ESAIM: PS
February 2007, Vol. 11, p. 3–22
DOI: 10.1051/ps:2007002

ESAIM: Probability and Statistics
www.edpsciences.org/ps

REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH TWO RCLL BARRIERS

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Abstract. In this paper we consider BSDEs with Lipschitz coefficient reflected on two discontinuous (RCLL) barriers. In this case, we prove first the existence and uniqueness of the solution, then we also prove the convergence of the solutions of the penalized equations to the solution of the RBSDE. Since the method used in the case of continuous barriers (see Cvitanic and Karatzas, *Ann. Probab.* **24** (1996) 2024–2056 and Lepeltier and San Martín, *J. Appl. Probab.* **41** (2004) 162–175) does not work, we develop a new method, by considering the solutions of the penalized equations as the solutions of special RBSDEs and using some results of Peng and Xu in *Annales of I.H.P.* **41** (2005) 605–630.

Mathematics Subject Classification. 60H10, 60G40.

Invited paper accepted September 2005.

1. Introduction

Non-linear backward stochastic differential equations (BSDE's in short) were firstly introduced by Pardoux and Peng ([10], 1990), who proved the existence and uniqueness of the adapted solution, under smooth square-integrability assumptions on the coefficient and the terminal condition, and when the coefficient $g(t,\omega,y,z)$ is Lipschitz in (y,z) uniformly in (t,ω). Then El Karoui, Kapoudjian, Pardoux, Peng and Quenez introduced the notion of reflected BSDE (RBSDE in short) ([6], 1997) with one continuous lower barrier. More precisely, a solution for such equation associated with a coefficient g, a terminal value ξ, a continuous barrier (L_t), is a triplet $(Y_t,Z_t,K_t)_{0\leq t\leq T}$ of adapted processes valued on $\mathbb{R}^{1+d+1}$, which satisfies a smooth square integrability condition,

$$Y_t=\xi+\int_t^T g(s,Y_s,Z_s)\mathrm{d}s+K_T-K_t-\int_t^T Z_s\mathrm{d}B_s,\ \ 0\leq t\leq T,\text{ a.s.,} \tag{1}$$

and $Y_t\geq L_t$ a.s. for any $0\leq t\leq T$, (K_t) is non-decreasing continuous, where B_t is a d-dimensional Brownian motion. The role of (K_t) is to push upward the process Y in a minimal way, in order to keep it above L. In this sense it satisfies

$$\int_0^T (Y_s-L_s)\mathrm{d}K_s=0. \tag{2}$$

Keywords and phrases. Reflected backward stochastic differential equation, penalization method, optimal stopping, Snell envelope, Dynkin game.

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In order to prove the existence and uniqueness of the solution, they used first a Picard-type iterative procedure, which requires at each step the solution of an optimal stopping problem. The second approximation is constructed by penalization of the constraint. At each step, they have the solution of a classical BSDE (Y^n, Z^n). The comparison theorem on the solutions of BSDEs ([10], 1990) gets to the convergence of the sequence (Y^n). For the sequence (Z^n), the fact that (L_t) is continuous is crucial (see [6], 1997, Lem. 6.1, and the proof using the Dini's theorem).

Following this paper, Cvitanic and Karatzas ([4], 1996) introduced the notion of reflected BSDE with two barriers. In this case a solution of such an equation associated with a coefficient g, a terminal value ξ, a continuous lower barrier (L_t) and a continuous upper barrier (U_t), with $L_t \le U_t$ and $L_T \le \xi \le U_T$ a.s. is a triplet $(Y_t, Z_t, K_t)_{0\le t\le T}$ of adapted processes, valued in $\mathbb{R}^{1+d+1}$, which satisfies

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}B_s, \ \ 0 \le t \le T, \text{ a.s.}, \tag{3}$$

$L_t \le Y_t \le U_t$, a.s. for any $0 \le t \le T$, (K_t) is a finite variation continuous process, $K = K_t^+ - K_t^-$, where K^+, K^- are increasing; the role of (K_t) is to keep the process Y between L an U in such a way that

$$\int_0^T (Y_s - L_s)\mathrm{d}K_s^+ = 0 \text{ and } \int_0^T (Y_s - U_s)\mathrm{d}K_s^- = 0. \tag{4}$$

In view to prove the existence and uniqueness of a solution, the method still bases on a Picard-type iteration procedure, which requires at each step the solution of a Dynkin game problem.

Then in the Section 6 of this paper ([4], 1996), an alternative method for proving the existence of a solution is presented, which still applies penalization of the constraints, under a condition which roughly says that the barrier can be approximated (uniformly) by semi-martingales whose finite variation part process is absolutely continuous with respect to the Lebesgue measure. Furthermore, the existence result is only obtained when the coefficient g does not depend on z.

In [8], 2004, Lepeltier and San Martin relaxed in some sense the condition on the barriers, proving by a penalization method an existence result, without any assumption (except square integrability assumption) on L and U, but only when there exists a continuous semi-martingale with terminal value ξ, between L and U. They proved also the existence result in the general case (where g may depend also on z). In [8], (see Lems. 5 and 6), the fact that L and U are continuous is also crucial.

In this paper, we consider the reflected BSDE's with right continuous left limit (RCLL) barriers. In this case the process Y may have jumps, and is RCLL. The role of $K_t = K_t^+ - K_t^-$ is to keep in a minimal way the process Y between two barriers L and U; it is then natural to replace (4) by

$$\int_0^T (Y_{s-} - L_{s-})\mathrm{d}K_s^+ = 0 \text{ and } \int_0^T (Y_{s-} - U_{s-})\mathrm{d}K_s^- = 0. \tag{5}$$

In Section 2 we set up accurately the problem and we present one "monotonic limit" theorem which will play an important role in the penalization method for the RBSDEs with two RCLL barriers.

In Section 3, we generalize the existence and uniqueness result for a RBSDE with two discontinuous barriers, using like in [4], a Picard iteration method and a Dynkin game problem.

In Section 4, we consider the penalization method for the RBSDEs. We prove that the solutions of penalized equations

$$Y_t^{m,n} = \xi + \int_t^T g(s, Y_s^{m,n}, Z_s^{m,n})\mathrm{d}s + n\int_t^T (Y_s^{m,n} - L_s)^- \mathrm{d}s - m\int_t^T (U_s - Y_s^{m,n})^- \mathrm{d}s - \int_t^T Z_s^{m,n}\mathrm{d}B_s$$

converge to the solution of the RBSDE. We use the idea that the solution of the RBSDE with one lower barrier, penalized with respect to an upper barrier, may be considered as the solution of a RBSDE with two barriers.

We also use a generalization of the "monotonic limit theorem" (see [11]). Some definitions and important results about the Snell envelope and Dynkin game are listed in the Appendix (Sect. 5).

2. Definitions and assumptions for reflected BSDE with two RCLL barriers

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $B = (B_1, B_2, \cdots, B_d)'$ be a d-dimensional Brownian motion defined on the finite interval $[0, T]$. Denote by $\{\mathcal{F}_t; 0 \le t \le T\}$ the natural filtration generated by the Brownian motion B:

$$\mathcal{F}_t = \sigma\{B_s; 0 \le s \le t\},$$

augmented with all P-null sets of $\mathcal{F}$.

We shall need the following notations. For any given $m \in \mathbf{N}^*$ and $t \in [0, T]$, let us introduce the following spaces:

- $\mathbf{L}^2_m(\mathcal{F}_t) := \{\xi : \Omega \to \mathbb{R}^m$, $\mathcal{F}_t$–measurable random variables ξ with $E[|\xi|^2] < \infty\}$;
- $\mathbf{H}^p_m(0, t) := \{\varphi : \Omega \times [0, t] \to \mathbb{R}^m$; $\mathcal{F}_t$–predictable processes with $E \int_0^t |\varphi_t|^p \mathrm{d}t < \infty\}$;
- $\mathbf{D}^2_m(0, t) :=\{\varphi \in L^p_{\mathcal{F}}(0, t; \mathbb{R}^m)$; $\mathcal{F}_t$–progressively measurable RCLL processes with $E[\sup_{0\le t\le\tau} |\varphi_t|^2] < \infty\}$;
- $\mathbf{A}^2(0, t) :=\{K\colon \Omega \times [0, t] \to \mathbb{R}$, $\mathcal{F}_t$–progressively measurable increasing RCLL processes with $K(0) = 0$, $E[(K_T)^2] < \infty$ $\}$.

In the real–valued case, *i.e.*, $m = 1$, the three first spaces will be simply denoted by $\mathbf{L}^2(\mathcal{F}_t)$, $\mathbf{H}^p(0, t)$, $\mathbf{D}^2(0, t)$ respectively. We shall denote by $\mathcal{P}$ the σ–algebra of predictable sets in $[0, T] \times \Omega$.

We suppose the following assumptions:

Assumption 2.1. *The terminal value ξ is a given random variable in $\mathbf{L}^2(\mathcal{F}_T)$.*

Assumption 2.2. *The coefficient $g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \longmapsto \mathbb{R}$, is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, and satisfies*

$$\text{(i)} \quad E \int_0^T g^2(t, 0, 0)\mathrm{d}t < +\infty, \tag{6}$$

and (ii)

$$\begin{array}{rcl} |g(t, \omega, y_1, z_1) - g(t, \omega, y_2, z_2)| & \le & k(|y_1 - y_2| + |z_1 - z_2|) \\ \forall (t, \omega) & \in & [0, T] \times \Omega;\ \ y_1, y_2 \in \mathbb{R};\ \ z_1, z_2 \in \mathbb{R}^d \end{array} \tag{7}$$

for some $0 < k < \infty$.

Assumption 2.3. *The two barriers $\{L_t, 0 \le t \le T\}$ and $\{U_t, 0 \le t \le T\}$ are RCLL progressively measurable real-valued processes satisfying*

$$E(\sup_{0\le t\le T} (L^+_t)^2 + \sup_{0\le t\le T} (U^-_t)^2) < +\infty, \tag{8}$$

and $L_t \le U_t$ for $0 \le t \le T$, with $L_T \le \xi \le U_T$ a.s.

For the existence of the solution of the reflected BSDE with two RCLL barriers, we shall need:

Assumption 2.4. *(i) There exists a process $J_t = J_0 + \int_0^t \phi_s \mathrm{d}B_s - V^+_t + V^-_t$, with $\phi \in \mathbf{H}^2_d(0, T)$, $V^+, V^- \in \mathbf{A}^2(0, T)$, such that*

$$L_t \le J_t \le U_t \ \ P\text{-a.s. for } 0 \le t \le T.$$

(ii) For $t \in [0, T)$, $L_t < U_t$, a.s..

Now we present the definition of the solutions of the RBSDEs with two RCLL barriers.

Definition 2.1. A triplet (Y, Z, K) of $\mathcal{F}_t$-progressively measurable processes, where Y, K, are RCLL processes and $Y, K : [0,T] \times \Omega \longmapsto \mathbb{R}$, and $Z : [0,T] \times \Omega \longmapsto \mathbb{R}^d$ is called a solution of the RBSDE with two RCLL reflecting barriers $L(\cdot)$, $U(\cdot)$, a terminal condition ξ and a coefficient g, if the followings hold:

(i) $Y \in \mathbf{D}^2(0,T)$, $Z \in \mathbf{H}^2_d(0,T)$, and $K = K^+ - K^-$, with $K^+, K^- \in \mathbf{A}^2(0,T)$.

(ii) $Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s + K^+_T - K^+_t - (K^-_T - K^-_t) - \int_t^T Z_s \mathrm{d}B_s$, $0 \le t \le T$.

(iii) $L_t \le Y_t \le U_t$, $0 \le t \le T$, a.s.

(iv) $\int_0^T (Y_{s-} - L_{s-})\mathrm{d}K^+_s = \int_0^T (U_{s-} - Y_{s-})\mathrm{d}K^-_s = 0$, a.s.

So the state-process $Y(\cdot)$ is forced to stay between the barriers $L(\cdot)$ and $U(\cdot)$ by the cumulation action of the reflection processes $K^+(\cdot)$, $K^-(\cdot)$ respectively; they act only necessarily to prevent $Y(\cdot)$ from crossing the respective barrier, and in this sense, their actions can be considered minimal.

Now we present a generalized "monotonic limit" theorem, which will play an important role in the penalization method for the RBSDE with two RCLL barriers. It is proved in [11], Theorem 3.1.

Theorem 2.1. *We consider the following BSDE's associated with two increasing processes: for $i \in \mathbb{N}$,*

$$Y^i_t = \xi + \int_t^T g(s, Y^i_s, Z^i_s)\mathrm{d}s + A^i_T - A^i_t - (K^i_T - K^i_t) - \int_t^T Z^i_s \mathrm{d}B_s, \tag{9}$$

with $E[\sup_{0\le t\le T} |Y^i_t|^2] < \infty$. Here g satisfies the Assumption 2.2, and A^i, $K^i \in \mathbf{A}^2(0,T)$; we also assume that for each $i \in \mathbb{N}$,

(h1) (A^i) is continuous with $\mathbf{E}[(A^i_T)^2] < \infty$;

(h2) $K^j_t - K^j_s \ge K^i_t - K^i_s$, $\forall 0 \le s \le t \le T$, a.s. $\forall i \le j$;

(h3) for $t \in [0,T]$, $K^i_t \nearrow K_t$, in $\mathbf{L}^2(\mathcal{F}_t)$, with $\mathbf{E}[K^2_T] < \infty$;

(h4) (Y^i_t) converges increasingly to (Y_t) with $\mathbf{E}[\sup_{0\le t\le T} |Y_t|^2] < \infty$.

Then there exists $Z \in \mathbf{H}^2_d(0,T)$ and $A \in \mathbf{A}^2(0,T)$, such that

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s + A_T - A_t - (K_T - K_t) - \int_t^T Z_s \mathrm{d}B_s, \tag{10}$$

where Z is the weak (resp. strong) limit of $\{Z^i\}_{i=1}^\infty$ in $\mathbf{H}^2_d(0,T)$ (resp. $\mathbf{H}^p_d(0,T)$, for $p < 2$), for each $t \in [0,T]$, A_t is the weak limit of $\{A^i_t\}_{i=1}^\infty$ in $\mathbf{L}^2(\mathcal{F}_t)$, and $K \in \mathbf{A}^2(0,T)$.

3. The RBSDE with two RCLL barriers and Dynkin game

For the existence and uniqueness of the solution of the RBSDE with two RCLL barriers, we need the notions of stochastic game and Dynkin game, which are described in the Appendix. In the following proposition, we generalize Theorem 4.1 in ([4], 1996) to the case of RCLL barriers. Set $\mathcal{T}$ be the set of all $\mathcal{F}_t$-stopping times, and for all $0 \le t \le T$, define

$$\mathcal{T}_t = \{\tau \in \mathcal{T};\ t \le \tau \le T\}. \tag{11}$$

Proposition 3.1. *Let (Y, Z, K), with $K = K^+ - K^-$ and $K^\pm \in \mathbf{A}^2(0,T)$ be a solution of the RBSDE with two RCLL barriers. For any $0 \le t \le T$ and any stopping times σ, τ in $\mathcal{T}_t$, consider the payoff*

$$R_t(\sigma, \tau) = \int_t^{\sigma\wedge\tau} g(s, Y_s, Z_s)\mathrm{d}s + \xi 1_{\{\sigma\wedge\tau = T\}} + L_\tau 1_{\{\tau < T, \tau \le \sigma\}} + U_\sigma 1_{\{\sigma < \tau\}}, \tag{12}$$

as well as the upper and lower values, respectively,

$$\begin{aligned} \overline{V}_t &= ess \inf_{\sigma\in\mathcal{T}_t} ess \sup_{\tau\in\mathcal{T}_t} E[R_t(\sigma,\tau)|\mathcal{F}_t], \\ \underline{V}_t &= ess \sup_{\tau\in\mathcal{T}_t} ess \inf_{\sigma\in\mathcal{T}_t} E[R_t(\sigma,\tau)|\mathcal{F}_t], \end{aligned} \tag{13}$$

of the corresponding stochastic game. This game has a value V_t, given by the state-process Y_t solution of RBSDE, i.e.,

$$V_t = \overline{V}_t = \underline{V}_t = Y_t. a.s. \tag{14}$$

Proof. For any $\varepsilon > 0$, consider the stopping time $\sigma_t^\varepsilon = \inf\{s \geq t, Y_s \geq U_s - \varepsilon\} \wedge T$, then $Y_{\sigma_t^\varepsilon} \geq U_{\sigma_t^\varepsilon} - \varepsilon$ on the set $\{\sigma_t^\varepsilon < T\}$; and on the set $\{\sigma_t^\varepsilon = T\}$, we have $Y_s < U_s - \varepsilon$ for $t \leq s < T$. So $Y_{s-} < U_{s-}$ for $t < s \leq \sigma_t^\varepsilon$, and with (iv) of Definition 2.1, $K^-_{\sigma_t^\varepsilon} = K^-_t$ follows. For any stopping time $\tau \in \mathcal{T}_t$, notice that $\{\sigma_t^\varepsilon = T\} \subset \{\tau \leq \sigma_t^\varepsilon\}$, so $\{\sigma_t^\varepsilon < \tau\} \subset \{\sigma_t^\varepsilon < T\}$. On the set $\{\sigma_t^\varepsilon < \tau\}$, we have

$$\begin{aligned} R_t(\sigma_t^\varepsilon, \tau) &\leq \int_t^{\sigma_t^\varepsilon} g(s, Y_s, Z_s)\mathrm{d}u + Y_{\sigma_t^\varepsilon} - (K^-_{\sigma_t^\varepsilon} - K^-_t) + \varepsilon \\ &\leq \int_t^{\sigma_t^\varepsilon} g(s, Y_s, Z_s)\mathrm{d}u + Y_{\sigma_t^\varepsilon} + (K^+_{\sigma_t^\varepsilon} - K^+_t) - (K^-_{\sigma_t^\varepsilon} - K^-_t) + \varepsilon \\ &= Y_t + \int_t^{\sigma_t^\varepsilon} Z_u \mathrm{d}B_u + \varepsilon. \end{aligned}$$

On the set $\{\tau \leq \sigma_t^\varepsilon\}$, we have

$$\begin{aligned} R_t(\sigma_t^\varepsilon, \tau) &= \int_t^{\tau} g(s, Y_s, Z_s)\mathrm{d}u + \xi 1_{\{\tau = T\}} + L_\tau 1_{\{\tau < T\}} - (K^-_\tau - K^-_t) \\ &\leq \int_t^{\tau} g(s, Y_s, Z_s)\mathrm{d}u + \xi 1_{\{\tau = T\}} + Y_\tau 1_{\{\tau < T\}} + (K^+_\tau - K^+_t) - (K^-_\tau - K^-_t) \\ &= Y_t + \int_t^{\tau} Z_u \mathrm{d}B_u. \end{aligned}$$

Now compare the two inequalities; we have $R_t(\sigma_t^\varepsilon, \tau) \leq Y_t + \int_t^{\sigma_t^\varepsilon \wedge \tau} Z_u \mathrm{d}B_u + \varepsilon$, a.s., hence

$$E[R_t(\sigma_t^\varepsilon, \tau)|\mathcal{F}_t] \leq Y_t + \varepsilon. \tag{15}$$

On the contrary, we consider the stopping time $\tau_t^\varepsilon = \inf\{s \geq t, Y_s \leq L_s + \varepsilon\} \wedge T$, then $Y_{\tau_t^\varepsilon} \leq L_{\tau_t^\varepsilon} + \varepsilon$ on the set $\{\sigma_t^\varepsilon < T\}$, and $K^+_{\tau_t^\varepsilon} = K^+_t$. For an arbitrary stopping time $\sigma \in \mathcal{T}_t$, and with a similar proof, we get $R_t(\sigma, \tau_t^\varepsilon) \geq Y_t + \int_t^{\sigma \wedge \tau_t^\varepsilon} Z(u)\mathrm{d}B_u - \varepsilon$, a.s., then

$$E[R_t(\sigma, \tau_t^\varepsilon)|\mathcal{F}_t] \geq Y_t - \varepsilon. \tag{16}$$

So we deduce

$$E[R_t(\sigma_t^\varepsilon, \tau)|\mathcal{F}_t] - \varepsilon \leq Y_t \leq E[R_t(\sigma, \tau_t^\varepsilon)|\mathcal{F}_t] + \varepsilon. \tag{17}$$

Thanks to the Lemma 5.3 in the Appendix, this stochastic game has a value, *i.e.* there exists V_t s.t. $V_t = \overline{V}_t = \underline{V}_t$. In addition, with (13) and (17), we have

$$\overline{V}_t \leq Y_t \leq \underline{V}_t,$$

i.e. $V_t = \overline{V}_t = \underline{V}_t = Y_t$. The proof is complete. □

Now we begin to prove the existence and uniqueness of the solution of the RBSDE. First we consider the RBSDE with a coefficient g, independant of y and z. In this case, from the previous result, we know the necessary form of the state-process Y_t, then we look for Z, K^+ and K^-. For this we introduce the followings:

$$\begin{aligned} N_t &= E[\xi + \int_0^T g(s)\mathrm{d}s|\mathcal{F}_t] - \int_0^t g(s)\mathrm{d}s, \\ L_t^\xi &= L_t 1_{\{t<T\}} + \xi 1_{\{t=T\}},\ \widetilde{L}_t = L_t^\xi - N_t, \\ U_t^\xi &= U_t 1_{\{t<T\}} + \xi 1_{\{t=T\}}, \widetilde{U}_t = U_t^\xi - N_t. \end{aligned} \tag{18}$$

Obviously, N_t is a continuous process on $[0,T]$ and $N_t \in \mathbf{D}^2(0,T)$. Then $\widetilde{L}_t, \widetilde{U}_t$ are RCLL processes on $[0,T]$, belong to $\mathbf{D}^2(0,T)$, and

$$\begin{aligned} \widetilde{L}_t &\leq \widetilde{U}_t,\ 0 \leq t \leq T, \\ \widetilde{L}_{T-} &\leq \widetilde{L}_T = 0 = \widetilde{U}_T \leq \widetilde{U}_{T-}. \end{aligned}$$

Then from (12), we get

$$E[R_t(\sigma,\tau)|\mathcal{F}_t] = E[\widetilde{L}_\tau 1_{\{\tau\leq\sigma\}} + \widetilde{U}_\sigma 1_{\{\sigma<\tau\}}|\mathcal{F}_t] + N_t.$$

If we consider the Dynkin game problem with payoff $R_t(\sigma,\tau)$, with $t=0$, player 1 chooses the stopping time σ, player 2 chooses the stopping time τ, then $R_0(\sigma,\tau)$ represents the amount paid by player 1 to player 2. So player 1 tries to minimize the payoff while player 2 tries to maximize it. The game stops when one player decides to stop, that is, at the stopping time $\sigma\wedge\tau$, or at T if $\sigma=\tau=T$. From Proposition 3.1, if the value of the Dynkin game exists, then Y_t satisfies

$$\begin{aligned} Y_t &= ess \inf_{\sigma\in\mathcal{T}_t} ess \sup_{\tau\in\mathcal{T}_t} E[\widetilde{L}_\tau 1_{\{\tau\leq\sigma\}} + \widetilde{U}_\sigma 1_{\{\sigma<\tau\}}|\mathcal{F}_t] + N_t \\ &= ess \sup_{\tau\in\mathcal{T}_t} ess \inf_{\sigma\in\mathcal{T}_t} E[\widetilde{L}_\tau 1_{\{\tau\leq\sigma\}} + \widetilde{U}_\sigma 1_{\{\sigma<\tau\}}|\mathcal{F}_t] + N_t. \end{aligned} \tag{19}$$

Thanks to Theorem 5.2 in the Appendix, we turn to the following system to study the value of the Dynkin game

$$\begin{aligned} X^+ &= S(\widetilde{L} + X^-), \\ X^- &= S(-\widetilde{U} + X^+), \end{aligned} \tag{20}$$

where S denote the Snell envelope (see Def. 5.1 in Appendix). This system was introduced by Bismut ([3], 1977) and was studied by him and Alario-Nazaret (1982). In the Appendix, we remember some results of Alario-Nazaret in her thesis ([1], 1982) and in [2]. The following theorem is deduced from Theorem 5.1 in the Appendix.

Theorem 3.1. *The system (20) admits a solution* (X^+, X^-) *in* $\mathbf{D}^2(0,T)\times\mathbf{D}^2(0,T)$.

Proof. This theorem is the direct application of Theorem 5.1 in the Appendix; the only thing that we need to point out is that Assumption 2.4 leads to

$$\widetilde{L} \leq \widetilde{X} - \widetilde{X}' \leq \widetilde{U}$$

for some positive $\mathcal{F}_t$-supermartingales $(\widetilde{X}, \widetilde{X}')$ of class $\mathcal{D}[0,T]$. It's easily seen if we take

$$\begin{aligned} \widetilde{X}_t &= J_0^+ + \int_0^t \phi_s^+ \mathrm{d}B_s + E[\xi^+ + \int_0^T g^+(s)\mathrm{d}s|\mathcal{F}_t] - \int_0^t g^+(s)\mathrm{d}s - V_t^+ - (J_T-\xi)^+ 1_{\{t=T\}}, \\ \widetilde{X}'_t &= J_0^- + \int_0^t \phi_s^- \mathrm{d}B_s + E[\xi^- + \int_0^T g^-(s)\mathrm{d}s|\mathcal{F}_t] - \int_0^t g^-(s)\mathrm{d}s - V_t^- - (J_T-\xi)^- 1_{\{t=T\}}, \end{aligned}$$

where J^+, ϕ^+, ξ^+, g^+ and $(J_T-\xi)^+$ (resp. J^-, ϕ^-, ξ^-, g^- and $(J_T-\xi)^-$) are the positive (resp. negative) part of J, ϕ, ξ, g and $(J_T-\xi)$ respectively. Then $\widetilde{X}$ and $\widetilde{X}'$ belong to $\mathcal{D}[0,T]$ by the assumptions on ξ, g, J, ϕ and $V^{\pm}$. □

With these results, we get the following theorem, which gives the method to find the processes Z, K^+ and K^-. The proof of this theorem is in the same way like the continuous case in [4], even easier, since in the discontinuous case, we do not need to prove the continuity of Y.

Theorem 3.2. *Let us consider the equation*

$$\begin{aligned}\pi(K^+) &= S(\widetilde{L}+\pi(K^-)) \\ \pi(K^-) &= S(-\widetilde{U}+\pi(K^+))\end{aligned} \tag{21}$$

where S denotes the Snell envelope and $\pi_t(V)=E[V_T|\mathcal{F}_t]-V_t$. If we suppose the Assumption 2.4, this equation has a solution $(K^+,K^-)\in\mathbf{A}^2(0,T)\times\mathbf{A}^2(0,T)$; then the triple (Y,Z,K), where $K=K^+-K^-$,

$$Y := N+\pi(K^+)-\pi(K^-) \tag{22}$$

and $Z\in\mathbf{H}^2_d(0,T)$ uniquely determined via

$$E[\xi+\int_0^T g(s)\mathrm{d}s + A_T - K_T^-|\mathcal{F}_t] = N(0)+E[K_T^+]-E[K_T^-]+\int_0^t Z_s\mathrm{d}B_s, 0\le t\le T, \tag{23}$$

is the unique solution of the RBSDE.

Proof. Since Assumption 2.4 is satisfied, by Theorem 3.1 the system (20) admits a solution $(X^+,X^-)\in\mathbf{D}^2(0,T)\times\mathbf{D}^2(0,T)$. By Lemma 5.1 in the Appendix, there exists a pair $(K^+,K^-)\in\mathbf{A}^2(0,T)\times\mathbf{A}^2(0,T)$ which solves the equation (21). In fact, (21) is equivalent to (20) when we set $X^+=\pi(K^+)$, $X^-=\pi(K^-)$.

Then by Theorem 5.2 in the Appendix, $Y=N+\pi(K^+)-\pi(K^-)$ is the value of a Dynkin game as (19), and by (18), (22), and (23), we have

$$Y_t+\int_0^t g(s)\mathrm{d}s + K_t^+ - K_t^- = E[\xi+\int_0^T g(s)\mathrm{d}s + K_t^+ - K_t^-|\mathcal{F}_t] = Y(0)+\int_0^t Z_s\mathrm{d}B_s, \tag{24}$$

for $0\le t\le T$, where $Y(0)=N(0)+E[K_T^+-K_T^-]$; in particular, $Y_T=\xi$; thus

$$\xi+\int_0^T g(s)\mathrm{d}s + K_T^+ - K_T^- = Y(0)+\int_0^T Z_s\mathrm{d}B_s. \tag{25}$$

From (24) and (25), we deduce the part (ii) of the definition 2.1:

$$Y_t=\xi+\int_t^T g(s)\mathrm{d}s + K_T^+ - K_t^+ - (K_T^- - K_t^-) - \int_t^T Z_s\mathrm{d}B_s.$$

From the definition of the Snell envelope (21) we have

$$\begin{aligned}\pi(K^+) &\ge \widetilde{L}+\pi(K^-), \\ \pi(K^-) &\ge -\widetilde{U}+\pi(K^+).\end{aligned}$$

Then with (18) and (22), it follows

$$L\le N+\widetilde{L}\le Y = N+\pi(K^+)-\pi(K^-)\le \widetilde{U}+N\le U.$$

Since the process K^+ (resp. K^-) is the increasing process of the decomposition of the Snell envelope $S(\widetilde{L}+\pi(K^-))$ (resp. $S(-\widetilde{U}+\pi(K^+))$), by the Lemma 5.1 in the Appendix, we get

$$\begin{aligned} 0 &= \int_0^T (S_{t-}(\widetilde{L}+\pi(K^-)) - \widetilde{L}_{t-} - \pi_{t-}(K^-))\mathrm{d}K_t^+ = \int_0^T (Y_{t-}-L_{t-})\mathrm{d}K_t^+, \\ 0 &= \int_0^T (S_{t-}(-\widetilde{U}+\pi(K^+)) + \widetilde{U}_{t-} - \pi_{t-}(K^+))\mathrm{d}K_t^- = \int_0^T (U_{t-}-Y_{t-})\mathrm{d}K_t^-, \end{aligned}$$

almost surely, which shows that (iii) and (iv) of Definition 2.1 are satisfied.

Finally for (i) of Definition 2.1, we know that the equation (21) has a fixed point $(K^+,K^-)\in \mathbf{A}^2(0,T)\times \mathbf{A}^2(0,T)$, with $N_t\in \mathbf{D}^2(0,T)$; it follows that $Y_t\in \mathbf{D}^2(0,T)$, and $Z\in \mathbf{H}^2_d(0,T)$ comes from the Itô representation of the square-integrable martingale $E[\xi+\int_0^T g(s)\mathrm{d}s + K_T^+ - K_T^-|\mathcal{F}_t]$. Uniqueness follows from Proposition 3.1. □

Finally, we get the following theorem.

Theorem 3.3. *For a given $\xi\in \mathbf{L}^2(\mathcal{F}_T)$, a process $g(t,\omega)\in \mathbf{H}^2(0,T)$, and two RCLL progressively measurable real-valued processes L,U, which satisfy assumptions 2.3 and 2.4, there exists a unique (Y,Z,K), with $Y\in \mathbf{D}^2(0,T)$, $Z\in \mathbf{H}^2_d(0,T)$, $K=K^+-K^-$, with $K^+,K^-\in \mathbf{A}^2(0,T)$, which is solution of the RBSDE with barriers L and U.*

Now we will consider the general case that is when g may depend on (y,z); for this we shall use a fixed point method. This method was firstly introduced by Pardoux and Peng ([10], 1990), and also used by Cvitanic and Karatzas ([4], 1996) in the case of two continuous barriers.

Theorem 3.4. *Let ξ be a given random variable in $\mathbf{L}^2(\mathcal{F}_T)$, a coefficient g which satisfies Assumption 2.2, and two RCLL progressively measurable real-valued processes L and U, which satisfy Assumptions 2.3 and 2.4. Then there exists a unique triplet (Y,Z,K), with $Y\in \mathbf{D}^2(0,T)$, $Z\in \mathbf{H}^2_d(0,T)$, $K=K^+-K^-$ and $K^+,K^-\in \mathbf{A}^2(0,T)$, which is solution of the RBSDE with two barriers L,U. The uniqueness holds in the following sense: if there exists another (Y',Z',K') with $K'=K'^+-K'^-$ and $K'^{\pm}\in \mathbf{A}^2(0,T)$, satisfying (i)-(iv) of Definition 2.1, we have $Y_t=Y'_t$, $Z_t=Z'_t$, $K_t=K'_t$, for $0\le t\le T$.*

Proof. Denote by $\mathcal{S}$, the space of progressively measurable processes $\{(Y_t,Z_t), 0\le t\le T\}$ valued in $\mathbb{R}\times\mathbb{R}^d$, which satisfy $E\int_0^T |Y_s|^2+|Z_s|^2\,\mathrm{d}s<\infty$. Given $(\varphi,\psi)\in\mathcal{S}$, we define $\overline{g}(t,\omega)$ by setting $\overline{g}(t,\omega)=g(t,\omega,\varphi(t,\omega),\psi(t,\omega))$; then by the Theorem 3.3, there exists a unique solution (Y,Z,K), $K=K^+-K^-$to the RBSDE wih coefficient $\overline{g}$ and $(Y,Z,K^+,K^-)\in \mathbf{D}^2(0,T)\times \mathbf{H}^2_d(0,T)\times(\mathbf{A}^2(0,T))^2$. In particular, $(Y,Z)\in\mathcal{S}$. In this way, we construct a mapping

$$\Phi:\mathcal{S}\longmapsto\mathcal{S}\ ,\ \ via\ (Y,Z)=\Phi(\varphi,\psi).$$

In order to establish the unique solution of the RBSDE, it is sufficient to prove that the mapping Φ is a contraction with respect to an appropriate norm on $\mathcal{S}$, defined by

$$\|(Y,Z)\|_\beta := \left(E\left[\int_0^T \mathrm{e}^{\beta t}(|Y_t|^2+|Z_t|^2)\mathrm{d}t\right]\right)^{\frac{1}{2}},$$

for an appropriate $\beta\in(0,\infty)$ which will be determined later.

Let (φ^0,ψ^0) be another pair in the set $\mathcal{S}$, $(Y^0,Z^0)=\Phi(\varphi^0,\psi^0)$ with K^0, be the unique solution of the RBSDE with coefficient function $\overline{g}^0(t,\omega)=g(t,\omega,\varphi^0(t,\omega),\psi^0(t,\omega))$. We define

$$\overline{\varphi}=\varphi-\varphi^0, \overline{\psi}=\psi-\psi^0, \overline{Y}=Y-Y^0, \overline{Z}=Z-Z^0, \overline{K}=K-K^0.$$

Clearly, $d\overline{Y}_t=[g(t,\varphi_t,\psi_t)-g(t,\varphi^0_t,\psi^0_t)]dt-d\overline{K}_t+\overline{Z}_t dB_t$, and $Y_t-Y_{t-}=-(K_t-K_{t-})$, $Y^0_t-Y^0_{t-}=-(K^0_t-K^0_{t-})$, so $\overline{Y}_t-\overline{Y}_{t-}=-(\overline{K}_t-\overline{K}_{t-})$. Applying Itô's formula to $e^{\beta t}\overline{Y}^2_t$, and taking expectation on the two sides, we get

$$
\begin{aligned}
& E[e^{\beta t}\overline{Y}^2_t]+E[\int_t^T e^{\beta s}(\beta\left|\overline{Y}_s\right|^2+\left|\overline{Z}_s\right|^2)ds]+E[\sum_{s\in[t,T]}((\overline{K}_s-\overline{K}_{s-})^2] \qquad (26)\\
= & \ 2E\int_t^T e^{\beta s}\overline{Y}_{s-}d\overline{K}_s-2E\int_t^T e^{\beta s}\overline{Y}_s\overline{Z}_s dB_s+2E\int_t^T e^{\beta s}\overline{Y}_s[g(s,\varphi_s,\psi_s)-g(s,\varphi^0_s,\psi^0_s)]dt\\
\le & \ 2kE\int_t^T e^{\beta s}\left|\overline{Y}_s\right|(\left|\overline{\varphi}_s\right|+\left|\overline{\psi}_s\right|^2)dt\\
\le & \ 4k^2E\int_t^T e^{\beta s}\left|\overline{Y}_s\right|^2 ds+\frac{1}{2}E\int_t^T e^{\beta s}(\left|\overline{\varphi}_s\right|^2+\left|\overline{\psi}_s\right|^2)dt,
\end{aligned}
$$

where k is the Lipschitz constant in (7). For the Itô integral term in the second line, we have

$$
\begin{aligned}
E\left(\int_0^T e^{2\beta s}(\overline{Y}_s)^2\left|\overline{Z}_s\right|^2 ds\right)^{\frac{1}{2}} & \le e^{\beta T}E\left[\sup_{t\le s\le T}\left|\overline{Y}_s\right|\left(\int_0^T\left|\overline{Z}_s\right|ds\right)^{\frac{1}{2}}\right]\\
& \le \frac{1}{2}e^{\beta T}E\left[\sup_{0\le t\le T}(\overline{Y}_s)^2+\int_0^T\left|\overline{Z}_s\right|ds\right]<\infty,
\end{aligned}
$$

since from well-known inequalities for semi-martingales $\sup_{s\le T}\left|\overline{Y}_s\right|\in\mathbf{L}^2(F_T)$. Then we know that this term is P-integrable with zero expectation.

For the term $E\int_t^T e^{\beta s}\overline{Y}_{s-}d\overline{K}_s=E\int_t^T e^{\beta s}\overline{Y}_{s-}d(\overline{K}^+_s-\overline{K}^-_s)$, notice that since (Y,Z,K), (Y^0,Z^0,K) satisfy (iii) and (iv) in Definition 2.1, we have

$$
\int_t^T e^{\beta s}\overline{Y}_{s-}d\overline{K}_s=\int_t^T e^{\beta s}\overline{Y}_{s-}d\overline{K}^+_s-\int_t^T e^{\beta s}\overline{Y}_{s-}d\overline{K}^-_s\le 0,
$$

in view of

$$
\begin{aligned}
\int_t^T e^{\beta s}\overline{Y}_{s-}d\overline{K}^+_s & = \int_t^T e^{\beta s}(Y_{s-}-Y^0_{s-})dK^+_s+\int_t^T e^{\beta s}(Y^0_{s-}-Y_{s-})dK^{0+}_s\\
& = \int_t^T e^{\beta s}(Y_{s-}-L_{s-})dK^+_s+\int_t^T e^{\beta s}(L_{s-}-Y^0_{s-})dK^+_s+\int_t^T e^{\beta s}(Y^0_{s-}-L_{s-})dK^{0+}_s\\
& \quad +\int_t^T e^{\beta s}(L_{s-}-Y_{s-})dK^{0+}_s\\
& \le 0,
\end{aligned}
$$

and similarly $\int_t^T e^{\beta s}\overline{Y}_{s-}d\overline{K}^-_s\ge 0$.

Now if we choose $t=0$ and $\beta=1+4k^2$ in the definition of the norm, we deduce from the inequality (26),

$$
E\left[\int_0^T e^{\beta s}\left(\left|\overline{Y}_s\right|^2+\left|\overline{Z}^2_s\right|\right)ds\right]\le\frac{1}{2}E\int_0^T e^{\beta s}\left(\left|\overline{\varphi}_s\right|^2+\left|\overline{\psi}^2_s\right|\right)dt,
$$

i.e. the mapping Φ is a contraction. The proof is complete. □

4. Dynkin game and the penalization method for the RBSDE with two RCLL barriers

In this section we will give another proof for the existence of a solution for reflected BSDEs with two RCLL barriers (Th. 3.4), which is based on a penalization method. For each $m, n \in \mathbb{N}$, since $g(s,y,z) + n(y - L_s)^- - m(U_s - y)^-$ is Lipschitz in (y, z), the following classical BSDE (*cf.* [10]) admits the unique solution $(Y^{m,n}, Z^{m,n})$

$$Y_t^{m,n} = \xi + \int_t^T g(s, Y_s^{m,n}, Z_s^{m,n})\mathrm{d}s + n\int_t^T (Y_s^{m,n} - L_s)^- \mathrm{d}s - m\int_t^T (U_s - Y_s^{m,n})^- \mathrm{d}s - \int_t^T Z_s^{m,n}\mathrm{d}B_s \quad (27)$$

when ξ and g satisfy Assumptions 2.1 and 2.2, L and U satisfy Assumptions 2.3 and 2.4. We set $K_t^{m,n,+} = n\int_0^t (L_s - Y_s^{m,n})^+ \mathrm{d}s$ and $K_t^{m,n,-} = m\int_0^t (U_s - Y_s^{m,n})^- \mathrm{d}s$.

We begin with establishing several basic estimates for $(Y^{m,n}, Z^{m,n}, K^{m,n,+}, K^{m,n,-})$. These estimates will be useful to prove the existence of a solution provided in this section.

Proposition 4.1. *We assume that Assumption 2.4 holds. Then there exists a constant C, independent of m and n, such that the following estimate holds:*

$$E\left[\sup_{0\le t\le T} (Y_t^{m,n})^2\right] + E\left[\int_0^T |Z_s^{m,n}|^2 \,\mathrm{d}s\right] + E[(K_T^{m,n,+})^2] + E[(K_T^{m,n,-})^2] \le C. \quad (28)$$

To prove this result, we need the following lemma.

Lemma 4.1. *There exists a triple (Y^*, Z^*, K^*), with $K^* = K^{*+} - K^{*-}$, and $Y^* \in \mathbf{D}^2(0,T)$, $Z^* \in \mathbf{H}_d^2(0,T)$ and $K^{*+}, K^{*-} \in \mathbf{A}^2(0,T)$, such that*

$$Y_t^* = \xi + \int_t^T g(s, Y_s^*, Z_s^*)\mathrm{d}s + K_T^{*+} - K_t^{*+} - (K_T^{*-} - K_t^{*-}) - \int_t^T Z_s^*\mathrm{d}B_s, \quad (29)$$

and $L_t \le Y_t^* \le U_t$, $\mathrm{d}P \otimes \mathrm{d}t$-a.s.

Proof. Let J defined in Section 1 and set $J_t^* = J_t + (\xi - J_T)1_{\{t=T\}}$, $v_t^+ = V_t^+ + (\xi - J_t)^- 1_{\{t=T\}}$, $v_t^- = V_t^- + (\xi - J_t)^+ 1_{\{t=T\}}$; then $v^\pm \in \mathbf{A}^2(0,T)$, J_t^* is still an RCLL semimartingale, and by BDG inequality $E[\sup_{0\le t\le T}(J_t^*)^2] \le C$, where C is a constant. Obviously, $L_t \le J_t^* \le U_t$ and

$$\begin{aligned} J_t^* &= \xi - \int_t^T \phi_s \mathrm{d}B_s + (v_T^+ - v_t^+) - (v_T^- - v_t^-) \\ &= \xi + \int_t^T g(s, J_s^*, \phi_s)\mathrm{d}s - \left(\int_t^T g(s, J_s^*, \phi_s)\mathrm{d}s + (v_T^+ - v_t^+) - (v_T^- - v_t^-)\right) - \int_t^T \phi_s \mathrm{d}B_s. \end{aligned}$$

Then if we set $K_t^{*+} = v_t^+ + \int_0^t g^+(s, J_s^*, \phi_s)\mathrm{d}s$, $K_t^{*-} = v_t^- + \int_0^t g^-(s, J_s^*, \phi_s)\mathrm{d}s$, so K^{*+}, $K^{*-} \in \mathbf{A}^2(0,T)$, $Y^* = J^* \in \mathbf{D}^2(0,T)$, $Z^* = \phi \in \mathbf{H}_d^2(0,T)$, and (Y^*, Z^*, K^*) satisfies (29). □

Proof of Proposition 4.1. Let (Y^*, Z^*, K^*) with $K^* = K^{*+} - K^{*-}$ be given as in Lemma 4.1. Then for $m, n \in \mathbb{N}$, the triplet also satisfies

$$\begin{aligned} Y_t^* &= \xi + \int_t^T g(s, Y_s^*, Z_s^*)\mathrm{d}s + K_T^{*+} - K_t^{*+} - (K_T^{*-} - K_t^{*-}) \\ &\quad + n\int_t^T (L_s - Y_s^*)^+ \mathrm{d}s - m\int_t^T (Y_s^* - U_s)^+ \mathrm{d}s - \int_t^T Z_s^* \mathrm{d}B_s. \end{aligned}$$

Set $(\overline{Y}^{m,n}, \overline{Z}^{m,n})$, $(\widetilde{Y}^{m,n}, \widetilde{Z}^{m,n})$ be respectively the solutions of the following equations,

$$\begin{aligned}\overline{Y}_t^{m,n} &= \xi + \int_t^T g(s, \overline{Y}_s^{m,n}, \overline{Z}_s^{m,n})\mathrm{d}s + K_T^{*+} - K_t^{*+} \\ &\quad + n\int_t^T (L_s - \overline{Y}_s^{m,n})^+\mathrm{d}s - m\int_t^T (\overline{Y}_s^{m,n} - U_s)^+\mathrm{d}s - \int_t^T \overline{Z}_s^{m,n}\mathrm{d}B_s.\end{aligned}$$

$$\begin{aligned}\widetilde{Y}_t^{m,n} &= \xi + \int_t^T g(s, \widetilde{Y}_s^{m,n}, \widetilde{Z}_s^{m,n})\mathrm{d}s - (K_T^{*-} - K_t^{*-}) \\ &\quad + n\int_t^T (L_s - \widetilde{Y}_s^{m,n})^+\mathrm{d}s - m\int_t^T (\widetilde{Y}_s^{m,n} - U_s)^+\mathrm{d}s - \int_t^T \widetilde{Z}_s^{m,n}\mathrm{d}B_s.\end{aligned}$$

By the comparison theorem for BSDE's, we obtain that for any $m, n \in \mathbb{N}$, $\overline{Y}_t^{m,n} \geq Y_t^{m,n} \geq \widetilde{Y}_t^{m,n}$ and $\overline{Y}_t^{m,n} \geq Y_t^* \geq L_t$, $\widetilde{Y}_t^{m,n} \leq Y_t^* \leq U_t$, so $(\overline{Y}^{m,n}, \overline{Z}^{m,n})$ is also solution of

$$\overline{Y}_t^{m,n} = \xi + \int_t^T g(s, \overline{Y}_s^{m,n}, \overline{Z}_s^{m,n})\mathrm{d}s + K_T^{*+} - K_t^{*+} - m\int_t^T (\overline{Y}_s^{m,n} - U_s)^+\mathrm{d}s - \int_t^T \overline{Z}_s^{m,n}\mathrm{d}B_s, \tag{30}$$

and $(\widetilde{Y}^{m,n}, \widetilde{Z}^{m,n})$ is also solution of

$$\widetilde{Y}_t^{m,n} = \xi + \int_t^T g(s, \widetilde{Y}_s^{m,n}, \widetilde{Z}_s^{m,n})\mathrm{d}s - (K_T^{*-} - K_t^{*-}) + n\int_t^T (L_s - \widetilde{Y}_s^{m,n})^+\mathrm{d}s - \int_t^T \widetilde{Z}_s^{m,n}\mathrm{d}B_s. \tag{31}$$

Then let us consider the following BSDEs

$$Y_t^+ = \xi + \int_t^T g(s, Y_s^+, Z_s^+)\mathrm{d}s + K_T^{*+} - K_t^{*+} - \int_t^T Z_s^+\mathrm{d}B_s, \tag{32}$$

$$Y_t^- = \xi + \int_t^T g(s, Y_s^-, Z_s^-)\mathrm{d}s - (K_T^{*-} - K_t^{*-}) - \int_t^T Z_s^-\mathrm{d}B_s. \tag{33}$$

Since $\overline{K}_t^{m,n,-} = m\int_0^t (\overline{Y}_s^{m,n} - U_s)^+\mathrm{d}s$ and $\widetilde{K}_t^{m,n,+} = n\int_0^t (L_s - \widetilde{Y}_s^{m,n})^+\mathrm{d}s$ are increasing processes, then using the comparison theorem for (30) and (32), (31) and (33), with (27), we get

$$Y_t^+ \geq \overline{Y}_t^{m,n} \geq Y_t^{m,n} \geq \widetilde{Y}_t^{m,n} \geq Y_t^-, \tag{34}$$

for any $m, n \in \mathbb{N}, \forall t \in [0, T]$. Then we have

$$E[\sup_{0\leq t\leq T}(Y_t^{m,n})^2] \leq \max\{E[\sup_{0\leq t\leq T}(Y_t^+)^2], E[\sup_{0\leq t\leq T}(Y_t^-)^2]\}. \tag{35}$$

Since $K^{*\pm} \in \mathbf{A}^2(0,T)$, by Itô's formula and BDG inequality, it follows that

$$E[\sup_{0\leq t\leq T}(Y_t^+)^2] \leq c, E[\sup_{0\leq t\leq T}(Y_t^-)^2] \leq c.$$

Using (35), we get that there exists a constant c independent of m, n, such that

$$E[\sup_{0\leq t\leq T}(Y_t^{m,n})^2] \leq c. \tag{36}$$

Now we consider the last two terms of (28). First, since for any $m,n\in\mathbb{N}$, $\widetilde{Y}_t^{m,n}\leq Y_t^{m,n}$, then $\widetilde{K}_t^{m,n,+}\geq K_t^{m,n,+}\geq 0$. So if $E[(\widetilde{K}_T^{m,n,+})^2]\leq c$, then $E[(K_T^{m,n,+})^2]\leq c$. Rewrite (31) into the following form

$$\widetilde{K}_t^{m,n,+}=\widetilde{Y}_0^{m,n}-\widetilde{Y}_t^{m,n}-\int_0^t g(s,\widetilde{Y}_s^{m,n},\widetilde{Z}_s^{m,n})\mathrm{d}s+K_t^{*-}+\int_0^t\widetilde{Z}_s^{m,n}\mathrm{d}B_s. \tag{37}$$

Notice that from (34) we have

$$E[\sup_{0\leq t\leq T}(\widetilde{Y}_t^{m,n})^2]\leq\max\{E[\sup_{0\leq t\leq T}(Y_t^+)^2],E[\sup_{0\leq t\leq T}(Y_t^-)^2]\}\leq c,$$

and $E[(K_T^{*-})^2]\leq c_1$; then with the Lipschitz property of g, taking square and expectation on the both sides of (37), we get

$$E[(\widetilde{K}_T^{m,n,+})^2]\leq c+c_2E\int_0^T\left|\widetilde{Z}_s^{m,n}\right|^2\mathrm{d}s. \tag{38}$$

Then applying Itô's formula to $\left|\widetilde{Y}_t^{m,n}\right|^2$, with classical technics and (38), it follows that

$$E[(\widetilde{K}_T^{m,n,+})^2]\leq c,\text{then } E[(K_T^{m,n,+})^2]\leq c.$$

In the same way, we deduce that $E[(K_T^{m,n,-})^2]\leq c$. Applying Itô's formula to $|Y_t^{m,n}|^2$, then

$$\begin{aligned}E[|Y_t^{m,n}|^2]+E\left[\int_t^T|Z_s^{m,n}|^2\,\mathrm{d}s\right]&\\ \leq c\left(1+\int_t^T|Y_s^{m,n}|^2\,\mathrm{d}s+\alpha\int_t^T|Z_s^{m,n}|^2\,\mathrm{d}s\right)&+E\left[\sup_{0\leq t\leq T}(L_t^+)^2\right]+E\left[\sup_{0\leq t\leq T}(U_t^-)^2\right]\\ &+E[(K_T^{m,n,+})^2]+E[(K_T^{m,n,-})^2].\end{aligned}$$

Set $\alpha=\frac{1}{3c}$, finally, we get $E[\int_0^T|Z_s^{m,n}|^2\,\mathrm{d}s]\leq c$. □

In (27), for fixed m, we set $g^m(s,y,z)=g(s,y,z)-m(U_s-y)^-$; obviously, g^m is Lipschitz and

$$E\int_0^T(g^m(s,0,0))^2\mathrm{d}s\leq 2E\int_0^T(g(s,0,0))^2\mathrm{d}s+2m^2TE\sup_{0\leq t\leq T}(U_t^-)^2<\infty.$$

By the classical comparison theorem of BSDEs, we know that $(Y^{m,n})$ is increasing in n for any fixed m. Thanks to the results for the RBSDE with one RCLL barrier obtained in [9], when $n\to\infty$ we know that $(Y^{m,n})\nearrow Y^{m,\infty}$ in $\mathbf{H}^2(0,T)$, $(Z^{m,n})\to Z^{m,\infty}$ weakly in $\mathbf{H}_d^2(0,T)$, $K_t^{m,n,+}\to K_t^{m,\infty,+}$weakly in $\mathbf{L}^2(\mathcal{F}_t)$, and that $(Y^{m,\infty},Z^{m,\infty},K^{m,\infty,+})$ is the solution of the following RBSDE with one lower barrier L,

$$Y_t^{m,\infty}=\xi+\int_t^T g(s,Y_s^{m,\infty},Z_s^{m,\infty})\mathrm{d}s+K_T^{m,\infty,+}-K_t^{m,\infty,+}-m\int_t^T(U_s-Y_s^{m,\infty})^-\mathrm{d}s-\int_t^T Z_s^{m,\infty}\mathrm{d}B_s, \tag{39}$$

$Y_t^{m,\infty}\geq L_t$, $0\leq t\leq T$, and $\int_0^T(Y_t^{m,\infty}-L_t)\mathrm{d}K_t^{m,\infty,+}=0$, a.s.. Then set $K_t^{m,\infty,-}=m\int_0^t(U_s-Y_s^{m,\infty})^-\mathrm{d}s$; with (28) we have the following lemma.

Lemma 4.2. *There exists a constant C independent of m such that*

$$\sup_{0\leq t\leq T}E(Y_t^{m,\infty})^2+E\int_0^T|Z_t^{m,\infty}|^2\,\mathrm{d}t+E(K_T^{m,\infty,+})^2+E(K_T^{m,\infty,-})^2\leq C. \tag{40}$$

Using the BDG inequality, it follows

$$E\left(\sup_{0\le t\le T}(Y_t^{m,\infty})^2\right)\le C.$$

From the comparison Theorem 3.4 in [9], we have $Y_t^{m,\infty}\ge Y_t^{m+1,\infty}$; we conclude that there exists a process Y such that $Y^{m,\infty}\searrow Y$, and using Fatou's Lemma, we get

$$E\left(\sup_{0\le t\le T}(Y_t)^2\right)\le C. \tag{41}$$

By the dominated convergence theorem, it follows that $Y^{m,\infty}\to Y$ as $m\to\infty$, in $\mathbf{H}^2(0,T)$. Using Theorem 3.4 in [9] again, we know that

$$K_t^{m,\infty,+}\ge K_t^{m+1,\infty,+}, K_t^{m,\infty,+}-K_s^{m,\infty,+}\ge K_t^{m+1,\infty,+}-K_s^{m+1,\infty,+},$$

for $0\le s\le t\le T$. With (40), we deduce that there exists a process K^+ s.t., for $t\in[0,T]$, $K_t^{m,\infty,+}\searrow K_t^+$ in $\mathbf{L}^2(\mathcal{F}_t)$. Obviously K^+ is an increasing process and $E[(K_T^+)^2]\le c$. So the assumptions of Theorem 2.1 are satisfied, and we deduce that the limit Y satisfies

$$Y_t=\xi+\int_t^T g(s,Y_s,Z_s)\mathrm{d}s+K_T^+-K_t^+-(K_T^- -K_t^-)-\int_t^T Z_s\mathrm{d}B_s, \tag{42}$$

where K_t^- is the weak limit of $K_t^{m,\infty,-}$ in $\mathbf{L}^2(\mathcal{F}_t)$, and $Z^{m,\infty}$ strongly converges to Z in $\mathbf{H}_d^p(0,T)$, for $p<2$.

Similarly, $(Y^{m,n})$ is decreasing on m for any fixed n; let $m\to\infty$, then by the results of [9], $(Y^{m,n})\searrow Y^{\infty,n}$ in $\mathbf{H}^2(0,T)$, $(Z^{m,n})\to Z^{\infty,n}$ weakly in $\mathbf{H}_d^2(0,T)$, $K_t^{m,n,-}\to K_t^{\infty,n,-}$weakly in $\mathbf{L}^2(\mathcal{F}_t)$, and $(Y^{\infty,n},Z^{\infty,n},K^{\infty,n,-})$ is the solution of the following RBSDE with one upper barrier U, *i.e.*

$$Y_t^{\infty,n}=\xi+\int_t^T g(s,Y_s^{\infty,n},Z_s^{\infty,n})\mathrm{d}s+n\int_t^T(Y_s^{\infty,n}-L_s)^-\mathrm{d}s-K_T^{\infty,n,-}+K_t^{\infty,n,-}-\int_t^T Z_s^{\infty,n}\mathrm{d}B_s, \tag{43}$$

$Y_t^{\infty,n}\le U_t$, $0\le t\le T$, $\int_0^T(Y_t^{\infty,n}-U_t)\mathrm{d}K_t^{\infty,n,-}=0$. Set $K_t^{\infty,n,+}=n\int_0^t(Y_s^{\infty,n}-L_s)^-\mathrm{d}s$; then

$$\sup_{0\le t\le T}E(Y_t^{\infty,n})^2+E\int_0^T\left|Z_t^{\infty,n}\right|^2\mathrm{d}t+E(K_T^{\infty,n,-})^2+E(K_T^{\infty,n,+})^2\le C. \tag{44}$$

Then by the comparison Theorem 3.4 in [9], and the above estimation, we get that there exists a process $Y'\in\mathbf{S}^2(0,T)$ such that $Y^{\infty,n}\nearrow Y'$ and the convergence also holds in $\mathbf{H}^2(0,T)$. Finally with Theorem 2.1, we get that the limit Y' satisfies

$$Y_t'=\xi+\int_t^T g(s,Y_s',Z_s')\mathrm{d}s+K_T'^+-K_t'^+-(K_T'^- -K_t'^-)-\int_t^T Z_s'\mathrm{d}B_s. \tag{45}$$

Here $Z^{\infty,n}$ strongly converges to Z in $\mathbf{H}_d^p(0,T)$, for $p<2$, $K_t'^-$(resp. $K_t'^+$) is the weak limit of $K_t^{\infty,n,-}$ (resp. $K_t^{\infty,n,+}$) in $\mathbf{L}^2(\mathcal{F}_t)$. Now we want to prove that the two limits are equal.

Lemma 4.3. *The two limits Y and Y' are equal.*

Proof. Since $Y^{m,n}\nearrow Y^{m,\infty}$ and $Y^{m,n}\searrow Y^{\infty,n}$, so for $\forall m,n\in\mathbb{N}$, $Y^{\infty,n}\le Y^{m,n}\le Y^{m,\infty}$. Then with $Y^{m,\infty}\searrow Y$, $Y^{\infty,n}\nearrow Y'$, it follows $Y\ge Y'$. On the other hand, consider (27) and (43), due to $Y^{\infty,n}\le Y^{m,n}$, it follows that for $0\le s\le t\le T$,

$$K_t^{m,n,+}-K_s^{m,n,+}\le K_t^{\infty,n,+}-K_s^{\infty,n,+}.$$

Otherwise we know that $K_t^{m,n,+} \to K_t^{m,\infty,+}$weakly in $\mathbf{L}^2(\mathcal{F}_t)$, $K_t^{\infty,n,+} \to K_t^{'+}$ weakly in $\mathbf{L}^2(\mathcal{F}_t)$, as $n \to \infty$ and $K_t^{m,\infty,+} \to K_t^+$ strongly in $\mathbf{L}^2(\mathcal{F}_t)$, as $m \to \infty$. In the previous inequality, first let $n \to \infty$, then let $m \to \infty$, we get

$$K_t^+ - K_s^+ \leq K_t^{'+} - K_s^{'+}. \tag{46}$$

Then consider (27) and (39), since $Y^{m,n} \leq Y^{m,\infty}$, then for $0 \leq s \leq t \leq T$

$$K_t^{m,n,-} - K_s^{m,n,-} \leq K_t^{m,\infty,-} - K_s^{m,\infty,-}.$$

Similarly, in this inequality, first let $n \to \infty$, then let $m \to \infty$, we have

$$K_t^{'-} - K_s^{'-} \leq K_t^- - K_s^-. \tag{47}$$

With (46), it follows for $0 \leq s \leq t \leq T$

$$K_t^+ - K_s^+ - (K_t^- - K_s^-) \leq K_t^{'+} - K_s^{'+} - (K_t^{'-} - K_s^{'-})$$

i.e. the process $K_t^{'+} - K_t^{'-} - (K_t^+ - K_t^-)$ is increasing, and by the comparison theorem for BSDE, it follows $Y' \geq Y$. At last $Y' = Y$. □

We get immediately $Z = Z'$, $K^+ - K^- = K'^+ - K^{'-}$. We are now able to prove that the limit of the solutions of the penalized BSDE's is the solution of the RBSDE with two RCLL barriers.

Theorem 4.1. *The triple (Y, Z, K), $Y \in \mathbf{D}^2(0,T)$, $Z \in \mathbf{H}_d^2(0,T)$, $K = K^+ - K^-$, $K^+, K^- \in \mathbf{A}^2(0,T)$ is the unique solution of the RBSDEs with two RCLL barriers L, U.*

Proof. Let us remember that from Theorem 3.4, we have the uniqueness. By the discussion before, we know that $(Y_t^{m,\infty}, Z_t^{m,\infty}, K_t^{m,\infty,+})$ is the solution of the RBSDE with one lower barrier L_t. In (39), denote $K_t^{m,\infty} = K_t^{m,\infty,+} - K_t^{m,\infty,-}$; then $(Y_t^{m,\infty}, Z_t^{m,\infty}, K_t^{m,\infty})$ can be considered as the solution of the RBSDE with two barriers L and $U + (U - Y^{m,\infty})^-$. In fact it is easy to see that

$$L \leq Y^{m,\infty} \leq U + (U - Y^{m,\infty})^-,$$

$$\int_0^T (Y_t^{m,\infty} - L_t)\mathrm{d}K_t^{m,\infty,+} = 0$$

and

$$\int_0^T (Y_t^{m,\infty} - U_t - (U - Y^{m,\infty})_t^-)\mathrm{d}K_t^{m,\infty,-} = m\int_0^T (Y_t^{m,\infty} - U_t)^-(U_t - Y_t^{m,\infty})^-\mathrm{d}t = 0.$$

So by the Proposition 3.1, we get

$$
\begin{aligned}
Y_t^{m,\infty} &= ess\inf_{\sigma\in\mathcal{T}_t} ess\sup_{\tau\in\mathcal{T}_t} E\bigg[\int_t^{\sigma\wedge\tau} g(s,Y_s^{m,\infty},Z_s^{m,\infty})\mathrm{d}s+\xi 1_{\{\sigma\wedge\tau=T\}} \\
&\quad +L_\tau 1_{\{\tau<T,\tau\leq\sigma\}}+U_\sigma 1_{\{\sigma<\tau\}}+(U_\sigma-Y_\sigma^{m,\infty})^{-}1_{\{\sigma<\tau\}}|\mathcal{F}_t\bigg] \\
&\geq ess\inf_{\sigma\in\mathcal{T}_t} ess\sup_{\tau\in\mathcal{T}_t} E\bigg[\int_t^{\sigma\wedge\tau} g(s,Y_s^{m,\infty},Z_s^{m,\infty})\mathrm{d}s+\xi 1_{\{\sigma\wedge\tau=T\}}+L_\tau 1_{\{\tau<T,\tau\leq\sigma\}} \\
&\quad +U_\sigma 1_{\{\sigma<\tau\}}|\mathcal{F}_t\bigg] \\
&\geq ess\inf_{\sigma\in\mathcal{T}_t} ess\sup_{\tau\in\mathcal{T}_t} E\bigg[\int_t^{\sigma\wedge\tau} g(s,Y_s,Z_s)\mathrm{d}s+\xi 1_{\{\sigma\wedge\tau=T\}}+L_\tau 1_{\{\tau<T,\tau\leq\sigma\}} \\
&\quad +U_\sigma 1_{\{\sigma<\tau\}}|\mathcal{F}_t\bigg]-kE\bigg[\int_0^T |Y_s^{m,\infty}-Y_s|+|Z_s^{m,\infty}-Z_s|\,\mathrm{d}s|\mathcal{F}_t\bigg].
\end{aligned}
\tag{48}
$$

Since $Y^{m,\infty}\to Y$ in $\mathbf{H}^2(0,T)$, $Z^{m,\infty}\to Z$ in $\mathbf{H}_d^p(0,T)$ for $p<2$, as $m\to\infty$, we can choose a subsequence which satisfies $E[\int_0^T \left|Z_s^{m_j,\infty}-Z_s\right|\mathrm{d}s|\mathcal{F}_t]\to 0$ a.s., so we deduce

$$
E[\int_0^T (|Y_s^{m,\infty}-Y_s|+|Z_s^{m,\infty}-Z_s|)\mathrm{d}s|\mathcal{F}_t]\to 0,\ a.s.
$$

In (48), let $m\to\infty$, we obtain

$$
Y_t\geq ess\inf_{\sigma\in\mathcal{T}_t} ess\sup_{\tau\in\mathcal{T}_t} E\left[\int_t^{\sigma\wedge\tau} g(s,Y_s,Z_s)\mathrm{d}s+\xi 1_{\{\sigma\wedge\tau=T\}}+L_\tau 1_{\{\tau<T,\tau\leq\sigma\}}+U_\sigma 1_{\{\sigma<\tau\}}|\mathcal{F}_t\right]. \tag{49}
$$

On the other side, in the same way, we know that $(Y^{\infty,n},Z^{\infty,n},K_t^{\infty,n,-})$ is the solution of the RBSDE with the upper barrier U_t, in (43). Denote $K_t^{\infty,n}=K_t^{\infty,n,+}-K_t^{\infty,n,-}$; $(Y_t^{\infty,n},Z_t^{\infty,n},K_t^{\infty,n})$ is solution of the RBSDE with two barriers $L-(Y^{\infty,n}-L)^-$ and U. Similarly by proposition 3.1, we deduce that

$$
\begin{aligned}
Y_t^{\infty,n} &\leq ess\sup_{\tau\in\mathcal{T}_t} ess\inf_{\sigma\in\mathcal{T}_t} E[\int_t^{\sigma\wedge\tau} g(s,Y_s,Z_s)\mathrm{d}s+\xi 1_{\{\sigma\wedge\tau=T\}}+L_\tau 1_{\{\tau<T,\tau\leq\sigma\}} \\
&\quad +U_\sigma 1_{\{\sigma<\tau\}}|\mathcal{F}_t]+kE[\int_0^T |Y_s^{\infty,n}-Y_s|+|Z_s^{\infty,n}-Z_s|\,\mathrm{d}s|\mathcal{F}_t].
\end{aligned}
\tag{50}
$$

Since $Y^{\infty,n}\to Y$ in $\mathbf{H}^2(0,T)$, $Z^{\infty,n}\to Z$ in $\mathbf{H}_d^p(0,T)$ for $p<2$, as $n\to\infty$, like above, let $n\to\infty$, we get

$$
Y_t\leq ess\sup_{\tau\in\mathcal{T}_t} ess\inf_{\sigma\in\mathcal{T}_t} E[\int_t^{\sigma\wedge\tau} g(s,Y_s,Z_s)\mathrm{d}s+\xi 1_{\{\sigma\wedge\tau=T\}}+L_\tau 1_{\{\tau<T,\tau\leq\sigma\}}+U_\sigma 1_{\{\sigma<\tau\}}|\mathcal{F}_t]. \tag{51}
$$

Comparing (49) and (51), in view of $ess\sup ess\inf\leq ess\inf ess\sup$, we deduce finally

$$
\begin{aligned}
Y_t &= ess\sup_{\tau\in\mathcal{T}_t} ess\inf_{\sigma\in\mathcal{T}_t} E[\int_t^{\sigma\wedge\tau} g(s,Y_s,Z_s)\mathrm{d}u+\xi 1_{\{\sigma\wedge\tau=T\}}+L_\tau 1_{\{\tau<T,\tau\leq\sigma\}}+U_\sigma 1_{\{\sigma<\tau\}}|\mathcal{F}_t] \\
&= ess\inf_{\sigma\in\mathcal{T}_t} ess\sup_{\tau\in\mathcal{T}_t} E[\int_t^{\sigma\wedge\tau} g(s,Y_s,Z_s)\mathrm{d}u+\xi 1_{\{\sigma\wedge\tau=T\}}+L_\tau 1_{\{\tau<T,\tau\leq\sigma\}}+U_\sigma 1_{\{\sigma<\tau\}}|\mathcal{F}_t].
\end{aligned}
$$

Using (18) in Section 2.3, we can rewrite Y in the following form

$$\begin{aligned} Y_t &= ess \inf_{\sigma\in\mathcal{T}_t} ess \sup_{\tau\in\mathcal{T}_t} E[\widetilde{L}_\tau 1_{\{\tau\le\sigma\}} + \widetilde{U}_\sigma 1_{\{\sigma<\tau\}}|\mathcal{F}_t] + N_t \\ &= ess \sup_{\tau\in\mathcal{T}_t} ess \inf_{\sigma\in\mathcal{T}_t} E[\widetilde{L}_\tau 1_{\{\tau\le\sigma\}} + \widetilde{U}_\sigma 1_{\{\sigma<\tau\}}|\mathcal{F}_t] + N_t, \end{aligned}$$

where $N_t = E[\xi + \int_0^T g(s)\mathrm{d}s|\mathcal{F}_t] - \int_0^t g(s)\mathrm{d}s$, $\widetilde{L}_t = L_t 1_{\{t<T\}} + \xi 1_{\{t=T\}} - N_t$, $\widetilde{U}_t = U_t 1_{\{t<T\}} + \xi 1_{\{t=T\}} - N_t$. That is, the process $Y_t - N_t$ is the value of the stochastic game problem, whose payoff is $J_t(\sigma,\tau) = E[\widetilde{L}(\tau)1_{\{\tau\le\sigma\}} + \widetilde{U}(\sigma)1_{\{\sigma<\tau\}}|\mathcal{F}_t]$. To go further, we need to check if $\widetilde{L}$ and $\widetilde{U}$ are also in $\mathbf{D}^2(0,T)$, which can be easily seen by using Doob's inequality. In fact

$$\begin{aligned} E\left[\sup_{0\le t\le T}(N_t)^2\right] &\le 2E\left[\sup_{0\le t\le T}\left(E\left[\xi + \int_0^T g(s,Y_s,Z_s)\mathrm{d}s|\mathcal{F}_t\right]\right)^2 + \left(\int_0^T g(s,Y_s,Z_s)\mathrm{d}s\right)^2\right] \\ &\le C\left(1 + E\int_0^T |Y_s|^2\,\mathrm{d}s + E\int_0^T \|Z_s\|^2\,\mathrm{d}s\right) < \infty, \end{aligned}$$

$$E[\sup_{0\le t\le T}(\widetilde{L}_t)^2] \le E[\sup_{0\le t\le T}(L_t)^2] + E[\sup_{0\le t\le T}(N_t)^2] + E[(\xi)^2] < \infty,$$

$$E[\sup_{0\le t\le T}(\widetilde{U}_t)^2] \le E[\sup_{0\le t\le T}(U_t)^2] + E[\sup_{0\le t\le T}(N_t)^2] + E[(\xi)^2] < \infty.$$

Thanks to the Theorem 5.2 in the Appendix, we know that $Y_t - N_t = X_t^+ - X_t^-$, where (X^+, X^-) is a pair of supermartingales in $\mathbf{D}^2(0,T)\times\mathbf{D}^2(0,T)$, solution of the system

$$\begin{aligned} X^+ &= S(\widetilde{L} + X^-) \\ X^- &= S(-\widetilde{U} + X^+) \end{aligned} \tag{52}$$

(notice that $\widetilde{L}_T = \widetilde{U}_T = 0$). Then by the Doob-Meyer decomposition theorem, we get

$$X_t^+ = E[K_T^{+,1}|\mathcal{F}_t] - K_t^{+,1}, X_t^- = E[K_T^{-,1}|\mathcal{F}_t] - K_t^{-,1},$$

where $K_t^{+,1}, K_t^{-,1}$ are predictable increasing processes and by Lemma 5.2, $K^{\pm,1}\in\mathbf{A}^2(0,T)$. With the representation theorem for the martingale part, it follows

$$\begin{aligned} Y_t &= N_t + X_t^+ - X_t^- \\ &= E[\xi + \int_0^T g(s,Y_s,Z_s)\mathrm{d}s + K_T^{+,1} - K_T^{-,1}|\mathcal{F}_t] - \int_0^t g(s,Y_s,Z_s)\mathrm{d}s - K_t^{+,1} + K_t^{-,1} \\ &= Y_0 + \int_0^t Z_s^1\mathrm{d}B_s - \int_0^t g(s,Y_s,Z_s)\mathrm{d}s - K_t^{+,1} + K_t^{-,1}. \end{aligned} \tag{53}$$

Finally rewrite (42) in forward form and compare with (53); similarly to the case of the RBSDE with one RCLL barrier [9], we get, $Z_t - Z_t^1 = 0$, $K_t^- - K_t^+ = K_t^{-,1} - K_t^{+,1}$. Then by (52), and the properties of the Snell envelope, since $X^+ \ge \widetilde{L} + X^-$ and $X^- \ge -\widetilde{U} + X^+$, we see easily that

$$L \le N + \widetilde{L} \le N + X^+ - X^- = Y \le N + \widetilde{U} \le U,$$

so (iii) of Definition 2.1 is satisfied.

Finally, (iv) of Definition 2.1 also comes from the theory of the Snell envelope, Lemma 5.1. Indeed

$$\begin{aligned} 0 &= \int_0^T (X^+ - (\widetilde{L} + X^-))_{t-} \mathrm{d}K_t^{+,1} = \int_0^T (X^+ - X^- - L + N)_{t-} \mathrm{d}K_t^+ \\ &= \int_0^T (Y_{t-} - L_{t-}) \mathrm{d}K_t^+, \end{aligned}$$

and

$$\begin{aligned} 0 &= \int_0^T (X^- - (-\widetilde{U} + X^+))_{t-} \mathrm{d}K_t^{-,1} = \int_0^T (X^- - X^+ + U - N)_{t-} \mathrm{d}K_t^- \\ &= \int_0^T (U_{t-} - Y_{t-}) \mathrm{d}K_t^-. \end{aligned}$$

The proof is complete. □

5. Appendix

5.1. Some remarks about the Snell envelope

Any $\mathcal{F}_t$-adapted RCLL process $\eta = (\eta_t)_{0\le t\le T}$, is called of class $\mathcal{D}[0,T]$, if the family $\{\eta(\tau)\}_{\tau\in\mathcal{T}}$ is uniformly integrable, where $\mathcal{T}$ is the set of all $\mathcal{F}_t$-stopping times, such that $0 \le \tau \le T$.

Definition 5.1. Let $\eta = (\eta_t)_{0\le t\le T}$ be of class $\mathcal{D}[0,T]$, with $\eta_T \ge 0$, then its Snell envelope $S_t(\eta)$ is defined as

$$S_t(\eta) = ess \sup_{\tau\in\mathcal{T}_t} E[\eta(\tau)|\mathcal{F}_t], 0 \le t \le T \tag{54}$$

where $\mathcal{T}$ is the set of all $\mathcal{F}_t$-stopping times, and for all $0 \le t \le T$, $\mathcal{T}_t = \{\tau \in \mathcal{T};\ t \le \tau \le T\}$.

From Theorems 2.28 and 2.29 of [5] (El Karoui, 1979), the Snell envelope has the following properties:

Proposition 5.1. *$S_t(\eta)$ is a RCLL positive process and is the smallest supermartingale, which dominate the process η. In addition, if η satisfy*

$$\eta^* := \sup_{0\le t\le T} |\eta_t| \in \mathbf{L}^1(\Omega), \tag{55}$$

then $S(\eta)$ is a potential of class $\mathcal{D}[0,T]$. (Indeed it's dominated by the martingale $E[\eta^*|\mathcal{F}_t]$.)

Proposition 5.2. *There exists a unique decomposition of the Snell envelope:*

$$S_t(\eta) = M_t - A_t^c - A_t^d \tag{56}$$

where M_t is a $\mathcal{F}_t$-martingale, A_t^c is a continuous integrable increasing process with $A_0^c = 0$, A_t^d is a pure-jumps integrable increasing predictable RCLL process with $A_0^d = 0$.

We need also the following results, whose proofs can be found in [5] and [7].

Lemma 5.1. *Relatively to the decomposition in the Proposition 5.2, we have*

$$\int_0^T (S_{t-}(\eta) - \eta_{t-}) \mathrm{d}A_t = 0, \tag{57}$$

where $A_t = A_t^c + A_t^d$.

Lemma 5.2. *Let $X = (X_t)_{0\le t\le T}$ be a supermartingale in the space $\mathbf{D}^2(0,T)$, and A be the increasing process of the Doob-Meyer decomposition of X. Then we have $E[A_T^2] < \infty$.*

Then easily, we have the following corollary.

Corollary 5.1. *Let $\eta = (\eta_t)_{0\le t\le T}$ be in the space $\mathbf{D}^2(0,T)$, $\eta_T = 0$, and $A = A^c + A^d$ where A^c, A^d are the increasing processes of the decomposition of the Snell envelope $S_t(\eta)$. Then A satisfies $E[A_T^2] < \infty$.*

5.2. Stochastic game and the Dynkin game problem

Definition 5.2. For a probability space $(\Omega, \mathcal{F}, P)$, let $\mathcal{U}$ (resp. $\mathcal{V}$) be the set of the strategies for the first (resp. second) player. We consider a family of random variables $J(u,v)$, indexed by the set $\mathcal{U}\times\mathcal{V}$. The rule of the game is the following:
(i) The first player wants to minimize $J(u,v)$ acting on $u \in \mathcal{U}$.
(ii)The second player wants to maximize $J(u,v)$ acting on $v \in \mathcal{V}$.
We call such a system a stochastic game.

Definition 5.3. A pair $(u^*, v^*) \in \mathcal{U}\times\mathcal{V}$ is called saddle point for the game, if for all $(u,v)\in\mathcal{U}\times\mathcal{V}$, we have:

$$J(u^*, v) \le J(u^*, v^*) \le J(u, v^*), \text{ a.s.}$$

Definition 5.4. We denote by $\overline{V}$ (resp.$\underline{V}$) the upper (resp. lower) value of the game, *i.e.*

$$\begin{aligned} \overline{V} &= ess\inf_{u\in\mathcal{U}} ess\sup_{v\in\mathcal{V}} J(u,v) \\ \text{resp. } \underline{V} &= ess\sup_{v\in\mathcal{V}} ess\inf_{u\in\mathcal{U}} J(u,v). \end{aligned}$$

Definition 5.5. If $\overline{V} = \underline{V} = V$ a.s., then V is called the value of the stochastic game.

Then we give a sufficient condition for the existence of a value in a stochastic game problem.

Lemma 5.3. *For a stochastic game with payoff $J(u,v)$, if for all $\varepsilon > 0$, there exist $u^\varepsilon \in \mathcal{U}$, $v^\varepsilon \in \mathcal{V}$, such that*

$$J(u^\varepsilon, v) - \varepsilon \le J(u, v^\varepsilon) + \varepsilon \text{ a.s. for all } u \in \mathcal{U}, v\in\mathcal{V}, \tag{58}$$

then this game has the value.

It is easy to prove this lemma, so we omit it.

Definition 5.6. The Dynkin game problem is a kind of stochastic game. Given a probability space equipped with a filtration $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, where $\mathcal{F}_t$ satisfies the general conditions of Dellacherie, $\mathcal{T}$ the set of $\mathcal{F}$-stopping times dominated by a fixed time T, two RCLL $\mathcal{F}_t$-progressive processes L^*, U^* of class $\mathcal{D}$, with $L^* \le U^*$, for any $(\tau,\sigma)\in\mathcal{T}\times\mathcal{T}$, the payoff $J(\tau,\sigma)$ is defined by

$$J(\tau,\sigma) = E[{L^*}_\tau 1_{\{\tau\le\sigma\}} - {U^*}_\sigma 1_{\{\sigma<\tau\}}].$$

The first player wants to choose a stopping time τ in view to get the maximum of the payoff $J(\tau,\sigma)$ on $\mathcal{T}$, while the second player wants to choose a stopping time σ in view to get the minimum of the payoff $J(\tau,\sigma)$ on $\mathcal{T}$.

In order to find the value of the Dynkin game problem, we need the following system, which was firstly introduced by Bismut ([3], 1977) then by Alario-Nazaret ([1], 1982).

$$\begin{aligned} X &= S(L^* + X') \\ X' &= S(U^* + X), \end{aligned} \tag{59}$$

where S is the Snell envelope (Def. 5.1). In ([3], 1977), the thesis of Alario-Nazaret ([1], 1982) and ([2], 1982), we can find the following result, which gives conditions for the existence of a solution for this system.

Theorem 5.1. *There exists a pair (X, X'), of positive $\mathcal{F}_t$-supermartingales of class $\mathcal{D}$ with $X_T = X'_T = 0$, which satisfies the system (59), if we have the followings:*
(i) $L^*_T = U^*_T = 0$.
(ii) There exist two positive $\mathcal{F}_t$-supermartingales $(\widetilde{X}, \widetilde{X}')$ of class $\mathcal{D}$, such that $L^* \leq \widetilde{X} - \widetilde{X}' \leq -U^*$.

The detailed proof can be found in [3] and [2], so we omit it. The following theorem gives the relation between this system and the value of the Dynkin game.

Theorem 5.2. *Suppose that (X, X') is a the solution of the system (59), and consider for any $0 \leq t \leq T$, the stochastic game with payoff*

$$R_t(\tau, \sigma) = E[L^*(\tau)1_{\{\tau \leq \sigma, \tau \leq T\}} - U^*(\sigma)1_{\{\sigma < \tau\}}|\mathcal{F}_t]$$

as well as its upper and lower values

$$\begin{aligned} \overline{V}_t &= ess \inf_{\sigma \in \mathcal{T}_t} ess \sup_{\tau \in \mathcal{T}_t} R_t(\tau, \sigma) \\ \underline{V}_t &= ess \sup_{\tau \in \mathcal{T}_t} ess \inf_{\sigma \in \mathcal{T}_t} R_t(\tau, \sigma), \end{aligned}$$

where $\mathcal{T}_t = \{\tau \in \mathcal{T};\ t \leq \tau \leq T\}$. Then we have almost surely

$$X_t - X'_t = \overline{V}_t = \underline{V}_t. \tag{60}$$

In the special case $t = 0$, then we get the existence of the value for the classical Dynkin game problem.

Proof. For any $t \in [0, T]$, $\varepsilon > 0$, consider the stopping time $\tau^\varepsilon_t = \inf\{s \geq t, X_s \leq X'_s + L^*_s + \varepsilon\} \wedge T$; then by the theory of the Snell envelope $X_{t \wedge \tau^\varepsilon_t}$ is a martingale ([5], 2.16 and 2.17). Notice that X' is a supermartingale and $X' \geq X + U^*$. Then for any stopping time $\sigma \in \mathcal{T}_t$, and notice that $\{\tau^\varepsilon_t < \sigma\} \subset \{\tau^\varepsilon_t < T\}$, we have

$$\begin{aligned} X_t - X'_t &\leq E[X_{\sigma \wedge \tau^\varepsilon_t} - X'_{\sigma \wedge \tau^\varepsilon_t}|\mathcal{F}_t] \\ &\leq E[(X - X')_{\tau^\varepsilon_t} 1_{\{\tau^\varepsilon_t < \sigma\}} + (X - X')_\sigma 1_{\{\sigma \leq \tau^\varepsilon_t\}}|\mathcal{F}_t] \\ &\leq E[(L^*_{\tau^\varepsilon_t} + \varepsilon)1_{\{\tau^\varepsilon_t < \sigma\}} - U^*_\sigma 1_{\{\sigma \leq \tau^\varepsilon_t\}}|\mathcal{F}_t] \\ &\leq E[L^*_{\tau^\varepsilon_t} 1_{\{\tau^\varepsilon_t < \sigma\}} - U^*_\sigma 1_{\{\sigma \leq \tau^\varepsilon_t\}}|\mathcal{F}_t] + \varepsilon = R_t(\tau^\varepsilon_t, \sigma) + \varepsilon \text{ a.s.} \end{aligned}$$

On the other hand, we consider the stopping time $\sigma^\varepsilon_t = \inf\{s \geq t, X'_s \leq X_s + U^*_s + \varepsilon\} \wedge T$; then $X'_{t \wedge \sigma^\varepsilon_t}$ is a martingale, and X is a supermartingale s.t. $X \geq X' + L^*$. Similarly, for any stopping time $\tau \geq t$, we get that

$$X_t - X'_t \geq E[L^*_\tau 1_{\{\tau \leq \sigma^\varepsilon_t\}} - U^*_{\sigma^\varepsilon_t} 1_{\{\sigma^\varepsilon_t < \tau\}}|\mathcal{F}_t] - \varepsilon = R_t(\tau, \sigma^\varepsilon_t) - \varepsilon \text{ a.s.}$$

Then from the Lemma 5.3, we deduce the result

$$X_t - X'_t = \overline{V}_t = \underline{V}_t, \text{ a.s.}$$

□

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