F. CONSTANTINESCU
J. G. TAYLOR

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EQUIVALENCE BETWEEN NON-LOCALIZABLE AND LOCAL FIELDS

by

J.G. Taylor,
Department of Mathematics,
King's College, London

F. Constantinescu,
Department of Applied Mathematics,
University of Frankfurt

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E R R A T A   T O

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(1) Page 7, line 5 from below should be $\Sigma = \underline{C} \oplus S(\underline{R}^4) \oplus \ldots \oplus S(\underline{R}^{4n}) \oplus \ldots$

(2) Page 8, lines 1,2 and 4 from below to read ϕ instead of f.

(3) Page 8, lines 1 and 10 from below
Page 9, lines 5,11,12,13, and 17 from below
Page 10, line 6 from below
Page 11, lines 3 and 5 from below
Page 14, lines 4,5 and 13 from bottom
Page 15, line 2
Page 25, line 12 from bottom

to read localisable instead of local.

(4) Page 15, lines 7 and 8 from below to Z instead of z

0. Abstract

We discuss the nature of non-localisable fields constructed as certain limits of sequences of local fields. For sequences for which the corresponding Wightman functions converge we construct a PCT operator; if the sequences converge strongly in a given Hilbert space then a scattering theory can be constructed for the non-localisable limit field. Such fields are shown to have the same S-operator as any local field which has the defining sequence of local fields in its Borchers class, and has the same in field. We give non-trivial examples of this equivalence between local and non-localisable fields.

1. Introduction

The problem of describing all relativistic quantum fields corresponding to a given S-matrix has not been yet solved. An important result in this direction was obtained by Borchers [1] in the frame of the (Wightman) axiomatic quantum field theory. According to this result of Borchers, fields are S-equivalent (i.e. correspond to the same S-matrix) if they are relatively local (or weakly relatively local). The relative locality (or the weak relative locality) is a relation of equivalence among quantum fields, so that all fields in a Borchers class (i.e. a class of relatively local or weak relatively local fields) are S-equivalent. The converse is not true: a Borchers class does not exhaust all fields with the same S-matrix (see for instance [2] p.170) but we do not consider this problem here.

The S-equivalence of relativistic quantum fields was also studied in perturbation theory; we refer the reader to [3] and references quoted there for detailed results.

Roughly speaking the above results (in the axiomatic or in perturbation theory) are known to physicists in the following form: two fields, one of them being a local function of the other one, have the same S-matrix.

We will show in this paper that quantum fields can be equivalent also in the "non-local" case. In particular we will show that a local quantum field can be S-equivalent to a non-localizable field. We think that this result can be of some interest because it shows that non-localizable fields can have a well-behaved S-matrix which is actually the S-matrix of a local quantum field. Some trivial and non-trivial examples are also given.

2. The non-localizable fields

We will adopt in this paper the Wightman formulation of a local quantum field theory. We will construct non-localizable fields along the general lines given recently by one of us [4]. We remark that there are also other interesting approaches to non-localizable (or non-local) fields [5,6] or to a scattering theory for non-local fields [7]. We hope to discuss the connexion of the limiting approach [4] to other approaches, especially to [7], in a separate publication. In [4] we look at non-localizable fields as limits of local ones in a topology considered already by Borchers [8] and Jaffe [9].

Let S_α and S^α ($\alpha > 0$) be spaces of test functions considered by Gelfand and Shilov ([10] Chapter IV). The test functions in S_α are roughly speaking infinitely differentiable functions $\phi(p)$ vanishing like $e^{-a|p|^{1/\alpha}}$ for $|p| \to \infty$ where $|p|$ is the Euclidean norm of $p(p^0,\bar{p})$ and a a positive constant which may depend on ϕ. A natural topology can be put on these test functions as in [10] Chapter IV. The elements of S^α are Fourier transforms of the functions in S_α. The spaces S_α and S^α are both nuclear and dense in S (the Schwartz space of infinitely differentiable functions vanishing at infinity, stronger than any polynomial). The set of functions which belong to S^α ($\alpha > 1$) and have compact support is dense in the Schwartz space $\mathcal{D}$

(infinitely differentiable functions with compact support).

Let us now consider a Wightman-Jaffe [11] type theory (over test functions in S^{α} in coordinate space) for the scalar neutral field $A(x)$. If for a given field $A(x)$, α can be chosen larger than one $A(x)$ can be localized in any finite region of space-time and local commutativity can be formulated as usual: $[A(\phi), A(\psi)]_{-} \Phi = 0$ for supp ϕ and supp ψ space-like separated ($\phi, \psi \in S^{\alpha}$) and Φ in the dense domain of definition for A).

For $\alpha < 1$ the functions in S^{α} are analytic, the field $A(x)$ is no longer localizable in a finite region of space-time and local commutativity for these fields cannot be formulated, at least in the usual fashion.

Such a behavior of $A(x)$ is generated evidently by a high energy behavior of these fields like $e^{\varepsilon |p|^{1/\alpha}}$ (with $\varepsilon \to 0+$) ($\alpha < 1$). We can look at the (non-localizable) fields $A(x)$ in this case (imposing a reasonable <u>generalized</u> local commutativity) as follows.

Let $S_n^{\alpha} = S^{\alpha}(\underline{R}^{4n})$ the S^{α} - space of test functions depending on $4n$ independent variables ($S_o^{\alpha} = \underline{C}$ = complex numbers). We construct the (locally convex) direct sum of S_n^{α}, $n = 0,1,2,\ldots$ (see for instance [12] p.214)

$$\sum^{\alpha} = \bigoplus_{n=o}^{\infty} S_n^{\alpha} \tag{1}$$

The elements of $\sum^{\alpha}$ are of the form $\phi = (\phi_o, \phi_1, \ldots, \phi_n, \ldots)$ with $\phi_n \in S_n^{\alpha}$ and ϕ has only a finite number of components. The locally convex topology in $\sum^{\alpha}$ can be given by a set of non-denumerable seminorms [12] and can be characterized by convergence as follows: $\{\phi^m\}$ is convergent to ϕ in $\sum^\alpha$ if and only if

i) there exists an integer N so that $\phi_n^m = 0$ for all $n > N$ and

ii) each component ϕ_n^m; $n = 0,1,2,\ldots$ converges to ϕ_n for $m \to \infty$ in the topology of S_n^α.

Let now $A_k(x)$, $k = 1,2,\ldots$ be a set of tempered scalar neutral relativistic quantum fields. This means that the $A_k(x)$ are tempered Schwartz distributions satisfying the following (Wightman) requirements |2|

a) Hilbert space of states

b) covariance under the inhomogeneous Lorentz group

c) positive energy

d) local commutativity

Evidently the $A_k(x)$'s are also operator valued generalized functions over the spaces S^α. For a given field $A_k(x)$ we consider all its vacuum expectation values W^k with test functions from S^α for a given α. For all $k = 1,2,\ldots$ we have (compare with [8] p.220)

1) $W_k \in \sum^{\alpha'}$, $k = 1,2,\ldots$

2) $W_k((a,\Lambda)\phi) = W(\phi)$, $\phi \in S^\alpha\ (\underline{R}^4)$

3) $W_k(\phi) = 0$ if $\phi \in M_{s_p}^{\alpha}$

4) $W_k(\phi) = 0$ if $g \in I_c$

5) $W_k(\phi^+ \times \phi) \geq 0$

In 1) $\sum^{\alpha'}$ is the dual space of $\sum^{\alpha}$, i.e. the space of all continuous linear functionals on $\sum^{\alpha}$. $\sum^{\alpha'}$ can be identified with the direct product of the spaces $S_n^{\alpha'} = S^{\alpha'}(\underline{R}^{4n})$ [12] but this fact will not be of special interest for us. We remark only that the space $\sum^{\alpha'}$ is complete [12] (see also [8] p. 235). The condition 2) represents the covariance under the inhomogeneous Lorentz transforms (a,Λ).
In 3) $M^{\alpha}_{s_p}$ is defined as

$$M^{\alpha}_{s_p} = \{\phi;\ \phi \in \textstyle\sum^{\alpha},\ D^{\gamma}(F\phi_n)(p_1,\dots,p_n) = 0 \text{ if}$$

$$p_n,\ p_{n-1} + p_n,\dots,p_2 + \dots + p_n \in \bar{V}_+ \text{ and } p_1 + p_2 + \dots p_n = 0\ \} \qquad (2)$$

$$\text{where } D^{\gamma} = \frac{\partial^{|\gamma|}}{\partial^{\gamma_1} p_1 \dots \partial^{\gamma_n} p_n}\ ,\ |\gamma| = \gamma_1 + \dots + \gamma_n;\ \gamma_1,\dots\gamma_n = 0,1,\dots$$

; $\bar{V}_+$ is the closed forward light cone and F stands for Fourier transform. Condition 3) represents the positivity of the energy.

In 4) (which is the local commutativity) I_c is defined as a linear subspace of $\sum = \underline{C} \oplus S(\ ^4) + \dots \oplus S(\ ^{4n}) \oplus \dots$ with the following base: $\phi(x_1,\dots,x_n)$ is an element of the base if f can be decomposed as the difference

$$f(x_1 \dots x_{i-1}\ x_i \dots x_k,\ x_{k+1} \dots x_n) - f(x_1 \dots x_{i-1},\ x_{i'} \dots x_{k'},\ x_{k+1} \dots x_n)$$

and $f(x_1 \dots x_i \dots x_k \dots x_n) = 0$ if $x_j - x_\ell$ is timelike for $j \neq \ell$;

$j,\ell = i, i+1,\ldots$; $(i'\ldots k')$ is any permutation of $(i \ldots k)$.

Finally, in 5) - which is the Hilbert space requirement, ϕ^+ is defined as

$$\phi^+ = (\bar{\phi}_o, \ldots, \overline{\phi_n(x_n,\ldots,x_1)}, \ldots)$$

where the bar means complex conjugation.

It is clear that $\sum^\alpha$ becomes an algebra with involution if the following definition for the product is chosen

$$\phi \times \psi = (\phi_o \psi_o, \phi_o \psi_1 + \phi_1 \psi_o, \ldots \sum_{i+k=n} \phi_i(x_1,\ldots,x_i)\, \psi_k(x_{i+1},\ldots,x_n), \ldots)$$

One can study the algebraic structure of $M^\alpha_{s_p}$ and I_c but we don't need to know this explicitly for the purposes of this paper, and we send the interested reader to [8].

All the (tempered) fields $A_k(x)$ were taken as local and the local commutativity given through 4) for $k = 1,2,\ldots$. Now we turn to the construction of the non local field $A(x)$ from the given tempered local fields $A_k(x)$. Let the sequence $W_k \in \sum^{\alpha'}$ be convergent in $\sum^{\alpha'}$. It is not difficult to prove (see for instance [8]) that the limit W of the sequence $\{W_k\}$ satisfies the conditions 1) - 5) with the possible exception of 4). Moreover, 4) is satisfied if $\alpha > 1$ but it cannot be satisfied for $\alpha < 1$.

Certainly from W we can reconstruct the fields and for $\alpha > 1$ we get a Wightman-Jaffe field. If $\alpha < 1$ $A(x)$ will continue to be a field in the Wightman sense which we call a non-local field (for a discussion of the case $\alpha = 1$ see [13]).

The crucial point in this construction is that the local commutativity of the fields $A_k(x)$, $k = 1,2,\ldots$ induces in the limit $k \to \infty$ a general local commutativity which we understand as a fast decrease of the commutator for $A(x)$ in space-like directions (for only a partial discussion of the extended commutator in space like directions see [14]. The result of [14] doesn't apply to a massive field in which case the decrease of the commutator in space like directions is much stronger; a more complete discussion is given in [15]. Certainly the (small) acausal effects we have described here depend on $\alpha < 1$ and tend to vanish in the limit $\alpha \to 1$ (for $\alpha > 1$ the commutor vanishes exactly for space time directions). The non-local fields constructed as above have many important properties as for instance PCT-invariance. A scattering theory for such fields exist and was discussed in [4].

Before going into the discussion of the equivalence between a local and a non local field let us consider a little further the limit $A_k \to A$ which we required as the basis of constructing non local fields. The non-local field $A(x)$ was recovered by the reconstruction theorem, which gives us also the Hilbert space $\underline{H}$ in which $A(x)$ acts. The reconstruction theorem gives us unfortunately not too much information about $\underline{H}$, which is constructed as the completion of a pre-Hilbert space in whose formation condition 5) enters essentially. It is therefore difficult to compare the Hilbert spaces in which two non local (or a local and a non-local) fields constructed as above are acting. Because fields which have the same PCT operator are good candidates for S-equivalence and because we have not yet enough control on the reconstructed Hilbert space, in looking for fields having the same PCT operator we will assume that all the fields A_k, $k = 1,2, \ldots$

and A are acting in the same Hilbert space (see examples in §6).

On the other hand it will be helpful for deriving certain results to consider also the following strong form of the limit $A_k \to A$:

Let $A_k(x)$ $k = 1,2,\ldots$ be (tempered) local fields acting in the same Hilbert space $\underline{H}$, having the same invariant domain of definition D and the same vacuum Ω and let $A(x)$ be a non-local fields constructed from $A_k(x)$. We assume that $A(x)$ has D as invariant domain and Ω as vacuum and moreover

$$s - \lim_{k\to\infty} A_k(x_1) \ldots A_k(x_n)\, \Omega = A(x_1) \ldots A(x_n)\, \Omega \tag{3}$$

where s - lim stands for the strong limit (i.e. the norm limit in $\underline{H}$). More precisely (3) means that for all $\phi_1,\ldots,\phi_n \in S^\alpha(\underline{R}^{4n})$ we have

$$s - \lim_{k\to\infty} A_k(\phi_1(x_1)) \ldots A_k(\phi_n(x_n))\, \Omega$$

$$= A(\phi_1(x_1)) \ldots A(\phi_n(x_n))\, \Omega \tag{4}$$

The convergence in (3) (or (4)) is stronger than the convergence of W_k in the sense of $\sum^{\alpha'}$. Writing (3) in the form

$$A_k(x_1) \ldots A_k(x_n)\Omega - A(x_1) \ldots A(x_n)\Omega \quad \to \quad 0 \text{ for}$$

$k \to \infty$, we can write (3) in terms of vacuum expectation values of the fields A_k, $k = 1,2,\ldots$ and A.

3. The PCT Operator

Let $A(x)$ be a (tempered) local scalar neutral field satisfying the Wightman axioms a) - d) of §2. Let D be the invariant dense domain in $\underline{H}$ on which $A(x)$ is defined. Let $U(a,\Lambda)$ be the unitary representation of the inhomogeneous group $P_+^\uparrow$ which corresponds to the field $A(x)$:

$$U(a,\Lambda) \quad A(\phi) \; U^{-1}(a,\Lambda) = \quad (\phi_{(a,\Lambda)}) \tag{6}$$

where $\phi \in S(\underline{R}^4)$ and $\phi_{(a,\Lambda)}(x) = \phi(\Lambda^{-1}(x - a))$. We assume that the vacuum $\Omega \in D$ for $A(x)$ is a cyclic vector (the field $A(x)$ is irreducible). The field $A(x)$ determines the Borchers class of all fields relatively local (or weak relatively local) to $A(x)$, corresponding to the same unitary representation of $U(a,\Lambda)$ of the inhomogeneous Lorentz group, the same invariant domain D (in the Hilbert space H), the same vacuum and also the same PCT operator.

Let now $A_k(x)$, $k = 1,2,\ldots$ be a sequence of fields in the Borchers class of $A(x)$ (there are always infinitely many fields equivalent to a given field!). We assume that the sequence $\{A_k(x)\}$ is convergent in the sense of (3) to a non-local field $B(x)$ defined in the same domain D as $A(x)$ and having the same vacuum Ω as a cyclic vector.

We have

Theorem 1 The non-local field $B(x)$ has a PCT operator θ_1 and $\theta_1 = \theta$ where θ is the PCT operator for the local (tempered) field $A(x)$.

Proof: The PCT theorem for $A_k(x)$, $k = 1,2,\ldots$ gives

$$(\Omega,\ A_k(x_1)\ \ldots\ A_k(x_n)\Omega) = (\Omega,\ A_k(-\ x_n)\ \ldots\ A_k(\ -\ x_1)\Omega) \tag{7}$$

or

$$(\Omega,\ A_k(\phi_1(x_1))\ \ldots\ A_k(\phi_n(x_n)\Omega) = (\Omega,\ A_k(\phi_n(\ -\ x_n)\ \ldots\ A_k(\phi_1(-\ x_1))\Omega). \tag{8}$$

Taking in (8) $k \to \infty$ for $\phi_1,\ldots,\phi_n \in S^\alpha(\underline{R}^4) \subset S(\underline{R}^4)$ one gets

$$(\Omega,\ B(\phi_1(x_1))\ \ldots\ B(\phi_n(x_n))\Omega) = (\Omega,\ B(\phi_n(-\ x_n)\ \ldots\ B(\phi_1(-\ x_1))\Omega) \tag{9}$$

i.e. the PCT theorem is valied for $B(x)$ (a fact which was already proved in [4]).

Moreover (9) implies the existence of an antiunitary operator θ_1 (the PCT operator for $B(x)$) such that

$$\theta_1\ B(\phi(x))\ \theta_1^{-1} = B(\bar{\phi}(-\ x)),\quad \theta_1\Omega = \Omega,\ \phi \in S^\alpha \tag{10}$$

(see for instance [2] p.143). We will write (10) as

$$\theta_1\ B(x)\ \theta_1^{-1} = B(-\ x) \tag{11}$$

We have also

$$\theta\ \ A(x)\ \theta^{-1} = \ \ A(-\ x),\ \theta\ A_k(x)\theta^{-1} = A_k(-\ x);\ k = 1,2,\ldots,\ \theta\Omega = \Omega \tag{12}$$

We can write now

$$\theta\, A_k(x_n) \ldots A_k(x_1)\Omega = \theta\, A_k(x_n)\, \theta^{-1}\theta\, A_k(x_{n-1})\theta^{-1}\theta \ldots \theta^{-1}\theta\, A_k(x_1) \times$$

$$\times \quad \theta^{-1}\theta\Omega = A_k(-x_n)\, A_k(-x_{n-1}) \ldots A_k(-x_1)\Omega \tag{13}$$

and

$$\theta_1\, B(x_n) \ldots B(x_1)\Omega = B(-x_n) \ldots B(-x_1)\Omega \tag{14}$$

Taking in (13) the strong limit for $k \to \infty$ we get

$$\theta\, B(x_n) \ldots B(x_1)\Omega = B(-x_n) \ldots B(-x_1)\Omega \tag{15}$$

The vector $\phi = B(x_n) \ldots B(x_1)\Omega$ runs over a dense set in $\underline{H}$ ($B(x)$ has Ω as cyclic vector) so that from (14) and (15) follows $\theta_1 = \theta$ and the theorem 1 is proved.

Remark Theorem 1 is also valied if the (irreducible) field $A(x)$ is only weak local. The temperedness of $A(x)$ can be weakened by requiring $A(x)$ to be only strictly localizable in the sense of Jaffe [11].

The meaning of the theorem 1 is that we have associated to a given Borchers class also some non-local fields constructed as (strong) limits of local ones belonging to the given Borchers class. The (local) fields in the Borchers class and the associated non-local fields are acting in the same Hilbert space, corresponds to the same unitary representation of the Lorentz group, have the same vacuum and the same PCT operator.

4. Asymptotic states and the S-matrix

We consider now (tempered) local fields A_k, $k = 1,2,\ldots$ which produce in the weak limit (i.e. the W_k's are convergent in the sense of $\sum^{\alpha'}$) a non local field A(x). The first step in achieving a scattering theory for our non local field A(x) is proving the cluster property. Let

$$\underline{\Phi} \equiv \underline{\Phi}\ (\underline{a}) = \int d\underline{x}\ W^T\ (\underline{x} + \underline{a})\ \Phi(\underline{x}) \quad , \qquad \Phi(\underline{x}) \ \varepsilon\ S^\alpha$$

an averaged translated truncated vacuum expectation value of A(x) (we use here the notation of Jost [16] Chap. VI). The cluster property is

$$\lim_{d\to\infty} d^M \underline{\Phi} = 0 \quad ; \quad M = 0, 1, \ldots \tag{16}$$

uniformly in $\underline{a}$ where $d = \max_{i,j} \| \underline{a}_i - \underline{a}_j \|$. Ruelle was able to derive (16) from locality [16]. For the non-local field A(x) we can write

$$|d^M \underline{\Phi}| \leq |d^M (\underline{\Phi} - \underline{\Phi}^k)| + |d^M \underline{\Phi}^k| \tag{17}$$

where

$$\underline{\Phi}^k \equiv \underline{\Phi}^k\ (\underline{a}) = \int d\underline{x}\ W^{kT}\ (\underline{x} + \underline{a})\ \Phi(\underline{x}), \quad \Phi\ \varepsilon\ S^\alpha \subset S \tag{18}$$

and from Ruelle's theorem we get

$$\lim_{d \to o} d^M \underline{\Phi}_k = 0 , \qquad k = 1, 2, \ldots$$

In order to have (16) for the non-local field A(x) it is enough to require that the limit $d^M (\underline{\Phi} - \underline{\Phi}^k) \to 0$ is achieved uniformly in $\underline{a}$ for $k \to \infty$ and M = 0, 1, 2, ... but fixed. This condition is satisfied for instance if (uniformely in $\underline{a}$)

$$\lim_{k \to \infty} \underline{a}^m \left| W^n(\underline{x} + \underline{a}) - W^n_k (\underline{x} + \underline{a}) \right| = 0, \quad n = 0, 1, 2, \ldots \tag{19}$$

where $\underline{a}^m = \prod_{\substack{1 \leq j \leq 3 \\ o \leq i \leq n}} (a^j_i)^{m^j_i}$; $\underline{a}(a_o\ a_1, \ldots, a_n)$, $a_i(0, a^1_i, a^2_i, a^3_i$

$\underline{x}\ (x_o, x_1, \ldots, x_n)$, $\quad x_i(x^o_i, x^1_i, x^2_i, x^3_i)$

The condition (19) was imposed in [4] in order to assure the existence of the cluster property. Once having the cluster property the Haag-Ruelle scattering theory can be now derived following Steinmann [7]. We are able to give another proof of the asymptotic condition for quantum fields considered as operator valued distributions on z (the functions in z are Fourier transforms of functions in $\mathcal{D}$) based on the existence of the cluster property. This proof works exactly also for our spaces S^α, $\alpha < 1$. The condition (19) looks rather technical . A discussion of the content of (19) will be given in §7 of this work.

In the rest of §4 we will show that the asymptotic condition for a non-local field follows also from the assumption that the limit $A_k \to A$ is achieved in the strong sense described in §2 in the same

Hilbert space denoted by $\underline{H}$. Exactly this result will be used in §5 in order to prove an equivalence theorem (between local and some non-localizable fields).

Let A_k be (tempered) local fields satisfying the requirements described in [16] Chap. 4. In particular apart from the usual (Wightman) requirements we admit that the spectrum of the energy-momentum operator $\underline{P}_k$ corresponding to $A_k(x)$; $k = 1,2,...$ coincides with the corresponding spectrum for the free field of mass μ. The representation $U_1(a,\Lambda)$ which is the restriction of $U(a,\Lambda)$ to the Hilbert space $\underline{H}_1$ corresponding to the one particle hyperboloid is assumed to be irreducible with spin zero (it corresponds to the mass μ). Let $\underline{P}_1$ the projection on $\underline{H}_1$; we assume

$$(\Omega,\ A_k(x)\ \ \underline{P}_1\ A_k(y)\Omega) = i\Delta_+\ \ (\mu^2;\ x - y) \tag{21}$$

Let $h(\lambda)$ be the well known cut-off function (in momentum space) in the Haag-Ruelle theory.

We consider the fields

$$\tilde{B}_k(p) \equiv \tilde{A}_k(p)\ h(p^2), \qquad k = 1,2,\ \ldots \tag{22}$$

so that we have

$$B_k(\phi)\Omega \in \underline{H}_1 \quad \text{for } \phi \in S(\underline{R}^4)$$

and

$$(\Omega,\ B_k(x)\ B_k(y)\Omega) = i\ \Delta_+\ \ (\mu^2;\ x - y) \tag{23}$$

for all $k = 1,2,\ \ldots$

From (3) we get

$$s - \lim_{k \to \infty} B_k(x_1) \ldots B_k(x_n)\Omega = B(x_1) \ldots B(x_n)\Omega \tag{24}$$

and

$$(\Omega,\ B(x)\ B(y)\Omega) = i\ \Delta_+\ (\mu^2,\ x - y)\ . \tag{25}$$

Let $\phi \in S(\underline{R}^3)$; we construct the operators [4]

$$B_{k\phi}(t) = \int B_k(t,\bar{r})\ \phi(\bar{r})\ d\bar{r} \tag{26}$$

where $\bar{r} = \bar{r}(x^1, x^2, x^3)$.

From [16] Chap. VI, lemma 6 we know that $B_{k\phi}(t)$ are defined on the common domain D, $B_{k\phi}(t)\Phi$, $\phi \in D$ are vectors in D which are C^{∞} in t and continuous in ϕ .

The proof of lemma 6 Chap VI [16] can be applied in order to prove the same properties for the operator

$$B_{\phi}(t) = \int B(t,\ \bar{r})\ \phi(\bar{r})\ d\bar{r} \tag{27}$$

with $\phi \in S^{\alpha}(\underline{R}^3) \quad S(\underline{R}^3)$.

Let now f_m, $m = 0, 1, \ldots, n$, be n + 1 smooth solutions of the Klein-Gordon equation

$$f_m(x) = \frac{1}{(2\pi)^{3/2}} \int \theta(p^0)\ \delta(p^2 - \mu^2)\ \left[e^{-ipx}\ g_+^m\ (\bar{p}) + e^{ipx}\ g_-^m\ (\bar{p})\right] \times$$

$$\times\ d^4p$$

with $g_{\pm}^{m}(\bar{p}) \in S_{\alpha}(\underline{R}^3) \subset \mathcal{D}(\underline{R}^3)$; $m = 0,1,2,\ldots,n$. We will have certainly

$$f_m(0,\bar{x}) \in S^{\alpha}(\underline{R}^3), \quad \frac{\partial f_m}{\partial x^o}(0,\bar{x}) \in S^{\alpha}(\underline{R}^3) \tag{28}$$

From (26), (27) and (28) follows that the operators

$$B_{kf_m}(t) = i \int_{x^o=t} \bar{f}_m \overleftrightarrow{\partial}_o B_k \, d\bar{r}, \quad B_{f_m}(t) = i \int_{x^o=t} \bar{f} \overleftrightarrow{\partial}_o B \, d\bar{r}$$

are defined on D and can be applied successively on the vacuum.

We construct now

$$\Phi_k(t) = B_{kf_o}(t)\, B_{kf_1}(t) \ldots B_{kf_n}(t)\Omega \quad , \quad k = 1,2,\ldots \tag{29}$$

and

$$\Phi(t) = B_{f_o}(t)\, B_{f_1}(t) \ldots B_{f_n}(t)\Omega \tag{30}$$

From the Haag-Ruelle theory follows (as a consequence of locality) that the strong limits of $\Phi_k(t)$ for $t \to \mp\infty$ exist

$$s - \lim_{t \to \mp\infty} \phi_k(t) = \phi_k^{\substack{in \\ out}} \quad , \quad k = 1,2, \ldots \tag{31}$$

$\phi_k^{\substack{in \\ out}}$ are the asymptotic states for the fields $A_k(x)$, $k = 1,2, \ldots$

We have

Theorem 2 The vectors

$$\Phi(t) = B_{f_o}(t)\, B_{f_1}(t) \ldots B_{f_n}(t)\Omega$$

have strong limits for $t \to \mp \infty$:

$$\underset{t\to\mp\infty}{s-\lim}\; B_{f_o}(t)\, B_{f_1}(t) \ldots B_{f_n}(t)\Omega = \Phi^{\substack{in\\out}}(f_o, f_1, \ldots, f_n) \tag{32}$$

These limits are independent, with respect to $L_+^\uparrow$, of the special coordinate system in which the various entities have been defined.

Proof: We write

$$\| \phi(t_2) - \phi(t_1) \| \leq \| \phi(t_2) - \phi_k(t_2) \| + \| \phi_k(t_2) - \phi_k(t_1) \| + + \| \phi(t_1) - \phi_k(t_1) \| \tag{33}$$

We have for each k (see (31))

$$\| \phi_k(t_2) - \phi_k(t_1) \| \to 0 \quad \text{for } |t_1|, |t_2| \to \infty$$

It remains to prove that

$$\lim_{k\to\infty} \| \phi(t) - \phi_k(t) \| = 0 \tag{34}$$

and this limit takes place uniformly in t .

We have

$$\phi(t) - \phi_k(t) = \int_{x_o^o=t} \bar{f}_o(x_o^o,\bar{r}_o) \overleftrightarrow{\partial}_{oo} B(x_o^o,\bar{r}_o)\, d\bar{r}_o \;\dots\; \times$$

$$\times \int_{x_n^o=t} \bar{f}_n(x_n^o,\bar{r}_n) \overleftrightarrow{\partial}_{on} B(x_n^o,\bar{r}_n)\, d\bar{r}_n\, \Omega - \int_{x_o^o=t} \bar{f}_o(x_o^o,\bar{r}) \overleftrightarrow{\partial}_{oo} B_k(x_o^o,\bar{r}_o) \;\dots\; \times$$

$$\times \int_{x_n^o=t} \bar{f}_n(x_n^o,\bar{r}_n) \overleftrightarrow{\partial}_{on} B(x_n^o,\bar{r}_n)\, d\bar{r}_n \quad \Omega$$

$$= \int_{x_o^o=t} \dots \int_{x_n^o=t} \{(\bar{f}_o(x_o^o,\bar{r}_o) \overleftrightarrow{\partial}_{oo} B(x_o^o,\bar{r}_o)) \;\dots\; (\bar{f}_n(x_n^o,\bar{r}_n) \overleftrightarrow{\partial}_{on} B(x_n^o,\bar{r}_n))\Omega$$

$$- (\bar{f}_o(x_o^o,\bar{r}_o) \overleftrightarrow{\partial}_{oo} B_k(x_o^o,\bar{r}_o)) \;\dots\; (f_n(x_n^o,\bar{r}_n) \overleftrightarrow{\partial}_{on} B_k(x_n^o,\bar{r}_n)\Omega\}\, d\bar{r}_o \dots dr_n$$

where $\overleftrightarrow{\partial}_{o_i}$; $i = 0,1,\dots,n$ involves differentiation with respect to x_i^o.
This expression is a finite sum of terms of the form

$$\int_{x_o^o=t} \dots \int_{x_n^o=t} g_o(x_o^o,\bar{r}_o) \dots g_n(x_n^o,\bar{r}_n) \;\{B(x_{i_o}^o, r_{i_o}) \dots B(x_{i_j}, \bar{r}_{i_j}) \;\times$$

$$\times \frac{\partial}{\partial x_{i_{j+1}}^o} B(x_{i_{j+1}}^o, \bar{r}_{i_{j+1}}) \dots \frac{\partial}{\partial x_{i_n}^o} B(x_{i_n}^o, \bar{r}_{i_n})\Omega \quad -$$

$$- B_k(x_{i_o}^o, \bar{r}_{i_o}) \dots B_k(x_{i_j}^o, \bar{r}_{i_j}) \frac{\partial}{\partial x_{i_{j+1}}^o} B_k(x_{i_{j+1}}^o, \bar{r}_{i_{j+1}}) \dots \quad \times$$

$$\times \frac{\partial}{\partial x_{i_n}^o} B_k(x_{i_n}^o, \bar{r}_{i_n})\}\quad d\bar{r}_o \dots d\bar{r}_n \qquad (36)$$

where $(i_o,\dots,i_n)$ is a permutation of $(0,1,\dots,n)$ and $0 \leq j \leq n$ and $g_\ell\ (x_\ell^o, \bar{r}_\ell)$; $\ell = 0,1,\dots,n$ is equal to $f_\ell(x_\ell^o, \bar{r}_\ell)$ or to $\frac{\partial}{\partial x_\ell^o}\ f_\ell(x_\ell^o, \bar{r}_\ell)$. We go over to momentum space in (39) and get that $\| \ \phi(t) - \phi_k(t) \|$ is smaller than a sum of terms of the form

$$\int d^4p_o \dots \int d^4p_n \quad || \tilde{A}(p_{i_o})h\ (p_{i_o}^2) \dots \tilde{A}(p_{i_j})h\ (p_{i_j}^2)\ p_{i_{j+1}}^o\ \tilde{A}(p_{i_{j+1}}) \times$$

$$\times\ h(p^2_{i_{j+1}}) \dots p_{i_n}^o\ \tilde{A}(p_{i_n})\ h^2(p_{i_n})\ \Omega \quad -$$

$$-\ \tilde{A}_k(p_{i_o})\ h(p_{i_o}^2) \dots \tilde{A}_k(p_{i_j})\ h\ (p_{i_j}^2)\ p_{i_{j+1}}^o\ \tilde{A}_k(p_{i_{j+1}}) \times$$

$$\times\ h\ (p_{i_{j+1}}^2) \dots p_{i_n}^o\ \tilde{A}_k\ (p_{i_n})\ h^2(p_{i_n})\ \Omega \quad \chi(\bar{p}_o,\dots,\bar{p}_n)\ || \qquad (37)$$

where $\chi(\bar{p}_o,\dots,\bar{p}_n)$ is a function in $S_\alpha\ (\underline{R}^{3(n+1)})$ (see 28).

In deriving (37) we have inserted in (36) a δ-function $\delta(x_\ell^o - t)$ $\ell = 0,1,\dots,n$ with the corresponding integration on x_ℓ^o.

We remark that (37) is independent of time.

Taking (3) into account we get that (34) is achieved for $k \to \infty$ uniformly in t.

The second part of the theorem 2 follows also from this uniformity in t. Indeed let $\Lambda \in L_+^\uparrow$, $f_{\ell,\Lambda}(x) = f_\ell(\Lambda^{-1}\ x)$, $\ell = 0,1,\dots,n$, $B^\Lambda(x) = B(\Lambda^{-1}\ x)$, $B_k^\Lambda\ (x) = B_k(\Lambda^{-1}\ x)$ and

$$B_{f\ell}^\Lambda\ (x) = i \int_{x^o=t} \bar{f}_{\ell,\Lambda}\ \overleftrightarrow{\partial_o}\ B^\Lambda\ d\bar{r}$$

$$B_{k,f\ell}^\Lambda(x) = i \int_{x^o=t} \bar{f}_{\ell,\Lambda}\ \overleftrightarrow{\partial_o}\ B_k^\Lambda\ d\bar{r}$$

$$\Phi_\Lambda(t) = B^\Lambda_{f_o}(t) \ldots B^\Lambda_{f_n}(t)\,\Omega \tag{38}$$

$$\Phi_{k,\Lambda}(t) = B^\Lambda_{k,f_o}(t) \ldots B^\Lambda_{k,f_n}(t)\,\Omega \tag{39}$$

We have to prove that for $t \to \mp \infty$, $\| \phi_\Lambda(t) - \phi(t) \| \to 0$.
We write

$$\| \phi_\Lambda(t) - \phi(t) \| \leq \| \phi_\Lambda(t) - \phi_{k,\Lambda}(t) \| + \| \phi_{k,\Lambda}(t) - \phi_k(t) \| +$$

$$+ \| \phi_k(t) - \phi(t) \| \tag{40}$$

The first and the third term on the righthand side of (40) can be made smaller than $\varepsilon/_3$ (for $\varepsilon > 0$ given) independently of t. Now we have only to take $|t|$ large enough in order that $\| \phi_{k,\Lambda}(t) - \phi_k(t) \| < \frac{\varepsilon}{3}$ for the local fields ϕ_k, $k = 1,2,\ldots$. This completes the proof of theorem 2.

Let $\underline{H}_{in}$ ($\underline{H}_{out}$) be the norm closure (in $\underline{H}$) of the linear combinations of elements ϕ^{in} (ϕ^{out}) including the vacuum.

We have

Theorem 3 Define the linear operator $A^{ex}(f)$ on the vectors $\phi^{ex}(f_o,f_1,\ldots,f_n)$ as

$$A^{ex}(f)\, \phi^{ex}(f_o,\ldots,f_n) = \phi^{ex}(f,f_o,\ldots,f_n) \tag{41}$$

where ϕ^{ex} stands for ϕ^{in} respectively ϕ^{out} and f is a smooth solution of the Klein-Gordon equation satisfying (27). Then the operators $A^{ex}(f)$

correspond to the free scalar field of mass μ which we denote by $A^{ex}_{(x)}$

$$A^{ex}(f) = i\int_{x^o=t} f(x) \overleftrightarrow{\partial}_o A^{ex}(x)\, d^3\bar{r} \tag{42}$$

and

$$U(a,\Lambda)\, A^{ex}(x)\;\; U^{-1}\,(a,\Lambda) = A^{ex}\,(\Lambda x + a) \tag{43}$$

$$\theta_1\, A^{in}\,(x)\,\theta_1 = A^{out}\,(-\,x) \tag{44}$$

<u>Proof</u> We have to prove that $(\phi^{ex}\,(f_o,\ldots,f_n),\ \phi^{ex}(g_1,\ldots,g_m))$ is the corresponding scalar product for a free field. We write

$$\|\ \phi^{ex} - \phi^{ex}_k\ \| \le \|\ \phi^{ex} - \phi(t)\| + \|\ \phi(t) - \phi_k(t)\ \| +$$
$$\|\ \phi_k(t) - \phi^{ex}_k\ \|$$

By taking into account (34) (uniformly in t) we get

$$s - \lim_{k\to\infty} \phi^{ex}_k = \phi^{ex} \tag{45}$$

It follows that

$$\begin{aligned}&(\phi^{ex}\,(f_o,\ldots,f_n),\quad \phi^{ex}(g_o,\ldots,g_m)) = \\ &= \lim_{k\to\infty}\ (\phi^{ex}_k\,(f_o,\ldots,f_n),\ \phi^{ex}_k\,(g_o,\ldots,g_m))\end{aligned} \tag{46}$$

By the Haag-Ruelle theory $(\phi_k^{ex}(f_o,\ldots,f_n),\ \phi_k^{ex}(g_o,\ldots,g_n))$ is (for all $k = 1,2,\ldots$) equal to the corresponding scalar product for the case of a free field and this proves the first part of the theorem 3. The relation (43) follows from the second part of theorem 2 and (44) follows from $\theta_1\ A(x)\ \theta_1 = A(-x)$ (see (10)).

Concluding this section we remark that we have been able to prove the asymptotic condition (under asymptotic condition we mean theorems 1 and 2) for (hopefully!) a large class of non localizable fields. We have two types of results: In the first part of the section the asymptotic condition was shown to be valid for non localizable fields defined as weak limits of local fields, the limit being achieved in a uniform sense (see (19)). In this case the cluster property is trivially satisfied and the cluster induces the asymptotic condition. In the second part of this section we have taken the strong variant (3) for defining non-localizable fields. This condition enables us to prove uniformity in t for $k \to \infty$ and in this way the theorems 2 and 3 can be given a direct proof.

In order to discuss the S-matrix (in the next section) we assume that the field A(x) satisfies asymptotic completeness. The S matrix operator can be now defined by

$$A^{out}(x) = S^{-1}\ A^{in}(x)\ S$$

5. Equivalence between a non-localizable and a local field

In order to discuss the equivalence of fields along the line of Borchers, we have to discuss fields having the same PCT operator. Because of some Hilbert space difficulties in the case of a non-localizable field constructed as a weak limit of local ones we have not discussed the existence of a PCT operator in this case. But we have proved that if the limit $A_n \to B$ is achieved strongly in the same Hilbert space the PCT operator for B exists (see theorem 1) and equals the PCT operator of a local field A which generates the Borchers class containing A_k; $k = 1,2,...$. We have

Theorem 4 If $A(x)$ is a local (tempered) field and $B(x)$ a non-local field constructed as in theorem 1 then the asymptotic fields $A^{ext}(x)$ and $B^{ext}(x)$ exists. Moreover if $A^{in}(x) = B^{in}(x)$ then the two fields $A(x)$ and $B(x)$ have the same S-matrix.

Proof The proof follows immediately from the fact that $A(x)$ and $B(x)$ have the same PCT operator and this operator takes in fields to out fields.

Theorem 4 shows that a non-localizable field can be S-equivalent to a local one. In the next section we discuss some examples in which theorem 4 applies.

6. Examples

Let us consider $A(x)$ as being the scalar neutral massive free field. We construct Wick series of this field (in four dimensions)

$$B(x) = \sum_{n=1}^{\infty} a_n : A(x)^n : \qquad (47)$$

A result of Jaffe [11] (see also [17] and [18]) shows that B(x) is strictly localizable if the series $\sum_{n=1}^{\infty} a_n z^n$ has an order of growth smaller than 2. For an order of growth equal to 2 and type zero we still get a localizable theory in the sense of [13]. If the order of growth of $\sum_{n=1}^{\infty} a_n z^n$ is larger than two B(x) will be non-localizable. We remark that the same thing also happens for the massless case though for this case in the region of non-locality the high energy behaviour of B(x) is very different from the high energy behaviour in the massive case, because of a "contraction" of the phase space volume by passing from $m = 0$ to $m \neq 0$. A discussion of the extension of the commutator of B(x) outside the light-cone ("acausal effects") are discussed partially in [14] for the massless case; a full discussion of this question for $m \neq 0$ is the subject of [15].

Now coming back to S-equivalence, it is well-known that the series (47) is in the Borchers class of A(x) if $\sum_{n=1}^{\infty} a_n z^n$ has order of growth smaller than two. It follows that in this case B(x) is trivial. Theorem 4 applies to B(x) in the case in which $\sum_{n=1}^{\infty} a_n z^n$ has an order of growth larger than two (we leave the reader to convince himself that this is the case) and therefore A(x) will be trivial also in the non-localizable case.

Another (non-trivial) example is given by taking A(x) to be a tempered non-trivial field (we assume that a such field exists!) and considering

$$B(x) = A(x) + g(\Box_x)\, K_x \quad (x) \qquad (48)$$

where $g(z)$ is an entire function (with real coefficients), $\Box_x = \frac{\partial^2}{\partial \vec{x}^2} - \frac{\partial^2}{\partial x_o^2}$ and $K_x = \Box_x - m^2$. If $g(z)$ has order of growth smaller than $1/2$, $B(x)$ is (strictly) localizable. If the order of growth of $B(x)$ is larger than one half $B(x)$ will be non-localizable. In the localizable case it is easy to prove that $B(x)$ is in the same Borchers class as $A(x)$ and is S-equivalent but to $B(x)$. In the non-localizable case the theorem 4 applies and shows that $B(x)$ is still S-equivalent to $A(x)$. Indeed it is easy to prove that the convergence of partial sums in (48) to $B(x)$ takes place in the strong sense (3) and that (21) is also satisfied if a similar condition holds for $A(x)$.

In the next section we will discuss in detail the cluster property in the weak approach $A_k(x) \to B(x)$ as described in §2.

7. The cluster property in the "weak" approach local → non-localizable

In §4 we have formally remarked that the cluster property is valid for $B(x)$ if (19) is valid uniformly in a. We would like to discuss this question here replacing the technical condition (19) by a condition which is more connected to the existence of a scattering theory for the approximating fields (which follows from the fact that the approximating fields are local).

Let us consider that all the (local) fields $A_k(x)$ have the spectrum of free field and that (21) is satisfied (we remark that these conditions are weaker than those imposed on $A_k(x)$ in §4; there it was for instance assumed $U(a,\Lambda)$ is the same representation of the Lorentz group for all $A_k(x)$).

Let $\bar{\alpha}_i = a_i - a_o$; $i = 1,2,\dots,n$ (see the notation after (19)). Then the second auxiliary theorem in §5 Chapt. VI [16] tells us that

$$\int W_k^T (x + a)\, \phi(x)\, dx, \qquad \phi \in S(\underline{R}^{4(n+1)}) \subset S^\alpha(\underline{R}^{4(n+1)})$$

is in $S(\underline{R}^{3n})$ with respect to $\bar{\alpha}$. Let $\tilde{W}_k^T(p)$ be the Fourier transform of $W_k^T(x)$; the translation invariance of $W_k^T(x)$ gives in momentum space

$$\tilde{W}_k^T(p_o,p_1,\dots,p_n) = \delta^4(p_1 + \dots + p_n) \times$$

$$\times \quad \tilde{W}_k^{\prime T}(p_1,\dots,p_n) \tag{49}$$

The fact that $\int W_k^T (x + a)\, \phi(x)\, dx$ is in $S(\underline{R}^{3n})$ in $\bar{\alpha}_i = a_i - a_o$, $i = 1,2,\dots,n$ means that after integrating $\tilde{W}_k^{\prime T}(p)$ with a test function from $S(\underline{R}^n_{p^o})$ in p^o we get a C^∞ function in $\bar{p}$ which is in $\theta_M(\underline{R}^{3n}_{\bar{p}})$ *). But $\theta_M(\underline{R}^{3n}_{\bar{p}}) \subset \theta_{\alpha M}(\underline{R}^{3n}_{\bar{p}})$. Now the cluster property for $B(x)$ constructed as a weak limit of $A_k(x)$ is valid of $\tilde{W}^{T\prime}(p) \in \theta_{\alpha M}(\underline{R}^{3n}_{\bar{p}}) \otimes$ $\otimes\, S_\alpha^\prime(\ ^n_{\bar{p}^o})$. Therefore we are faced with the following problem:

let $\{f_\nu(p,\bar{p})\}$ be a sequence of generalized functions in $S_\alpha^\prime(\underline{R}^n_{p^o}) \otimes$ $\otimes\, \theta_{\alpha M}(\underline{R}^{3n}_{\bar{p}})$ which converges in the sense of $S_\alpha^\prime(\underline{R}^{4n}_p)$. We have to

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*) In fact a result of Borchers [19] allows us to find some stronger properties in $\bar{p}$ but we are not interested in this problem here.

require that the limit $f^{(p^o, \bar{p})}$ of this sequence also lies in $S'_\alpha(\underline{R}^n_{p^o}) \otimes \theta_{\alpha M}(\underline{R}^{3n}_{\bar{p}})$. It follows that a simple condition we can impose for the existence of a scattering theory for $A(x)$ is the following

Condition S: The field $A(x)$ must be approximated by $A_k(x)$ in such a way that the truncated Wightman functions $W'^T_k(p)$ converge in the sense of $S'_\alpha(\underline{R}^n_{p^o}) \otimes \theta_{\alpha M}(\underline{R}^{3n}_{\bar{p}})$. The condition S can be interpreted as a regularity condition. Indeed the C^∞-regularity of $\tilde{W}'^T(p)$ in $\bar{p}$ (which is responsible for the existence of the cluster property) must be retained in the limit (with the same effect) and this is, roughly speaking, the content of condition S.

8. Conclusions

We have shown how it is possible to formulate a notion of equivalence between non-localizable fields, or in the special case of this paper between a non-localizable and a local one. Indeed we may define this equivalence by a natural extension of the idea of a Borcher's class as follows.

Definition.

Two fields are PCT-equivalent if and only if they have the same PCT operator.

Such a relation between two fields is evidently an equivalence, and so divides the class of all fields into equivalence sub-classes. Each sub-class will be composed of Borchers classes of local fields possibly together with some non-localizable fields. Fields in the same

PCT-class will have the same S-operator if they have the same in fields, so will be S-equivalent. We showed in detail how a non-localizable field constructed as a strong limit of local fields can be shown to be PCT-equivalent to a given local field in terms of conditions on the local fields of the approximating sequence. However we haven't shown that there exist any non-localizable fields which are not PCT-equivalent to some local field. Our discussion in the paper has shown that non-localizable fields can be as physically reasonable as local ones in describing a given S-matrix. Apart from computational advantages there seems to be nothing gained in using a local field equivalent to a non-localizable one. The idea of imposing localizability on fields in non-polynomial Lagrangian theories would seem to be unnecessary from this point of view.

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REFERENCES

1. Borchers, H.J.: Nuovo Cim. 15, 784 (1960).

2. Streater, R.F. and Wightman, A.S.: "PCT Spin and Statistics and all That", W.A. Benjamin, New York, 1964.

3. Kamefuchi, S., O'Raifeartaigh, L. and Salam A.: Nucl. Phys. 28, 529 (1961)

4. Taylor, J.G.: Ann. Phys. (N.Y.) 68, 484 (1971)

5. Iofa, M.Z. and Fainberg, V.Ya.: Soviet Physics - JETP 29, 880 (1969); Kiev reports ITF 69-23, 25 (1969); TMP, 1, 187 (1969).

6. Hindle, C.M.: Forschritte d. Phys. 19, 369 (1971)

7. Steinmann, O.: Comm. Math. Phys. 18, 179 (1970).

8. Borchers, H.J.: Nuovo Cim. 24, 214 (1962)

9. Jaffe, A.: J. Math. Phys. 6, 1172 (1965).

10. Gelfand I.M. and Shilov G.E.: "Generalised Functions", Vol. 2. Academic Press, New York, 1968.

11. Jaffe, A.M.: Phys. Rev. 158 (1967), 1454.

12. Köthe, G.: "Topologische lineare Raume", Springer, Heidelberg (1960)

13. Constantinescu, F.: J. Math. Phys., 12, 293 (1971).

14. Blomer R.F. and Constantinescu, F.: Nuovo Cim., 6A, 62 (1971)

15. Constantinescu, F. and Taylor J.G.: Preprint King's College London, April, 1972.

16. Jost, R.: "The general theory of quantized fields", Amer. Math. Soc., Providence, Rhode Island, 1965.

17. Jaffe, A.M.: Ann. Phys. (N.Y.) 32, 127 (1965)

18. Rieckers, Int. Journ. Theor. Phys. 4, 55 (1971).

19. Borchers, H.J.: Nuovo Cim, 33 (1964) 1600.