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1. Introduction.

The classical approach to the foundations of quantum theory, pioneered by Birkhoff and von Neumann [1], sought axioms for quantum logic all of whose realizations could be represented by orthogonal projection operators in suitable Hilbert spaces. B. Mielnik [4] objected that this program would not fully justify the use of Hilbert spaces in quantum theory, for that theory deals not only with the yes-no measurements considered in quantum logic, but also with statistical predictions. Accordingly, he introduced the following concept of probability space, and showed that not all of its realizations could be obtained from Hilbert spaces in the usual quantum-theoretic manner.

A *probability space* (S,p) consists of a nonempty set S together with a function p from S×S into the closed interval [0,1] of the real line, satisfying the following three axioms.

(A) p(a,b) = 1 if and only if a = b.

(B) p(a,b) = p(b,a).

To state the last axiom concisely, we first define two members a and b of S to be *orthogonal* if p(a,b) = 0. A maximal set of pairwise orthogonal elements of S is called a basis of S. The existence of bases is an easy consequence of Zorn's lemma.

(C) For any $a \in S$ and any basis B of S,

$$\sum_{b \in B} p(a,b) = 1,$$

where the (possibly infinite) or even uncountable sum is interpreted, as usual, as the supremum of all its finite partial sums.

The elements of S are to be viewed as possible states of a physical system, and p(a,b) as the probability that a system is found to be in state b after known to be in state a. In the usual quantum-mechanical formalism, S is the set of unit vectors of a Hilbert space (modulo identification of a vector a with all its scalar multiples λa, $|\lambda| = 1$) and $p(a,b) = |\langle a,b \rangle|^2$. A basis for an S of this sort is just an orthonormal basis for the Hilbert space. Mielnik [4] showed that, in any probability space (S,p), all bases have the same cardinality, called the dimension of (S,p). Thus, the dimension of the probability spaces used in quantum mechanics is the same as the dimension of the associated Hilbert space.

There is another way to associate a probability space with any given inner product space N. Let S be the unit sphere of N (without identifying vectors that differ by a phase factor), and let

$$p(a,b) = \tfrac{1}{4}||a+b||^2. \qquad (1)$$

Although only the norm of N, not the inner product, appears explicitly in the definition of (S,p), the fact that the norm is associated to an inner product is used in verifying that (S,p) satisfies Mielnik's axiom (C) (the other two axioms being immediate). Indeed, the only vector orthogonal to a in (S,p) is -a, so (C) reads

$$(2) \qquad \tfrac{1}{4}||a+b||^2 + \tfrac{1}{4}||a-b||^2 = 1,$$

which follows from the parallelogram law. Notice that (S,p) has dimension 2; among these spaces are Mielnik's examples of probability spaces not obtainable from Hilbert spaces in the usual quantum-mechanical way.

In [5], it was shown that the formula (1) defined a probability space structure on the unit sphere of a normed real linear space N if and only if N is an inner product space. In [6], this result was generalized to show that, if S is the unit sphere of a normed real linear space and if (S,p) is a probability space of dimension 2 in which p(a,b) is any reasonable function of $||a+b||$, then N is an inner product space. To state this result, and our later results, precisely, we introduce, as in [6], the class

$$(3) \quad F = \{f | f:[0,2] \to [0,1],\ f \text{ continuous and strictly increasing},\ f(0) = 0,\ f(2) = 1\}.$$

Then Theorem 3.1 of [6] asserts that a normed linear space N is an inner product space if and only if its unit sphere S, equipped with some p of the form $p(a,b) = f(||a+b||)$, $f \in F$, is a Mielnik probability space (necessarily of dimension 2).

In this paper, we shall introduce the notion of a partial probability space and use it to obtain a characterization of uniformly convex spaces analogous to the characterization of inner product spaces just quoted.

2. Partial probability spaces.

A partial probability space is a pair (S,p), where S is a nonempty set and p maps S×S into [0,1] in such a way that axioms (A) and (B) for probability spaces and the following weakened form of axiom (C) hold.

(C*) For any a ε S and any basis B of S,

$$\sum_{b\varepsilon B} p(a,b) \leq 1.$$

Partial probability spaces abound, for any nonempty subset of a probability space is a partial probability space. To see this, simply observe that a basis for such a subset is a subset of a basis for the whole probability space. It is also easy to construct partial probability spaces which are not subspaces of any probability space. We shall consider later the problem of finding conditions under which a partial probability space can be embedded in a probability space.

Although there is no immediate physical interpretation of (C*), it is related to the behavior of quantum-mechanical transition probabilities between states of an unstable system. If we let S consist of the states of such a system, say a neutron, and if we let p(a,b) be the probability that a neutron known to be in state a at a certain time is found to be in state b ten minutes later, then axiom (C) is not satisfied but axiom (C*) is, because there is a non-zero probability that the neutron will, during the ten minute interval, decay into a proton, an electron, and an anti-neutrino. Notice the relation between this observation and the remarks in the preceding paragraph: Axiom (C) would hold if we added to S all the states of the systems into which the neutron can decay. Notice also that our (S,p) fails to be a partial probability space because the "if" part of axiom (A) is false (and axiom (B) expresses the rather strong assumption of time-reversal invariance). It would perhaps be reasonable to weaken Mielnik's axioms further by omitting the "if" part of (A) and possibly also omitting (B). This would make no difference in our main result, the characterization of uniformly convex spaces below, since the additional hypotheses used there are sufficient to imply the omitted axioms.

3. Uniform convexity.

A normed real linear space N is uniformly convex if for every positive ε there is a positive δ such that, for $x,y \in N$,

$$||x|| \leq 1,\ ||y|| \leq 1, \text{ and } ||x-y|| \geq \varepsilon \text{ imply } ||\tfrac{1}{2}(x+y)|| \leq 1-\delta.$$

An equivalent condition is that, for sequences $\{x_n\}$, $\{y_n\}$ of vectors in N of norm ≤ 1, if $\lim_{n\to\infty}||\frac{1}{2}(x_n+y_n)|| = 1$, then $\lim_{n\to\infty}||x_n-y_n|| = 0$. In the paper [2] in which he introduced the concept of uniform convexity, Clarkson showed that the classical Banach spaces L_p $(1 < p < \infty)$ are uniformly convex. For $p \geq 2$, the proof is based on the inequality

$$(4) \qquad \left|\left|\frac{a+b}{2}\right|\right|^p + \left|\left|\frac{a-b}{2}\right|\right|^p \leq \frac{1}{2}||a||^p + \frac{1}{2}||b||^p,$$

where the norm is the L_p-norm. When a and b are on the unit sphere, the right side of (4) reduces to 1, and we obtain axiom (C*) for the probability function $\left|\left|\frac{a+b}{2}\right|\right|^p$, a function for which axioms (A) and (B) are easily seen to hold as well. Thus, Clarkson's inequality (4) implies that the unit sphere of L_p is a partial probability space with the above probability function. Note that, when $p = 2$, we obtain again the probability function (1) of Section 1. For other values of p, this partial probability space is not a probability space because L_p is not a Hilbert space.

To relate uniform convexity of a space N to partial probability space structures on its unit sphere S, we need the following characterization of uniformly convex spaces.

Lemma. A normed real linear space N is uniformly convex if and only if, for all sequences $\{a_n\}$, $\{b_n\}$ of unit vectors,

$$\lim_{n\to\infty} ||a_n+b_n|| = 2 \text{ implies } \lim_{n\to\infty} ||a_n-b_n|| = 0.$$

<u>Theorem</u>. Let N be a normed real linear space, and let S be its unit sphere. N is uniformly convex if and only if, for some $f \in F$, the probability function $p(a,b) = f(||a+b||)$ makes S a partial probability space.

<u>Lemma</u>. Suppose g maps [0,2] into [0,1], is monotone non-decreasing, takes the value 0 at 0 and nowhere else, maps 2 to 1, and is continuous at 2. Then there is a $g_1 \in F$ such that $g_1(t) \leq g(t)$ for all $t \in [0,2]$.

4. Additional Remarks.

The comments we made after the proof of the "if" part of our theorem show that the theorem remains true if the class F is replaced by either

$$F_1 = \{f \mid f:[0,2] \to [0,1],\ f \text{ strictly increasing, continuous at } 2,\ f(0) = 0,\ f(2) = 1\}$$

or

$$F_2 = \{f \mid f:[0,2] \to [0,1],\ f \text{ continuous, } f(2) = 1, \text{ and } f(t) = 0 \text{ iff } t = 0\}.$$

Furthermore, the theorem remains true if axioms (A) and (B) are omitted from the definition of partial probability spaces. Although this observation permits us to redefine partial probability spaces so as to include examples like the one discussed in the last paragraph of Section 2 (where the "if" part of (A) was false and (B) not immediately clear), the added generality would be irrelevant in our theorem as both the "if" part of (A) and (B) hold for any p of the assumed form $f(||a+b||)$, $f \in F$ (or F_1 or F_2). Notice, however, that the theorem remains true if we put the inequality $p(a,b) \geq f(||a+b||)$ in place of equality, and then (B) no longer follows.

We close this paper with a brief and incomplete discussion of conditions under which a partial probability space (S,p) is a subspace of a probability space. We define orthogonality and bases exactly as in probability spaces. It is no longer true, in general, that all bases have the same cardinality. Note, however, that in the spaces (S,p) occurring in the theorem of

Section 3, all bases have cardinality 2, for a and b are orthogonal iff a = -b. If $a \in S$ and $B \subseteq S$, we define

$$p(a,B) = p(B,a) = \sum_{b \in B} p(a,b),$$

and if $C \subseteq S$ also,

$$p(B,C) = \sum_{b \in B} p(b,C) = \sum_{c \in C} p(B,c) = \sum_{b \in B} \sum_{c \in C} p(b,c).$$

Thus, for example, (S,p) is a probability space iff p(a,B) = 1 for all $a \in S$ and all bases B of S. For orthogonal systems (i.e. sets of pairwise orthogonal elements) B and C, we write $B \leq C$ to mean that p(b,C) = 1 for all $b \in B$. Axiom (A) guarantees that $\leq$ is reflexive; transitivity of $\leq$ follows immediately from the following condition (D) which holds in some, but by no means all, partial probability spaces:

(D) If $B \leq C$ (where B and C are orthogonal systems) then, for all $a \in S$, $p(a,B) \leq p(a,C)$.

<u>Proposition</u> <u>1</u>. If the partial probability space (S,p) is a subspace of a probability space $(\overline{S},\overline{p})$, then (S,p) satisfies condition (D).

Condition (D), which we have just shown to be necessary for embeddability into a probability space, in some circumstances is also sufficient. We probe here just one result of this sort; the additional hypothesis about the bases is unnecessarily restrictive, but it holds in the spaces (S,p) considered in Section 3.

Proposition 2. If condition (D) holds in a partial probability space (S,p) all of whose bases have the same finite cardinality n, then (S,p) is a subspace of a probability space (of dimension 2n).

Proofs and details will appear in the Proceedings of AMS.

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