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SOME ANALYTICAL RESULTS ON THE ORNSTEIN-UHLENBECK SEMIGROUP IN INFINITELY MANY DIMENSIONS

by P. A. Meyer

The infinite dimensional Ornstein-Uhlenbeck process and the corresponding calculus with << laplacians >> and << gradients >> on Wiener space has recently been used by Malliavin, to prove results on hypoellipticity of second order p.d.e.'s (Hörmander's theorem) by probabilistic methods. The classical approach to the most elementary problems of this kind uses singular integrals. So one may wonder whether there aren't on the infinite dimensional space itself some underlying << singular integral theorems >> . We are going to show that indeed this is the case though the results don't seem to simplify Malliavin's method of proof. At the same time, we shall relate these results to the logarithmic Sobolev inequality of Gross, which can be interpreted as a << Riesz potential inequality >> in infinitely many dimensions, relative to the Ornstein-Uhlenbeck semigroup.

The results will be presented and commented, but the reader will be referred to $[4]_a$ and $[4]_b$ for the proofs, except at the end of the paper where an improvement of the results will be given in full detail.

I must express here my regrets that my mathematical education has been restricted to martingale theory, and has left me ignorant about gaussian measures on general infinite dimensional spaces. It is obvious for me that the results have little to do with the particular structure of Brownian motion, and should be extended to abstract Wiener spaces. My ignorance also makes me feel uneasy about giving credit to other authors. I consider that all the definitions and results for the case of $\underline{\underline{L}}^2$ should be considered as known, thanks to the work of Feissner, Gross, Hida, Kuo, Nelson... but that $\underline{\underline{L}}^p$ results, $p\neq 2$, aren't likely to be known - though there is some overlap with the work of Shikegawa, also motivated by the methods of Malliavin.

Finally, let me mention that the work concerning the logarithmic Sobolev inequality has been done jointly with Dominique Bakry.

I. STANDARD SINGULAR INTEGRAL RESULTS IN $\mathbb{R}^d$

Let us first recall some quite classical results on Fourier multipliers, whose proofs can be found in Stein's book [6]. The analogy will guide us throughout the paper, and we'll carefully keep the same notations to help the comparison of results.

NOTATION	OPERATOR ON FUNCTIONS ON $E = \mathbb{R}^d$	CORRESPONDING FOURIER MULTIPLIER
D_k	Partial derivative	iu_k
L	Laplacian	$-\lvert u\rvert^2$
P_t	Brownian motion semigroup (analysts' normalization)	$e^{-t\lvert u\rvert^2}$
Q_t	Cauchy semigroup (also called half-space Poisson kernel)	$e^{-t\lvert u\rvert}$
C	Cauchy generator ($C=-\sqrt{-L}$	$-\lvert u\rvert$
R	Potential of (P_t), newtonian potential	$1/\lvert u\rvert^2$
V	Potential of (Q_t), Cauchy potential	$1/\lvert u\rvert$
R^α	Riesz potentials(usually α is restricted to $0\leq\alpha<d/2$, and our R^α is called $I^{2\alpha}$)	$c_\alpha\lvert u\rvert^{-2\alpha}$

The use of the probabilists' normalization for brownian motion would lead to factors 2 or $\sqrt{2}$ at many places. Let us state the main classical results.

RIESZ TRANSFORMS THEOREM. The multipliers $iu_k/|u|$ define bounded operators on $\underline{\underline{L}}^p(\mathbb{R}^d)$ for $1<p<\infty$ (Stein [6], chapter III).

These operators will be denoted by $\mathcal{R}_k$. If d=1, $\mathcal{R}$ is the Hilbert transform.

RIESZ POTENTIALS THEOREM. The operators R^α maps boundedly $\underline{\underline{L}}^p$ into $\underline{\underline{L}}^q$ ($1<p<\infty$, $0<\alpha<d/2p$, $1/q = 1/p - 2\alpha/d$ (Stein [6], p. 119).

Taking into account the first theorem, the second one is closely related to the classical Sobolev theorem (Stein, p. 125).

Let us state the first theorem in another way, which will extend more easily to a general set up. Let f belong to the Schwartz space $\underline{\underline{S}}$; then the Riesz transform $\mathcal{R}_k Cf$ is $-D_k f$, and the theorem tells that $\|D_k f\|_{\underline{\underline{L}}^p} \leq c_p \|Cf\|_{\underline{\underline{L}}^p}$ [1]. Let us set

$$\Gamma(f,g) = \Sigma_k\; D_k f D_k g$$

then we also have

$$\|\sqrt{\Gamma(f,f)}\|_p \leq c_p \|Cf\|_p \tag{1}$$

On the other hand, we have $Cf = \Sigma_k\, \mathcal{R}_k D_k f$, from which we can deduce that the above inequality in fact is a norm equivalence in $\underline{\underline{L}}^p$. The point here is the fact that we can forget now about the differentiable structure, partial derivatives, etc : $\Gamma(f,g)$ can be defined from the semigroup only,

1. It is understood that a constant like c_p may vary from place to place. Also we denote the L^p norm simply by $\|\cdot\|_p$ if no confusion can arise.

through the formula $\Gamma(f,g) = L(fg)-fLg-gLf$.

II. GENERAL SYMMETRIC SEMIGROUPS

Let now E be a nice measurable space, μ be a σ-finite measure on E, and (P_t) be a Markov semigroup on E, such that $P_t 1=1$, and symmetric with respect to μ :

$$< P_t f,g >_\mu = < f,P_t g >_\mu \qquad (f,g \text{ bounded and measurable }).$$

It is well known that P_t induces a contraction[1] on every space $\underline{\underline{L}}^p(\mu)$. As a bounded self-adjoint operator in $\underline{\underline{L}}^2$, it is well known that P_t has the following representation - with a spectral family independent of t

$$P_t = \int_0^\infty e^{-t\lambda} dE_\lambda$$

Let $\phi(\lambda)$ be a function on $[0,\infty[$. To stress the analogy with the preceding section, we shall say that the (possibly unbounded) self-adjoin operator $\int_0^\infty \phi(\lambda)dE_\lambda$ corresponds to the <u>spectral multiplier</u> ϕ. Then we may extend as follows the preceding list to the abstract situation : only the partial derivative operators have disappeared.

	OPERATOR	SPECTRAL MULTIPLIER
P_t	Semigroup	$e^{-t\lambda}$
L	Generator (<< laplacian >>)	$-\lambda$
Q_t	Cauchy semigroup	$e^{-t\sqrt{\lambda}}$
C	Cauchy generator ($-\sqrt{-L}$)	$-\sqrt{\lambda}$
R	Potential	λ^{-1}
V	Cauchy potential ($R^{1/2}$)	$\lambda^{-1/2}$
R^α	Riesz potential	$\lambda^{-\alpha}$

The last three multiplier functions $\phi(\lambda)$ are defined to be equal to 0, not to $+\infty$, for $\lambda=0$.

The problem of giving conditions on the multiplier so that the corresponding operator acts boundedly on $\underline{\underline{L}}^p$ ($1<p<\infty$) has been studied by Stein [7] using Littlewood-Paley methods (martingale methods !). The most important case for which the answer is positive is $\phi(\lambda)=\lambda^{iu}$, corresponding to a Riesz potential of purely imaginary order.

If the domain of L contains a sufficiently rich algebra, we may also define the bilinear operator $\Gamma(f,g)=L(fg)-gLf-fLg$, and it turns out that this is formally a <u>positive</u> bilinear function. So it has a meaning to raise the following problems

<< RIESZ TRANSFORM >> PROBLEM . Are the norms $\|\sqrt{\Gamma(f,f)}\|_p$ and $\|Cf\|_p$ equivalent, $1<p<\infty$?

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1. We assume the semigroup is strongly continuous on these spaces.

(The answer is trivially << yes >> for $p=2$).

<< RIESZ POTENTIALS >> PROBLEM . Which are the << smoothing properties >> of R^{α} ?

This certainly is a very vague problem, while for the first one some results are known : for instance, Stein's methods in [7] work for compact Lie groups. On the other hand, the answer is positive for all convolution semigroups in $\mathbb{R}^d$ for $p\geq 2$ ($[5]_a$ and $[5]_b$).

III. THE ORNSTEIN-UHLENBECK SEMIGROUP

From now on E will be the space of all continuous functions w from $[0,\infty[$ to $\mathbb{R}^d$ such that $w(0)=0$, with the usual Borel σ-field. The measure μ of section II will be the standard Wiener measure on E. We are going to define a Markov semigroup (P_t) on E, symmetric with respect to μ. Probably the shortest way to define it consists in throwing in the Mehler formula

$$(1) \qquad P_t(w,f) = \int f(we^{-t/2}+u\sqrt{1-e^{-t}})\mu(du)$$

which obviously defines a Markov kernel. One then must check the semigroup property and μ-symmetry by having P_t act on some simple functions. Though we shall not go into details, we'll need a few facts below.

Let E' be the space of all mappings α from $\mathbb{R}_+$ to $\mathbb{R}^d$, right continuous, such that $\alpha(0)=0$, with compact support and bounded variation. The duality form between E and E' is $\{w,\alpha\} = -\int_0^{\infty} w(s)d\alpha(s)$ ($w\in E$). We set

$$(2) \qquad q(\alpha) = \int_0^{\infty} |\alpha(s)|^2 ds \ , \ e_{\alpha}(w) = e^{i\{w,\alpha\}} \ , \ \varepsilon_{\alpha}(w) = e_{\alpha}(w)e^{-q(\alpha)/2}$$

The Fourier transform of μ is $\hat{\mu}(\alpha)=e^{-q(\alpha)/2}$. Mappings $\{\cdot,\alpha\}$ are called <u>linear</u>[1] and the algebra they generate is that of <u>polynomials</u> on E. On the other hand, finite linear combinations of functions e_{α} (or ε_{α}) are called <u>trigonometric polynomials</u>, and constitute an algebra by themselves. Both algebras are dense in all spaces $\underline{L}^p(\mu)$, $1\leq p<\infty$.

P_t acts very simply on trigonometric polynomials :

$$(3) \qquad P_t\varepsilon_{\alpha} = \varepsilon_{\alpha e^{-t/2}}$$

formula from which the semigroup property and symmetry can easily be deduced. On polynomials, the action of P_t can be described as follows. Let $\alpha_1,\dots,\alpha_n$ be a finite orthonormal system in E' (with respect to the quadratic form q) and let $H(x_1,\dots,x_n)$ be a Hermite polynomial of degree k on $\mathbb{R}^n$. Let $h(w)$ be the polynomial $H(\{w,\alpha_1\},\dots,\{w,\alpha_n\})$ on E. Then we have

$$(4) \qquad P_t h = e^{-kt/2}h$$

1. We could also consider linear functionals $\{w,\alpha\}=\int_0^{\infty}\alpha(s)dw(s)$ (stochastic integral) defined μ-a.e., for $\alpha\in L^2(\mathbb{R}_+,\mathbb{R}^d)$. 0

It follows that the algebra of polynomials is closed under the action of (P_t), and of its generator L . A consequence is the possibility of using the operator Γ .

From the point of view of spectral decomposition, P_t is very easily described. Let $\mathcal{E}_k$ be the orthogonal projection in $\underline{\underline{L}}^2(\mu)$ onto the k-th Wiener chaos ($\mathcal{E}_0 f$ is simply the integral $\mu(f)$). Then we have

(5) $$P_t f = \Sigma_k \; e^{-kt/2} \mathcal{E}_k$$

Therefore the spectral multipliers $\phi(\lambda)$ of section II are really multiplier sequences $\phi(k)$, and the gap between 0 and the first eigenvalue 1/2 greatly simplifies the theory of the Riesz multipliers : $(k/2)^{-\alpha}$ for $k>0$ (0 for $k=0$) is a bounded multiplier sequence for complex α of real part $\geqq 0$.

IV . LOGARITHMIC SOBOLEV INEQUALITIES

The well known logarithmic Sobolev inequality of Gross is the following. Let f belong to $\mathcal{D}^2(L)$, the domain of L in $\underline{\underline{L}}^2(\mu)$. Then f also belongs to the Orlicz space $\underline{\underline{L}}^2\log\underline{\underline{L}}$, and we have

(6) $$\mu(|f|^2\log|f|) \leqq \|f\|_2^2\log\|f\|_2 - 2\langle Lf,f\rangle_\mu$$

This is a very sharp inequality, and we are going to lose some information in its interpretation as follows : assume $\mu(f)=0$, and set $-Cf=g$ so that $f=Vg$. Then the last term on the right is just $2\langle g,g\rangle=2\|g\|_2^2$. On the other hand, V is given by a bounded multiplier sequence, so it is bounded from $\underline{\underline{L}}^2$ to $\underline{\underline{L}}^2$, and the first term on the right is negative for $\|g\|_2$ small. So we deduce from (6) that $V=R^{1/2}$ <u>is a bounded operator from $\underline{\underline{L}}^2$ to $\underline{\underline{L}}^2\log\underline{\underline{L}}$</u> .

The beautiful results of Feissner [3] extending the inequality of Gross can also be interpreted (less obviously) as a statement on V : <u>it maps boundedly $\underline{\underline{L}}^2\log^n\underline{\underline{L}}$ into $\underline{\underline{L}}^2\log^{n+1}\underline{\underline{L}}$ for integer</u> n (positive or negative). This gives at once regularization properties for $R^{k/2}$, $k\in\mathbb{N}$, and since we may define R^z for complex z, the natural idea is to try complex interpolation (as Feissner himself did). On the other hand, Stein's result mentioned in section II is a theorem of the same kind in $\underline{\underline{L}}^p$, for purely imaginary z. So the first work consists in extending Stein's theorem to Orlicz spaces (the Burkholder-Davis-Gundy inequalities of martingale theory will care for that), and the second part of the proof is complex interpolation. We get :

THEOREM 1. The Riesz potential operator R^z for z complex, $\mathrm{Re}(z)=\varepsilon\geqq 0$, maps boundedly the Orlicz space $\underline{\underline{L}}^p\log^s\underline{\underline{L}}$ into $\underline{\underline{L}}^p\log^{s+p\varepsilon}\underline{\underline{L}}$, for $1<p<\infty$, s real.

V. SINGULAR INTEGRALS

We introduce some notation . If k is a positive integrer, we denote by $\mathcal{P}^p(L^k)$ the closure of the space of polynomials under the norm

$$(7) \qquad \|f\|_{p,k} = (\|f\|_p^p + \|Lf\|_p^p + \dots + \|L^k f\|_p^p)^{1/p}$$

Given $\alpha \in E'$, we denote by $D_\alpha f$ the derivative of the polynomial f along $\bar{\alpha}(t) = \int_0^t \alpha(s)ds$

$$D_\alpha f(w) = \lim_{s \to 0} \frac{1}{s}(f(w+s\bar{\alpha})-f(w))$$

One checks quite easily that $D_\alpha\{\cdot,\beta\}$ is the constant $q(\alpha,\beta)$, the bilinear form corresponding to q . Therefore the derivative of a polynomial is again a polynomial of lower degree. We have the very important formula

$$(8) \qquad P_t D_\alpha = e^{t/2} D_\alpha P_t \ , \ LD_\alpha = D_\alpha L + \tfrac{1}{2} D_\alpha$$

and

$$(9) \qquad \Gamma(f,f) = \Sigma_n (D_{\alpha_n} f)^2$$

for any orthonormal basis of $\underline{\underline{L}}^2(\mathbb{R}_+,\mathbb{R}^d)$ consisting of elements of E'. The following statement corresponds exactly to the boundedness of the Riesz transforms in $\mathbb{R}^d$ in section I :

THEOREM 2. Let f be a polynomial, or more generally belong to $\mathcal{P}^1(L)$. Then we have a norm equivalence in every $\underline{\underline{L}}^p(\mu)$, $1<p<\infty$

$$(10) \qquad \|\sqrt{\Gamma(f,f)}\|_p \sim \|Cf\|_p \ .$$

The proof is quite technical, resting on Littlewood-Paley inequalities and properties (8), (9). The probability of a mistake in it seems to be small, but not 0 . However, a completely different proof by Muckenhoupt [1] confirms the result in dimension 1.

We are going to extend this result to higher order gradients. This extension is quite superficial, since we'll use the weakening of (10) which consists in using Lf instead of Cf on the right side ($Cf=-VLf$, and V is bounded from $\underline{\underline{L}}^p$ to itself), and only the $\lesssim$ half of the equivalence. The reason for this weakening is the difficulty in handling the commutator of D_α and C, while that of D_α and L is so simple (8).

We denote by (α_n) a fixed basis of $\underline{\underline{L}}^2(\mathbb{R}_+,\mathbb{R}^d)$ as above. Given $m=(n_k,\dots,n_1)\in\mathbb{N}^k$ we denote by D_m the operator $D_{\alpha_{n_k}}\dots D_{\alpha_{n_1}}$, and we define for any polynomial f [2]

$$(11) \quad \Gamma_0(f,f)=f^2 \ , \ \Gamma_1(f,f)=\Gamma(f,f)=\Sigma_{m\in\mathbb{N}} (D_m f)^2, \quad \Gamma_k(f,f)=\Sigma_{m\in\mathbb{N}^k} (D_m f)^2$$

THEOREM 3 . $\|\sqrt{\Gamma_k(f,f)}\|_p \le c_p \|f\|_{p,k}$ (f polynomial, $1<p<\infty$, $k\in\mathbb{N}$).

1. Hermite conjugate expansions. TAMS 139, 1969, p. 243-260.
2. We'll see below that these functions don't depend on the basis.

PROOF. Consider a system of Rademacher functions $r_m(t)$ indexed by $\mathbb{N}^k$ and set $f_m = D_m f$. We apply theorem 2 to the function $g_t = \Sigma_{m\in I}\, r_m(t) f_m$, with I a finite set of $\mathbb{N}^k$:

$$\|(\Sigma_n (D_{\alpha_n} g_t)^2)^{1/2}\|_p^p \leq c_p \|Lg_t\|_p^p$$

and we integrate in t. On the right side, we have $E[\int |\Sigma_{m\in I}\, r_m(t) Lf_m|^p dt]$, which (according to Khinchin's inequality) is equivalent to $\|(\Sigma_{m\in I} (Lf_m)^2)^{1/2}\|_p^p$. On the other hand , according to (8) we have $Lf_m = D_m Lf + \frac{k-1}{2} D_m f$, and we can dominate the right side by

$$c_p \| \sqrt{\Gamma_k(Lf + \tfrac{k-1}{2}f,\ Lf + \tfrac{k-1}{2}f)}\|_p^p$$

On the left side, denote by G the mapping $f \mapsto (D_{\alpha_n} f)_{n\in\mathbb{N}}$ from polynomials to sequences of polynomials. Then what we have is $E[\int \|\Sigma_{m\in I}\, r_m(t) Gf_m\|_{\ell^2}^p dt]$. Since Khinchin's inequality is also valid in a Hilbert space, this is equivalent to $\|(\Sigma_{m\in I} \|Gf_m\|_{\ell^2}^2)^2\|_p^p$. Letting now I increase to $\mathbb{N}^k$, we get that

$$\|\sqrt{\Gamma_{k+1}(f,f)}\|_p^p \leq c_p \|\sqrt{\Gamma_k(Lf + \tfrac{k-1}{2}f, Lf + \tfrac{k-1}{2}f)}\|_p^p$$

Theorem 3 follows at once by induction on k .

As a consequence, we get the finiteness of $\Gamma_k(f,f)$ a.e., hence the possibility of defining $\Gamma_k(f,g) = \frac{1}{2}(\Gamma_k(f+g,f+g) - \Gamma_k(f,f) - \Gamma_k(g,g))$.

The following result implies at once, by induction on k, the following results : if f is a polynomial, $\Gamma_k(f,f)$ doesn't depend on the choice of the basis (α_n), and is also a polynomial.

THEOREM 4. Let f be a polynomial. Then we have

$$\Gamma_{k+1}(f,f) = L\Gamma_k(f,f) - 2\Gamma_k(f,Lf) - k\Gamma_k(f,f)$$

PROOF. We have

$$\begin{aligned} L((D_m f)^2) &= 2D_m f\, LD_m f + \Gamma(D_m f, D_m f) \\ &= 2D_m f\, D_m Lf + k(D_m f)^2 + \Gamma(D_m f, D_m f) . \end{aligned}$$

Sum on $m\in I$, a finite subset of $\mathbb{N}^k$, and let I increase to $\mathbb{N}^k$. Then $\Sigma_{m\in I} (D_m f)^2$ increases to $\Gamma_k(f,f)$, and the convergence takes place in $\underline{\underline{L}}^1$ since the limit belongs to $\underline{\underline{L}}^1$ (theorem 3). To prove theorem 4, since L is a closed operator, we need only show that the right side converges in $\underline{\underline{L}}^1$. Now

$k\Sigma_{m\in I} (D_m f)^2$ converges in $\underline{\underline{L}}^1$ to $k\Gamma_k(f,f)$ (theorem 3)

$\Sigma_{m\in I} D_m f\, D_m Lf$ converges in $\underline{\underline{L}}^1$ to $\Gamma_k(f,Lf)$

(here use polarization to get monotone convergence). Finally

$\Sigma_{m\in I}\, \Gamma(D_m f, D_m f) = \Sigma_{m\in I,n} (D_{\alpha_n} D_m f)^2$ increases to $\Gamma_{k+1}(f,f)$

Theorem 4 follows.

With the help of the preceding results, we can prove completely a statement which was stated in [4] as a reasonable conjecture :

THEOREM 5. Let f and g belong to $\mathcal{D}^{2p}(L^k)$ ($1<p<\infty$). Then their product fg belongs to $\mathcal{D}^p(L^k)$.

COROLLARY. Let $\underline{\underline{T}}$ be $\bigcap_{p,k} \mathcal{D}^p(L^k)$. Then $\underline{\underline{T}}$ is an algebra.[1]

PROOF. This can be reduced to a problem on polynomials : show that $\|fg\|_{p,k} \leqq c_{p,k}$ if $\|f\|_{2p,k} \leqq 1$, $\|g\|_{2p,k} \leqq 1$. By polarization, we may reduce to f=g . Since everything is trivial at level 0, we use induction. Assume

$$\text{if } \|f\|_{2p,k} \leqq 1 \text{ , then for all } i\leqq k \ \|\Gamma_i(f,f)\|_{p,k-i} \leqq c_{p,k,i}$$

We prove that the same property holds at level k+1. The new inequalities to prove are that, if $\|f\|_{p,k+1} \leqq 1$

$$\|\Gamma_{k+1}(f,f)\|_{p,0} \leqq c_{p,k+1,0}$$

which is precisely theorem 3, and also that

$$\|L\Gamma_i(f,f)\|_{p,k-i} \leqq c_{p,k+1,i}$$

which follows from the induction hypothesis and theorem 4 :

$$L\Gamma_i(f,f) = 2\Gamma_i(f,Lf) + i\Gamma_i(f,f) + \Gamma_{i+1}(f,f) \text{ .}$$

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1. This corollary simplifies some of the logic in Stroock [8].