} : c \subset (\mathbb{Z}^d)^*\}$ must satisfy the constraint

$$\delta\varphi = 0 \quad .$$

This shows that the connected components of each configuration φ can be interpreted as closed, $(d-2)$-dimensional vortices with integer winding numbers prescribed by $\{\varphi_c\}$. (Back in the rotator model φ corresponds to vortices in the spin field.) By choosing appropriate, non-translation-invariant boundary conditions one can force an open vortex into the system which extends to the boundary (where it is "closed off" by the b.c.) and plays the role of an interface, Σ , in the Ising model. This vortex might cause a breakdown of translation invariance in the thermodynamic limit. In the next section we sketch theoretical arguments supporting the following

Conjecture. [7]

1) In $d \leq 3$ dimensions, all equilibrium states of the rotator model are translation-invariant and are given by formula (17) of Theorem 3.

2) In $d \geq 5$ dimensions, the rotator model has non-translation-invariant equilibrium states for all $\beta > \beta_c$.

3) In $d = 4$ dimensions there exists an inverse temperature $\beta_R > \beta_c$ such that for $\beta > \beta_R$ there exist non-translation invariant equilibrium states while for $\beta < \beta_R$ all states are of the form (17).

The idea behind this conjecture is that in dimension $d \leq 3$ vortices have dimension 0 or 1 and are therefore unstable against long wave length fluctuations, no matter how large β is. (For $d = 3$, results analogous to the ones of Gallavotti [14] should hold.) For $d = 4$, vortices are two-dimensional. They are therefore likely to be rigid for very large β, but are expected to have logarithmic fluctuations above a roughening temperature; see Sect. 2. Finally, vortices of dimension ≥ 3 are expected to have finite fluctuations, as long as $\beta > \beta_c$; (Sect. 2.)

2. The solid-on-solid model.

In this section we review some recent rigorous results on an approximate, statistical theory of (lattice) surfaces, like the interface in the Ising model, the vortex sheets in the four-dimensional rotator model or the electric flux "world sheets" in a lattice gauge theory. We also show that the same approximation yields an uninteresting theory of one- or three- and higher dimensional objects : One dimensional objects (strings) fluctuate on a scale of $\sqrt{L}$, as expected on the basis of the central limit theorem, while three-dimensional objects ("bags") have uniformly bounded fluctuations. See Theorem 4, below.

The approximation considered in this section involves the following elements :

1) Only surfaces (or strings, or bags) which are graphs of functions are admitted as elements of the statistical ensemble, E.

2) The statistical weight of a surface is a local functional of the surface, e.g. its area.

Specifically, the models which we consider are defined as follows : As our parameter space we choose some finite, rectangular array of sites, Λ, in the lattice $\mathbb{Z}^d$, $d = 1,2,3,\ldots$; (the interesting case is $d = 2$.) Each (hyper-) surface in our statistical ensemble $E \equiv E_\Lambda$ is given by the graph of a function, $\vec{\phi}_\Lambda$, assigning to each site $j \in \Lambda$ an m-tuple of integers, $\vec{\phi}_j = (\phi_j^1,\ldots,\phi_j^m)$ interpreted as the coordinates ("heights") of the (hyper-) surface in the directions transverse to the parameter directions, in such a way that $(j^1,\ldots,j^d, \phi_j^1,\ldots,\phi_j^m)$ are the coordinates of the center of a d-cell in the surface described by $\vec{\phi}_\Lambda$. We assume, temporarily, that

$$\vec{\phi}_j = 0 \ , \text{ for } \ j \notin \Lambda \ , \ (0 \text{ b.c.}).$$

The statistical weight, $w_\beta(\vec{\phi}_\Lambda)$, of the surface described by $\vec{\phi}_\Lambda$ is defined by

$$w_\beta(\vec{\phi}_\Lambda) = Z^{-1}_{\beta,\Lambda} e^{-\beta A(\vec{\phi}_\Lambda)} \quad , \tag{20}$$

where the "action" $A(\vec{\phi}_\Lambda)$ is given by the total d-dimensional volume of $\vec{\phi}_\Lambda$ (or an approximation thereof), in particular $A(\vec{\phi}_\Lambda)$ is the area of the surface when $d = 2$, and the partition function, $Z_{\beta,\Lambda}$, is chosen such that

$$\sum_{\vec{\phi}_\Lambda} w_\beta(\vec{\phi}_\Lambda) = 1 \ .$$

For $m = 1$,

$$A(\phi_\Lambda) = |\Lambda| + \sum_{(jj')} |\phi_j - \phi_{j'}| \quad , \tag{21}$$

where the sum ranges over all nearest neighbor pairs. The factor $\exp(-\beta|\Lambda|)$ can be absorbed in a redefinition of $Z_{\beta,\Lambda}$. The model so obtained is called the solid-on-solid (s-o-s) model [2]. It describes the statistical mechanics of the interface of a limiting, d-dimensional Ising model with $\pm$ b.c. which is obtained by letting the nearest neighbor couplings in the 1-direction tend to ∞ while keeping them fixed in the other directions.

When $m > 1$ it is difficult to analyze the models with actions given by the volume of d-dimensional hypersurfaces in $\mathbb{Z}^{d+m}$; (see Sect. 3.) We shall consider, instead, e.g. the small fluctuation approximation to the volume, given by

$$A(\vec{\phi}_\Lambda) \approx |\Lambda| + \frac{1}{2} \sum_{(jj')} (\vec{\phi}_j - \vec{\phi}_{j'})^2 \quad , \tag{22}$$

but approximate actions like

$$|\Lambda| + \sum_{(jj')} |\vec{\phi}_j - \vec{\phi}_{j'}| \tag{23}$$

can be analyzed, too.

We let $\langle(\cdot)\rangle_{\beta,\Lambda}$ denote the expectation defined by (20), with $A(\vec{\phi}_\Lambda)$ as in (21) or (22), (23). [We shall usually think of the s-o-s model corresponding to (21), but most results described in the following remain valid for the models with actions (22), (23), as follows from the analysis in [5].] Let $F(\vec{\phi})$ be an arbitrary continuous, polynomially bounded function of $\{\phi_j - \phi_{j'}\}$, where (jj') are nearest neighbor pairs belonging to some finite subset of $\mathbb{Z}^d$. On this class of functions a thermodynamic limit

$$\langle F\rangle_\beta = \lim_{j\to\infty} \langle F\rangle_{\beta,\Lambda_j} \quad ,$$

$\Lambda_j \nearrow \mathbb{Z}^d$, as $j \to \infty$, can be constructed by a compactness argument; (for the action in (22) it exists by correlation inequalities [18], and, in all cases, it exists for large enough values of β .) We have

Theorem 4.

Consider the models defined in (20) - (23). Then

1) For $d = 1$,

$$\langle(\vec{\phi}_o-\vec{\phi}_x)^2\rangle_\beta \sim c_1(\beta)|x| \ , \text{ as } \ |x| \to \infty \quad .$$

2) For $d = 2$,

$$\langle(\vec{\phi}_o-\vec{\phi}_x)^2\rangle_\beta \leq c_2(\beta) \quad ,$$

uniformly in x , provided β is large enough. When β is small enough,

$$c_3(\beta) \log|x| \leq \langle(\vec{\phi}_o-\vec{\phi}_x)^2\rangle_\beta \leq c_4(\beta) \log|x| \tag{24}$$

3) For $d \geq 3$,

$$\langle(\vec{\phi}_o-\vec{\phi}_x)^2\rangle_\beta \leq c_5(\beta) \ ,$$

for all β .

□

Remarks.

(1) Part 1) is a standard consequence of the central limit theorem : The random variables $\vec{\phi}_j-\vec{\phi}_{j'}$, where (jj') ranges over the bonds (nearest neighbor pairs) of $\mathbb{Z}$, are independently distributed!

The first half of part 2) follows from a standard low-temperature (Peierls contour) expansion, (as observed in [19].) The deepest result is the lower bound in (24) which was established in [5] by a rather difficult analysis. The upper bound in (24) and part 3) are standard consequences of infrared bounds [20] which are applicable, because the functions $\exp(-\beta|\phi|)$ and $\exp(-\frac{\beta}{2}\phi^2)$ are of positive type. Part 3) has recently been noticed in [13].

(2) The model with $d = 2$, $m = 1$ and $\Lambda(\phi_\Lambda)$ given by the r.s. of (22) is dual to the Villain approximation of the two-dimensional rotator model; see Sect. 1, (18), (19), etc. The behaviour described in part 2) of Theorem 4 is, in this case, related to the Kosterlitz-Thouless transition [5].

(3) The transition described in part 2) is a model of the roughening transition: For large β typical lattice surfaces are rigid, i.e. have uniformly bounded fluctuations. When β drops below some critical value, β_R, then typical surfaces are rough and exhibit logarithmic fluctuations. This is the universal behavior of continuum surfaces. A roughening transition occurs only in ensembles of lattice surfaces, because the lattice breaks the continuous group of translations transverse to the surface. At high temperatures, this symmetry is restored, i.e. "enhanced at large distances" in the models considered above, [5,21]; (see also [22].)

Next, we sketch a few ideas in the proofs of parts 2) and 3) of Theorem 4. For simplicity we consider the action (22) with $m = 1$, but the results hold in general [5]. We start with the lower bound in (24).

Let $d\mu_{\beta,\Lambda}(\phi)$ be the Gaussian measure with mean 0 and covariance $(-\beta\Delta_\Lambda)^{-1}$, where Δ_Λ is the finite difference approximation of the Laplacean with 0 Dirichlet data at the boundary of Λ. The equilibrium state of our model can be rewritten as follows :

$$\tilde{Z}_{\beta,\Lambda} dw_\beta(\phi_\Lambda) = \prod_{j\in\Lambda} \Big(\sum_{n\in\mathbb{Z}} \delta(\phi_j - n)\Big) d\mu_{\beta,\Lambda}(\phi)$$

$$= \prod_{j\in\Lambda} \Big(1+2 \sum_{q_j=1} \cos(2\pi q_j \phi_j)\Big) d\mu_{\beta,\Lambda}(\phi) . \qquad (25)$$

There are three basic steps in the proofs [5] of the lower bound in (24).

1° The first step is a combinatorial identity : Let ρ denote an arbitrary function on $\mathbb{Z}^2$ of finite support with values in $2\pi\,\mathbb{Z}$; ρ is called a "charge density". We say that ρ is neutral iff $\sum_j \rho_j = 0$. Let $\phi(\rho) \equiv \sum_j \phi_j \rho_j$.
It is proven in [5] by means of an inductive construction extending over all distance scales of 2^n, $n = 0,1,2,\ldots$, that, for all $\Lambda \subset \mathbb{Z}^2$,

$$\prod_{j\in\Lambda} \Big(1+2 \sum_{q_j=1}^{\infty} \cos(2\pi q_j \phi_j)\Big) = \sum_{N\in F_\Lambda} c_N \prod_{\rho\in N} (1+K(\rho)\cos\phi(\rho)) , \qquad (26)$$

where F_Λ is a finite family of collections, N, of neutral charge densities, ρ, with the property that two densities, ρ and $\rho' \neq \rho$, in each N have disjoint supports which are so far separated that $\cos\phi(\rho)$ and $\cos\phi(\rho')$ are "almost independent". Furthermore, $c_N > 0$ for all $N \in F_\Lambda$. The constant $K(\rho)$ is an entropy factor which can be bounded by $\exp(cA(\rho))$, where

$$A(\rho) = \sum_{n=o}^{\infty} (A_n(\rho)-1) ,$$

and $A_n(\rho)$ is the number of $2^n \times 2^n$ squares needed to cover the support of ρ .

2° The second step consists of a "block spin integration" which allows us to extract "self-energies" of the densities, ρ , providing convergence factors which compensate the constants $K(\rho)$. In the simplest case (namely for the partition function) it results in the following identity : For all $N \in F_\Lambda$,

$$\begin{aligned} &\int \prod_{\rho\in N} (1+K(\rho)\cos \phi(\rho))d\mu_{\beta,\Lambda}(\phi) \\ &= \int \prod_{\rho\in N} (1+e^{-\beta\widetilde{E}(\rho)}K(\rho)\cos \phi(\overline{\rho}))d\mu_{\beta,\Lambda}(\phi) , \end{aligned} \tag{27}$$

where $\widetilde{E}(\rho) \approx \text{const.} \sum_{i,j} \rho_i(-\Delta_\Lambda)^{-1}_{ij} \rho_j$ is related to the electrostatic energy of the charge density ρ , and the renormalized charge densities, $\overline{\rho}$, are still neutral but have "magnified" supports.

A key estimate consists in showing that

$$\widetilde{E}(\rho) \geq \varepsilon A(\rho) ,$$

for some $\varepsilon > 0$. Thus, for large β ,

$$z(\rho) \equiv e^{-\beta\widetilde{E}(\rho)}K(\rho) \leq e^{-\beta/2\, \ell n d(\rho)} << 1 ,$$

where $d(\rho)$ is the diameter of the support of ρ . Thus, for large β ,

$$dw_{\text{ren.}}(\phi) \equiv \widetilde{Z}^{-1}_{\beta,\Lambda} \sum_{N\in F_\Lambda} c_N \prod_{\rho\in N} (1+z(\rho)\cos \phi(\overline{\rho}))d\mu_{\beta,\Lambda}(\phi) , \tag{28}$$

is a <u>positive measure</u> which is, formally, <u>invariant under the continuous symmetry</u>

$$\phi_j \to \phi_j+c , \tag{29}$$

where c is an arbitrary real constant; for

$$\cos \phi(\overline{\rho}) = \cos[(\phi+c)(\overline{\rho})] ,$$

as $\sum c\, \overline{\rho}_j = 0$, by the neutrality of $\overline{\rho}$; moreover $d\mu_{\beta,\Lambda}(\phi)$ is clearly formally invariant under the symmetry (29), except that the b.c. imposed on $d\mu_{\beta,\Lambda}(\phi)$ break (29).

3° Since, for large β , the measure $dw_{ren.}$ given by (28) is positive and formally invariant under the continuous group of symmetries (29) which, however, is always broken by the b.c. imposed at $\partial\Lambda$, we may apply a Mermin-type argument to conclude that

$$\lim_{\Lambda\nearrow\mathbb{Z}^2} \int dw_{ren}(\phi_\Lambda)|\hat{\phi}(k)|^2 \geq \frac{\text{const.}}{k^2} \tag{30}$$

for all $k \neq 0$. Here $\hat{\phi}(k) \equiv (2\pi)^{-2} \sum_{j\in\mathbb{Z}^2} \phi_j e^{ik\cdot j}$. From this one can deduce the lower bound in (24), (by Fourier transformation.)

Next, we comment on the proof of the upper bound in (24) and part 3) of Theorem 4. Part 3) was previously proven in [13]. Here we sketch a slightly different argument which gives a stronger result. For technical convenience we interpret the state $\langle(\cdot)\rangle_\beta$ as a limit of finite volume states $\langle(\cdot)\rangle_{\beta,\Lambda}$ with <u>periodic b.c.</u> . [In order to define the periodic b.c. state, one replaces the counting measure on $\{\phi_j \in \mathbb{Z}\}$ by $\exp(-\varepsilon\phi_j^2)$ × the counting measure. One first takes $\Lambda\nearrow\mathbb{Z}^d$ and subsequently $\varepsilon\searrow 0$.] As explained in [20], the upper bound in (24) and part 3) follow from estimates of the form

$$\langle\exp(\varepsilon \sum_j h_j(\partial_\alpha\phi)_j)\rangle_{\beta,\Lambda} \leq \exp[c(\beta)\varepsilon^2\|h\|_2^2] \quad , \tag{31}$$

with ε small enough, $(\varepsilon\|h\|_2^2 \leq \varepsilon_o$, for some $\varepsilon_o > 0$.) Here ∂_α is the α^{th} component of the finite difference gradient, and h is an arbitrary real-valued function on Λ . Inequality (31) is proven by using a transfer matrix in the α-direction of the lattice. As explained in [20], the transfer matrix formalism reduces the problem to estimating the quadratic form with integral kernel

$$e^{\varepsilon h(x-x')} e^{-F_\beta(x-x')} \quad ,$$

where

$$F_\beta(x) = \begin{cases} \beta|x| \ , \text{ or} \\ (\beta/2)x^2 \end{cases}$$

from above in terms of the quadratic form with integral kernel $\exp[-F_\beta(x-x')]$. This is accomplished by using Fourier transformation : For $F_\beta(x) = \beta|x|$, the required bound follows by noticing the inequality

$$|[(k+i\varepsilon h)^2+\beta^2]^{-1}| \leq e^{c(\beta)\varepsilon^2 h^2} [k^2+\beta]^{-1} \quad ,$$

for $|\varepsilon h| < \beta/2$. For $F_\beta(x) = (\beta/2)x^2$, one uses

$$|\exp[-(1/2\beta)(k+i\varepsilon h)^2]| \leq e^{\varepsilon^2 h^2/2\beta} \exp[-(1/2\beta)k^2] ,$$

for arbitrary ε and h .

From these inequalities (31) follows. The upper bound in (24) and part 3) of Theorem 4 follow from (31) by expanding to second order in ε , dividing by ε^2 and taking ε to 0 .

In dimension $d \geq 3$, we expect that a result much stronger than part 3) of Theorem 4 holds. For <u>all</u> $\beta > 0$,

$$|\langle\vec{\phi}_o \cdot \vec{\phi}_x\rangle_\beta| \leq c(\beta)e^{-m(\beta)|x|} , \tag{32}$$

for some constants $c(\beta) < \infty$ and $m(\beta) > 0$. This would imply that all correlations between distant pieces of three- or higher dimensional random hypersurfaces decay exponentially. For $d = 3$, $(m = 1)$ and the action $A(\phi_\Lambda)$ given by (22), the bound (32) has recently been proven by Göpfert and Mack in [23].

Next, we review some results on <u>surface</u> (or <u>step</u>) <u>free energies</u> in the solid-on-solid models : Let $Z_{\beta,\Lambda}$ be the usual partition function of the model with 0 b.c. defined in (20). Let $Z_{\beta,\Lambda}(\vec{\xi})$, $\vec{\xi} \in \mathbb{Z}^m$, be the partition function of the same model, but with b.c.

$$\left.\begin{array}{ll} \vec{\phi}_j = \vec{\xi} , & \text{for } j \notin \Lambda , j_1 > 0 \\ \vec{\phi}_j = 0 , & \text{for } j \notin \Lambda , j_1 < 0 \end{array}\right\} \tag{33}$$

We set

$$\tau_d(\vec{\xi};\beta) \equiv \lim_{\Lambda\nearrow\mathbb{Z}^d} \log(Z_{\beta,\Lambda}/Z_{\beta,\Lambda}(\vec{\xi})) \tag{34}$$

We note that $\tau_d(\xi = 1;\beta)$ $(m = 1)$ <u>is expected to behave qualitatively similarly as the step free energy,</u> $\sigma(\beta)$, <u>of the</u> (d+1)-<u>dimensional Ising model</u>. We consider, for simplicity, only the case $m = 1$, assume that $\xi \neq 0$ and that the action is given by (21) or (22). We then have

<u>Theorem 5</u>.

1) $\tau_1(\xi;\beta) = 0$, <u>for all</u> ξ .

2) $\left.\begin{array}{ll} \tau_2(\xi;\beta) > 0 , & \underline{\text{for large}}\ \beta \\ \tau_2(\xi;\beta) = 0 , & \text{for small}\ \beta \end{array}\right\}$ <u>for all</u> $\xi \neq 0$.

3) For the models with action given by (22) and $d \geq 3$,

$$\tau_d(\xi;\beta) > 0 \text{ , for all } \beta > 0 \text{ , } \xi \neq 0 \text{ .}$$

Remarks.

Part 1) is trivial. The inequality in part 2) is a consequence of a standard low temperature expansion; e.g. [19]. The equation in part 2) follows from the results of Sects. 6 and 7 of [5]. Part 3) follows from the results of Göpfert and Mack [23] ($d = 3$) and correlation inequalities [18], ($d = 3 \rightarrow d > 3$) . For results related to the ones in [23] but established earlier see also [24].

We believe that Theorem 5 can be extended to all $m \geq 1$ and all actions (21) - (23) , but not all cases have been worked out.

Finally, some recent results in [5,25] suggest that the continuum limits of the two-dimensional models studied in this section are given by massless Gaussian measures, for $\beta < \beta_R$ and for arbitrary $m = 1,2,3,\ldots$. (This is trivial for $d = 1$.) In the next section, we study random surfaces with more complicated continuum limits.

3. Selfavoiding random surfaces and string theories.

In this section we restrict our discussion to two-dimensional random surfaces embedded in a lattice $\mathbb{Z}^d$ (or embedded in $\mathbb{E}^d$) , $d = 3,4,\ldots$. We propose to consider statistical theories of such surfaces which are geometrically more natural than the ones studied in the last section, but which are seemingly almost as simple as the s-o-s models. Our discussion is sketchy; (some details appear elsewhere.)

The models considered in Sect. 2 have a serious defect : All random surfaces admitted in the ensembles introduced in Sect. 2 are required to be graphs of functions. It is natural to study more general ensembles of lattice surfaces and their continuum limits. If one admits lattice surfaces which may pass through each plaquette (unit square) of $\mathbb{Z}^d$ an arbitrary number of times one cannot construct a mathematically meaningful statistical theory : The number of such surfaces of a given area - i.e. containing a given number of plaquettes counted with multiplicites - grows faster than exponentially in the area; see e.g. [26].

There are at least three ensembles of lattice surfaces which are physically natural :

a) Branched random surfaces arising in plaquette percolation models [27]. (They consist of arbitrary connected arrays of "occupied" plaquettes, each plaquette in $\mathbb{Z}^d$ being either "empty" or "occupied" once. The weight of such a surface, Σ, is given by $p^{A(\Sigma)}$, $0 < p < 1$, $A(\Sigma) = \#$ plaquettes belonging to Σ.)

b) Let γ be a closed curve in $\mathbb{Z}^d$, and let $E_\gamma^{s.a.}$ be the class of all "self-avoiding" connected lattice surfaces bounded by γ, i.e. surfaces, $\Sigma \subset \mathbb{Z}^d$, with the property that each link $b \in \Sigma$, $b \notin \gamma$, belongs to precisely two plaquettes of Σ and each $b \in \gamma$ to precisely one plaquette of Σ.

c) Let γ be a closed curve in $\mathbb{Z}^d$, and let E_γ be the class of all connected surfaces bounded by γ which pass through each plaquette of $\mathbb{Z}^d$ at most once.

The ensemble E_γ described in c) occurs naturally in the study of <u>interfaces</u>; (see Sect. 1), while the ensemble $E_\gamma^{s.a.}$ introduced in b) and the one introduced in a) (which we denote by $E_{perc.}$) arise in models which are limits of <u>gauge theories</u>; see Sect. 4, and [27,28]. For a somewhat detailed discussion of plaquette- (and general d-cell) percolation see [27] - we limit our review to a discussion of $E^{s.a.}$ and E_γ, ensembles which are also studied in connection with string theories.

Let $E_\gamma^{\#} = E_\gamma^{s.a.}$, or E_γ. Each surface $\Sigma \in E_\gamma^{\#}$ is assigned the statistical weight

$$w_{\beta,\mu}^{\gamma}(\Sigma/\gamma) = Z_{\beta,\mu}(\gamma)^{-1}\cdot\exp[-\beta(A(\Sigma)+\mu\chi(\Sigma))] , \tag{35}$$

where $\chi(\Sigma)$ counts the number of handles of Σ (Euler characteristic), $\beta > 0$, $\mu \geq 0$; ($A(\Sigma)$, the <u>area</u> of Σ, counts the number of plaquettes in Σ.)

It is an elementary combinatorial exercise to show that for each d and each γ

$$Z_{\beta,\mu}(\gamma) = \sum_{\Sigma\in E_\gamma^{\#}} \exp[-\beta(A(\Sigma)+\mu\chi(\Sigma))] < \infty , \tag{36}$$

for β large enough, while

$$Z_{\beta,\mu}(\gamma) \text{ diverges,} \tag{37}$$

for β small enough.

One can argue that there is some value, β_o, of β which only depends on d (and possibly on μ), but is independent of γ such that (36) holds for all $\beta > \beta_o$, while (37) holds for all $\beta < \beta_o$.

As a field theorist one is then interested in the question whether β_o is a critical point, in the sense that there is some divergent correlation length, as $\beta \searrow \beta_o$. This question can be investigated by considering, for example, the "string tension"

$$\alpha(\beta,\mu) \equiv -\lim_{d(\gamma)\to\infty} A(\gamma)^{-1} \log Z_{\beta,\mu}(\gamma) , \tag{38}$$

where γ is a square loop in a coordinate plane, $A(\gamma)$ the minimal area enclosed by γ, and $d(\gamma)$ its diameter. In a statistical mechanics context, e.g. in the s-o-s model, the string tension is interpreted as the surface tension. The point β_o is a critical point if

$$\alpha(\beta,\mu) \searrow 0 , \quad \text{as} \quad \beta \searrow \beta_o . \tag{39}$$

More refined methods to analyze the vicinity of β_o would involve the study of "correlations". We sketch one example; (but see [27] for a more detailed discussion): Let γ,γ' be two non-intersecting loops, and define

$$Z_{\beta,\mu}(\gamma,\gamma') \equiv \sum_{\substack{\Sigma\in E^{\#}_{\gamma\cup\gamma'} \\ \Sigma \text{ connected}}} \exp[-\beta(A(\Sigma)+\mu\chi(\Sigma))]$$

We define a "glue ball mass"

$$m(\beta,\mu) = \lim_{a\to\infty} -\frac{1}{a} \log Z_{\beta,\mu}(\gamma,\gamma'_a) , \tag{40}$$

where γ'_a is the loop that corresponds to a translation of γ' in the direction of a lattice axis by a distance a.

If β_o is a critical point in the sense of (39) one expects that

$$m(\beta,\mu) \searrow 0 , \text{ as } \beta \searrow \beta_o . \tag{41}$$

A more subtle question concerns the behaviour of the dimensionless quantity $m(\beta,\mu)^2/\alpha(\beta,\mu)$, as $\beta \searrow \beta_o$. In [23] a model is studied in which the analogue of this quantity tends to 0, as $\beta \searrow \beta_o$.

Finally, we want to ask whether the three models introduced in this section exhibit roughening. This question can be studied, for example as follows : We choose a square loop, γ, of diameter $d(\gamma)$ lying in a coordinate plane and define the probability

$$P(d|\gamma) = Z_{\beta,\mu}(\gamma)^{-1} \sum_{\substack{\Sigma\subset E^{\#}_{\gamma} \\ d(\Sigma,0)>d}} \exp[-\beta(A(\Sigma)+\mu\chi(\Sigma))] , \tag{42}$$

where $d(\Sigma,0)$ is the maximal distance of the set $\Sigma \cap \pi_o$ from the origin, and π_o is a $(d-2)$-dimensional plane perpendicular to the curve γ and containing the origin. We consider

$$P(d) = \lim_{d(\gamma)\to\infty} P(d|\gamma) . \tag{43}$$

It is easy to show that for β sufficiently large

$$P(d) \leq e^{-c(\beta)d} , \tag{44}$$

for some constant $c(\beta) > 0$.

The question then is whether there exists some value β_R of β , with

$$\beta_R > \beta_o , \tag{45}$$

such that for $\beta_o < \beta < \beta_R$

$$P(d) = 1 , \quad \text{for all} \quad d < \infty .$$

On the basis of results concerning the Ising model in three dimensions [6,13] and the s-o-s model [5] (see Sects. 1,2) one would conjecture that the percolation model of branched surfaces exhibits a roughening transition.

(If β_o is a critical point it might also be possible that (44) is valid for all $\beta < \beta_o$, with $c(\beta)\searrow 0$, as $\beta \searrow \beta_o$.)

Once all these preliminary questions (see (39) - (45)) - which actually seem to be very hard ones - are out of the way one can address the most interesting one : What are the continuum limits of these lattice models of random surfaces ? So far, there has not been much theoretical progress on these questions.

Until now, there is only one convincing attempt at constructing a continuum theory of random surfaces, the one by Polyakov [29], clarified in [30]. Presumably, this theory, too, can be obtained as a continuum limit of some "lattice theory" : It is essentially the continuum limit of discrete, imaginary-time quantum gravity of piecewise linear, simplicial (two-dimensional) surfaces; a straightforward, functional integral version of Regge calculus. Polyakov's theory is certainly very fascinating, but (as the remark above indicates) its relation to the physics of interfaces in statistical mechanics or to gauge theory is mysterious. In contrast, the other models, a), b) and c), discussed in this section and the s-o-s model discussed in Sect. 2 are related to gauge theory and to the physics of interfaces, respectively, in a definite way : They are obtained as limiting models, as some parameter tends to 0 or ∞ . See Sects. 2 and 4. Polyakov's theory is intended to represent a correct mathematical formulation of string theories, (dual resonance models.)

4. Lattice gauge theories.

There are (at least) two ways in which random geometrical objects, such as random loops or random surfaces, arise in the analysis of lattice gauge theories (in the imaginary time description.)

1) Sheets of chromo-electric flux.

Lattice gauge theories can be reformulated as gases of random geometrical objects in different ways : The best known such reformulation results from the strong coupling (high temperature) expansion which represents a lattice gauge theory as a gas of closed random surfaces - closed sheets of chromo-electric flux - which interact by hard core exclusion [31]. (For a somewhat different description of chromo-electric flux sheets, see also [32].)

2) Defect gas description of lattice gauge theory

We first consider a pure lattice gauge with a discrete gauge group on a d-dimensional lattice (or a Higgs theory with a non-trivial, discrete unbroken subgroup.) In such a theory, gauge field configurations can be characterized in terms of frustrated plaquettes, i.e. unit squares, where the curvature is non-vanishing. As a consequence of an integral form of the Bianchi identities, frustrated plaquettes form (d-2)-dimensional, closed surfaces which one calls (by an abuse of this name) vortices. They are labelled by group elements. The original lattice gauge theory can now be reformulated as a gas of vortices interacting by geometrical constraints. At weak coupling (low temperature) the vortices have small effective activities and form a dilute gas. This observation is the starting point for the low temperature analysis of lattice gauge theories : Vortices play the role of the Peierls contours in the Ising model and can be used to construct an analogue of the Peierls argument (or a contour expansion) which permits one to control the qualitative features of such lattice gauge theories at weak coupling, in three or more dimensions. See [21]. The upshot of this analysis is that gauge theories with discrete gauge groups exhibit deconfining transitions in dimension ≥ 3 .

Clearly, in theories with continuous gauge groups, vortices (as defined above) are not likely to provide us with a useful notion, although vortices of a somewhat different type appear to play an important role in a confinement mechanism in gauge theories with gauge groups containing a non-trivial, discrete center. As an example of a gauge theory where vortices are not a useful notion we consider the compact U(1) lattice model (compact QED.) This gauge theory permanently confines electric charge in two and three dimensions [23], but exhibits a deconfining transition in four or more dimensions [33,21]. Vortices are not among the "topologically stable defects" of the U(1) model and cannot be used to explain those facts. (This circumstance is analogous to the one that the interface is unstable in the rotator model; see Sect. 1.) The topologically stable defects of the U(1) model which are dilute at weak coupling are its magnetic excitations : Monopoles in three dimensions, monopole lines (magnetic currents) in four dimensions, etc. In the continuum limit, such excitations are labelled by first Chern classes of the field configurations at infinity (identified with $S^2 \times \mathbb{R}^{d-3}$.) Thus they have dimension $d-3$ and carry an integer magnetic charge. The corresponding magnetic excitations of the U(1) model on the lattice can be exhibited by applying a duality transformation (Fourier transformation in the gauge field variables) and a Poisson summation formula, (as explained at the end of Sect. 1 for the rotator model.) The interactions between different magnetic excitations have long range. This makes the analysis of these models, at weak coupling, interesting and mathematically non-trivial; see [21,23,33].

In the U(1) model, confinement breaks down if the magnetic excitations are bound in finite, neutral clusters which form a dilute gas, thus causing only small (infrared-irrelevant) corrections to Gaussian "spin wave" theory. This only happens in four or more dimensions.

In a non-abelian, pure gauge theory, e.g. one with gauge group SU(n) , there are two kinds of topological excitations, vortices, of co-dimension 2, and instantons, of co-dimension 4. Vortices are labelled by elements of the center of the gauge group, instantons by elements of $\pi_3(G)$. One can argue that, in four or more dimensions, it is the statistical mechanics of the instanton gas which determines whether, at long distances, the theory is in a perturbative or non-perturbative phase. In four dimensions, it is most likely that the instanton gas is always in a plasma phase, instantons are not stably bound in neutral clusters, the infrared behaviour is non-perturbative. However, in five or more dimensions, instantons form closed surfaces of dimension $d-4$, and a simple energy-entropy argument suggests that, at weak coupling, the effective activity of an instanton decreases exponentially in its volume (= length for $d = 5$,...). One is thus led to predict that non-abelian models exhibit a deconfining transition to a perturbative phase at weak coupling, in five or more dimensions. (In contrast to abelian gauge theories or ones with discrete gauge group, there are, however, no rigorous results for non-abelian lattice gauge theories at weak coupling, yet!)

Next, we briefly summarize some recent, rigorous results concerning random geometrical objects in lattice gauge theories and some limiting models of such theories :

As our lattice we choose $\mathbb{Z}^d$, the gauge group is assumed to be compact and is denoted by G. The gauge field, $g = \{g_{xy}\}$, is a map from oriented pairs of nearest neighbors, xy, in $\mathbb{Z}^d$ to elements, g_{xy}, of G such that

$$g_{yx} = g_{xy}^{-1} .$$

Formally

$$g_{xy} = P(e^{\int_x^y A_\mu(\xi)d\xi^\mu}) , \text{ for all } xy . \tag{46}$$

The Euclidean functional measure (vacuum functional) of a lattice gauge theory is defined by

$$d\mu_\beta(g) = \lim_{\Lambda \nearrow \mathbb{Z}^d} Z_{\beta,\Lambda}^{-1} e^{-\beta A(g_\Lambda)} \prod_{xy\subset\Lambda} dg_{xy} , \tag{47}$$

where Λ is a rectangular array of sites, dg_{xy} is the Haar measure on G, for all xy, $A(g_\Lambda)$ is the (Euclidean) lattice action for the model in Λ, $\beta = 1/e^2$ is the inverse square coupling ("inverse temperature"), and $Z_{\beta,\Lambda}$ is the usual partition function (making $d\mu_\beta$ a probability measure.) Given a loop γ in $\mathbb{Z}^d$, we let

$$g_\gamma = \prod_{xy\subset\gamma}^{\circlearrowleft} g_{xy} \tag{48}$$

denote the ordered product of gauge fields along γ, (i.e. the holonomy operator associated with γ.) We define the (Wegner-) Wilson loop observable by

$$W_\chi(\gamma) = \chi(g_\gamma) , \tag{49}$$

where χ is some (irreducible) character of G.

We shall consider the following examples of lattice actions :

(1) $$A(g_\Lambda) = -\sum_{p\subset\Lambda} \text{Re } \chi_o(g_{\partial p}) , \tag{50}$$

where p ranges over the plaquettes (unit squares) of Λ, ∂p is the oriented boundary of p, and χ_o is a faithful character of G, (e.g. the one of the fundamental representation for $G = SU(n)$.)

(2) $$A(g_\Lambda) = -\sum_{p\subset\Lambda} \delta_e(g_{\partial p}) , \tag{51}$$

where δ_e is the (Kronecker) δ-function on G concentrated at the unit element, e, and G is assumed to be discrete in this example.

Let Σ be a (d-2)-dimensional surface in the dual lattice, $(\mathbb{Z}^d)^*$, with boundary $\partial\Sigma$ (a closed, (d-3)-dimensional surface.) We define a disorder operator, $D_z(\partial\Sigma)$, where z is an element of the center of G, as follows :

$$D_z(\partial\Sigma) = \exp[-\beta(A(g\cdot z_\Sigma)-A(g))] \quad , \tag{52}$$

where

$$(g\cdot z_\Sigma)_{\partial p} = \begin{cases} g_{\partial p}\cdot z & \text{if } p \text{ is dual to a } d-2 \text{ cell in } \Sigma\ ; \\ \\ g_{\partial p}\ , & \text{otherwise.} \end{cases}$$

In order to analyze the behaviour of chromoelectric flux sheets we study expectation values like

$$\langle \prod_{j=1}^{n} W_\chi(\gamma_j)D_z(\partial\Sigma)\rangle_\beta \equiv \int \prod_{j=1}^{n} W_\chi(\gamma_j)D_z(\partial\Sigma)d\mu_\beta(g) \ , \tag{53}$$

$n = 1,2,3,\ldots$. In particular, the roughening transition for electric flux sheets can be analyzed in terms of $\langle W_\chi(\gamma)D_z(\partial\Sigma)\rangle_\beta$.

We also introduce bulk- and surface thermodynamic functions, (see Sect. 1, (6)-(8); Sect. 3, (38) for related definitions) :

(a) The free energy :

$$f(\beta) = \lim_{L,T\to\infty} (TL^{d-1})^{-1}\log Z_{\beta,\Lambda_{L,T}}$$

(b) The string tension :

$$\alpha(\beta,\chi) = \lim_{L\to\infty} -\frac{1}{L^2}\log\langle W_\chi(\gamma_L)\rangle_\beta \quad ,$$

where γ_L is a square loop in the 1-2 coordinate plane of diameter L . [The string tension corresponds to the surface tension, $\tau(\beta)$, in spin systems. In three-dimensional $\mathbb{Z}_2$ models they are related by a duality transformation.]

We define an expectation

$$\langle(\cdot)\rangle_\beta^\chi = \lim_{L\to\infty} \langle(\cdot)W_\chi(\gamma_L)\rangle_\beta/\langle W_\chi(\gamma_L)\rangle_\beta \quad . \tag{54}$$

One can also define the analogue of

(c) The step free energy :

$$\sigma(\beta) = \lim_{L\to\infty} - \frac{1}{L} \log \frac{\langle W_\chi(\gamma_L')\rangle_\beta}{\langle W_\chi(\gamma_L)\rangle_\beta} ,$$

where the loop γ_L' differs from γ_L by a step of height 1 in the middle of two opposite sides. The functions $\alpha(\beta,\chi)$ and $\sigma(\beta)$ serve to describe the thermodynamics of chromo-electric flux. In particular, $\sigma(\beta)$ is of interest in studies of the roughening transition of flux sheets [4] . It is natural to also introduce functions describing the thermodynamics of "magnetic flux" (vortex sheets), or more generally the thermodynamics of the gas of stable ("topological") excitations, like the magnetic excitations in the $U(1)$ model,... . As an example, we define a thermodynamic function for vortex sheets : Let Λ be some rectangular array of sites centered at 0 , and Σ a $(d-2)$-dimensional coordinate plane in $(\mathbb{Z}^d)^*$. We let Ω denote the set of plaquettes on $\partial\Lambda$ which are dual to some $d-2$ cell in Σ . We consider the following boundary conditions on $\partial\Lambda$:

(0 b.c.) $\quad g_{\partial p} = e$ (the unit element in G) , for all $p \subset \partial\Lambda$;

(twisted b.c.)

$$g_{\partial p} = \begin{cases} e , & p \subset \partial\Lambda , \quad p \notin \Omega \\ z , & p \in \Omega , \quad \text{for some } z \text{ in the center of } G . \end{cases}$$

Let $Z^0_{\beta,\Lambda}$, $\langle(\cdot)\rangle^0_{\beta,\Lambda}$ be the partition function and the expectation with 0 b.c. on $\partial\Lambda$, and $Z^z_{\beta,\Lambda}$, $\langle(\cdot)\rangle^z_{\beta,\Lambda}$ the corresponding quantities with twisted b.c. . We define

(d) The magnetic "string" tension

$$\varphi(\beta,z) = \lim_{L,T\to\infty} - \frac{1}{TL^{d-3}} \log\left(\frac{Z^z_{\beta,\Lambda}}{Z^0_{\beta,\Lambda}}\right) .$$

One will introduce analogous functions associated with other excitations of dimension ≥ 1 , in particular with the stable ones, (like the magnetic current lines in the four-dimensional $U(1)$ model.) Point-like excitations are studied in terms of "topological susceptibilities" and sum rules (like the Stillinger-Lovett sum rule for the gas of magnetic monopoles in the three-dimensional $U(1)$ model.)

These quantities will be studied in more detail elsewhere.

Next, we summarize some recent results.

Theorem 6.

1) [27] Let $G = \mathbb{Z}_n$, let $d\mu_\beta$ be given by eq. (47) with an action $A(g_\Lambda)$ defined as in (51). (This is the n-states Potts lattice gauge theory.) Then there exists an analytic interpolation in n (of thermodynamic functions and correlations) in a neighborhood of the positive real axis with the property that the model corresponding to the limit $n \to 1$ is the plaquette percolation model of branched random surfaces defined in Sect. 3, a), (ensemble $\mathcal{E}_{\text{perc.}}$)

2) [28] Let $G = SU(n)$, and renormalize the gauge fields such that

$$g_{xy}^{-1} g_{xy} = g_{yx} g_{xy} = n \, \mathbb{1} \; .$$

Then, for β small enough, there exists an analytic interpolation in n with the property that the $n \to 0$ limit yields the model of selfavoiding random surfaces defined in Sect. 3, b) (ensemble $\mathcal{E}_\gamma^{\text{s.a.}}$), in particular

$$\lim_{n \to o} n^{-|\gamma|} \langle W_{\chi_o}(\gamma) \rangle_\beta = Z_{\beta',o}(\gamma) \; ,$$

where $Z_{\beta,\mu}(\gamma)$ is defined in (36).

This result motivates the definition and analysis of the models introduced in Sect. 3.

Next, we discuss some results which are related to the ones in Sect. 1. They are based on the correlation inequalities in [7] which are only known to hold for abelian gauge groups and an action $A(g_\Lambda)$ given by expression (50), (i.e. the Wilson action.) The analogue of Theorem 3 is the statement that if the free energy $f(\beta)$ is continuously differentiable at some value $\beta = \beta_o$ then there exists only one translation invariant state, $\langle (\cdot) \rangle_{\beta_o}$, for $\beta = \beta_o$ [7]. Thus non-uniqueness of the vacuum functional in an (abelian) lattice gauge theory only occurs at a first order transition. A result analogous to Theorem 1 is

Theorem 7. [7]

1) If $\alpha(\beta,\chi) = 0$, and $\langle (\cdot) \rangle_\beta^\chi$ is invariant under translations in the 1-2 plane then

$$\langle (\cdot) \rangle_\beta^\chi = \langle (\cdot) \rangle_\beta \quad ,$$

(i.e. the electric flux sheet is completely rough.)

2) If $\varphi(\beta,z) = 0$, and ... then

$$\langle(\cdot)\rangle_\beta^z = \langle(\cdot)\rangle_\beta \quad ,$$

(i.e. the vortex sheet is rough or "fat".)

3) In the $U(1)$ models,

$$\varphi(\beta,z) = 0 \text{ , for all } \beta \quad ,$$

(i.e. $U(1)$-vortices are always fat. This is the analogue of the results for the rotator model described in Sect. 1.)

4) [6] In the three-dimensional $\mathbb{Z}_2$ model

$$\alpha(\beta) > 0 \Leftrightarrow \varphi(\beta) = 0 \quad . \qquad \square$$

Next, we would have to discuss roughening transitions in lattice gauge theories. The electric flux sheet bounded by an (infinitely extended) Wilson loop may, a priori, undergo a roughening transition which does not coincide with a deconfining transition [32,4]. That transition can be described by the following "order parameter" :

$$D(\beta,n) \equiv \langle D_z(\partial\Sigma)\rangle_\beta^\chi \quad , z \neq e \quad , \tag{55}$$

where Σ is a $(d-2)$-dimensional, rectangular array of sites with sides of length $2n$ which is centered at the origin and is perpendicular to the plane containing the Wilson loop. In the three-dimensional $\mathbb{Z}_2$ model the parameter $D(\beta,n)$ defined in (55) is dual to the parameter $D(\beta,n)$ introduced in Sect. 1, (10). For small β one expects that the phase of $D(\beta,n)$ approaches the value $\arg z$ (the phase of the central element z) exponentially fast, as $n \to \infty$. This can presumably be proven by a fairly straightforward extension of the arguments in [8,10]. The behaviour of the function $D(\beta,n)-D(\beta,\infty)$ is a measure for the fluctuations of the infinite flux sheet in directions perpendicular to the plane of the Wilson loop.

The roughening transition is characterized by the circumstance that, for all $\beta > \beta_R$,

$$\arg D(\beta,n) = 0 \quad , \quad \text{for all } n \quad , \tag{56}$$

while, for $\beta < \beta_R$,

$$\lim_{n\to\infty} \arg D(\beta,n) \neq 0 \,.^{1)} \tag{57}$$

(Here $e_R = \sqrt{\beta_R^{-1}}$ is the coupling constant at which the roughening transition occurs.)

It follows from Theorem 7 that, in abelian lattice gauge theories,

$$\alpha(\beta,\chi_o) = 0 \Rightarrow \arg D(\beta,n) = 0 \text{ , for all } n \text{ ,}$$

i.e.

$$\beta_c \geq \beta_R \quad , \tag{58}$$

where β_c is the point at which the deconfining transition occurs.

It is expected that the roughening transition can also be characterized by the vanishing of the step free energy, $\sigma(\beta)$, i.e.

$$\begin{aligned} \sigma(\beta) &> 0 \quad , \quad \text{for} \quad \beta < \beta_R \quad , \\ \sigma(\beta) &= 0 \quad , \quad \text{for} \quad \beta > \beta_R \quad . \end{aligned} \tag{59}$$

However, there are no rigorous results about roughening transitions known, yet.

Besides chromo-electric flux sheets there can exist other two-dimensional "topological" excitations, like vortex sheets, exhibiting a roughening transition. Such a transition should only occur in a phase characterized by a non-vanishing surface free energy of the excitations in question, (the analogue of the string- or surface tension. Recall that, in the four-dimensional $U(1)$ model, $\varphi(\beta,z) = 0$ implies that $\langle(\cdot)\rangle_\beta^z = \langle(\cdot)\rangle_\beta$ is translation invariant!) One expects that in the confinement phase of a (lattice) gauge theory only the string tension is non-vanishing, i.e. only the chromo-electric flux sheet may exhibit a roughening transition, while other two-dimensional defects, e.g. the vortex sheets in four-dimensional theories, are rough or "fat" throughout that phase.

Heuristically, roughening transitions in lattice gauge theories can be describe in terms of the models studied in Sect. 2, like the s-o-s model, but there is no rigorous justification of such approximate theories, yet.

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1) This characterization has been developed in collaboration with E. Seiler.

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