LAPLACE ASYMPTOTICS FOR
GENERALIZED K.P.P. EQUATION

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Abstract. Consider a one dimensional non linear reaction-diffusion equation (KPP equation) with non-homogeneous second order term, discontinuous initial condition and small parameter. For points ahead of the Freidlin-KPP front, the solution tends to 0, and we obtain sharp asymptotics (i.e., non logarithmic). Our study follows the work of Ben Arous and Rouault who solved this problem in the homogeneous case.

Our proof is probabilistic, and is based on the Feynman-Kac formula and the large deviation principle satisfied by the related diffusions. We use the Laplace method on Wiener space. The main difficulties come from the non-linearity and the possibility for the endpoint of the optimal path to lie on the boundary of the support of the initial condition.

1. Introduction

The purpose of this paper is to obtain precise (i.e., non logarithmic) asymptotics of \( u^r(T, x) \) for certain values of \( (T, x) \), where \( u^r(T, x) \) is the solution of generalized KPP equation

\[
\begin{align*}
\partial_t u^r &= \frac{\sigma^2}{2} \partial_x^2 u^r + \frac{c(x)}{2} u^r (1 - r(u^r)) \\
u^r(0, x) &= 1 \quad (x \leq 0)
\end{align*}
\]

where \( c \) is a non-negative \( C^2 \) function such that there exists \( \bar{k} > 0 \) satisfying

\[
c(x) \leq \bar{k} (1 + |x|), \quad |\sigma'(x)| \leq \bar{k} (1 + x^2),
\]

\( \sigma \) is \( C^2 \), with bounded derivatives, such that there exists \( M > m > 0 \) satisfying

\[
m \leq \sigma(x) \leq M,
\]

\( r \) is a one-to-one \( C^1 \) increasing function from \([0, 1]\) to \([0, 1]\).

Our study follows the work of Ben Arous and Rouault (1993) who solved this problem when \( r = 1 \) and \( r(u) = u \).
The Feynman-Kac formula gives an equivalent form of (1.1), that is

$$u^t(T,x) = E[\frac{1}{\sqrt{2\pi s}} \int_0^T e^{(X^s_t - X^u_t)} 1 - r(u(T,s,X^s_t))] ds$$  \hspace{1cm} (1.4)$$

where $X^s$ is the diffusion solution of the stochastic differential equation $X^s_t = x + \int_0^t X^s_r dW_r$. According to the large deviation principle satisfied by the laws of $(X^s_t)_{t \geq 0}$, the limit of $\frac{1}{\sqrt{2\pi s}} \int_0^T e^{(X^s_t - X^u_t)} 1 - r(u(T,s,X^s_t)) ds$ exists for many continuous functionals $f$ and $F$. By adapting this technique, Freidlin (1985 and 1990) derived the asymptotics of $\ln u^t(T,x)$ (the difficulty coming from the $u^t$ in the expectation of (1.1)). He proved the existence of a non-positive function $V^t(T,x)$ such that

i) if $V^t(T,x) < 0$, then $\exp(\frac{1}{\sqrt{2\pi s}} \int_0^T e^{(X^s_t - X^u_t)} 1 - r(u(T,s,X^s_t))) ds = O(\exp(\frac{1}{\sqrt{2\pi s}}))$,

ii) if $V^t(T,x)$ is in the interior of $V^t = 0$, then $u^t(T,x) \to 1$. This makes clear the propagation of a wave front: $[t,x]$ is ahead (resp. behind) of the front if $u^t(t,x) \to 0$ (resp. $u^t(t,x) \to 1$).

In order to get precise asymptotics of $E[f(X^s_t)] \exp(\frac{-1}{\sqrt{2\pi s}} F(X^s_t))$, one can use the Laplace method on Wiener space (see Doucres and Rouault (1983), Ben Arous (1988)) under the standard hypothesis: the maximum of $F - I$ on $[\{ \psi \mid f(\psi) \neq 0 \}$, where $I$ is the action functional of $(X^s_t)_{t \geq 0}$, is attained at a unique path $\varphi$, and this maximum is non-degenerate. When $f$ is not continuous at $\varphi$, new techniques are required (see Azencott (1985)).

However, precise asymptotics of $u^t(t,x)$ cannot be obtained by using directly Laplace method because of the presence of $u^t$ in the expectation of (1.1), presence related to the non-lineararity of (1.1). Nevertheless, $u^t(t,x)$ approaches 0 with exponential speed ahead of the front. Thus, if the path $s \to X_{T-t}^s(\omega)$ stays ahead of the front, we can neglect most of its contribution; i.e. for all $t < T$ and $\alpha > 0$,

$$\exp(\frac{-1}{\sqrt{2\pi s}} \int_0^T e^{(X^s_t(\omega))} r(u^t(T-s,X^s_t(\omega))) ds = O(\exp(\frac{-\alpha}{\sqrt{2\pi s}}))$$

More precisely, only $\exp(\frac{-1}{\sqrt{2\pi s}} \int_0^T e^{(X^s_t(\omega))} r(u^t(T-s,X^s_t(\omega))) ds$ contributes ($\alpha \in [0,1]$ if $T \to \infty$ and $T$ define a boundary layer, see Ben Arous and Rouault (1993)).

But what happens for other $\omega$? Under the Laplace method usual hypothesis, only paths $X^s(\omega)$ close to the optimal path $\varphi$ really contribute. So, if we want to approximate the case where the paths $s \to X_{T-t}^s(\omega)$ stay ahead of the front, we will assume that $\varphi$ stays, in reversed time, ahead of the front (i.e. $\forall s \in [0,T] \quad V^s(T-s,\omega_s) < 0$). This hypothesis is close to condition (N) of Freidlin (see Freidlin (1985 p. 408), and Freidlin (1990) where one can find several examples where this condition is satisfied). A recent result of Barles and Souganidis (1994) concerning the asymptotics of $\exp(\frac{-1}{\sqrt{2\pi s}} F(X^s_t))$ behind the front might enable us to carry the proof to the end without this hypothesis.

In section 2, we state our main results, give connections with branching diffusions and summarize the proof, which starts in section 3 where we
carry out the Laplace method. We prove in section 4 that only a boundary layer
contribute. In section 5, we construct diffusion bridges using the
Skorokhod integral. In section 6, we study the contribution of the non-linear
part and end the proof in section 7.

2. Results

Let \( T \in [0, +\infty] \) and \( x \in \mathbb{R} \). \( H_x \) stands for the Cameron-Martin space
\[
\{ \psi : [0, T] \to \mathbb{R} \mid \psi_0 = x; \psi \text{ absolutely continuous}, \int_0^T \psi_s^2 \, ds < \infty \}.
\]

If \( f \) is a continuous function on \( [0, T] \), we let \( \| f \| = \sup_{0 \leq t \leq T} |f(t)| \).
If \( f \) is continuous on \( [a, b] \), we let \( \| f \|_1 = \sup_{0 \leq t \leq 1} |f(t)| \).
\( O(1), O(x) \ldots \) and positive constants denoted \( \text{"const"} \) are all universal, i.e. may
only depend on \( \sigma, c, \epsilon, x \) and \( T \).

2.1. The Linear Problem

We introduce the linear problem related to (1.1)

\[
\begin{aligned}
\frac{\partial v^\epsilon}{\partial t} &= \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} v^\epsilon + \frac{c(x)}{2^*} v^\epsilon \\
v^\epsilon(0, x) &= 1 \{ x < 0 \}
\end{aligned}
\]

Precise asymptotics of \( v^\epsilon \) will help computing the ones of \( u^\epsilon \) since \( u^\epsilon \leq v^\epsilon \)
(consequence of the maximum principle or of the Feynman-Kac formula).
Let
\[
F(\psi) = \int_0^T c(\psi_s) \, ds.
\]

According to the Feynman-Kac formula,
\[
v^\epsilon(T, x) = E^1 \{ X_T^\epsilon \leq 0 \} \exp \epsilon^{-2} F(X_T^\epsilon).
\]

The laws of \( (X_T^\epsilon)_{\epsilon \geq 0} \) satisfy a large deviation principle with action functional
\[
I(\psi) = \frac{1}{2T} \int_0^T \psi_s^2 S(\psi_s) \, ds \text{ if } \psi \in H_x
\]
\[
= +\infty \text{ otherwise},
\]
where \( S = \sigma^{-2} \). Therefore, \( \lim_{\epsilon \to 0} \epsilon^2 \ln v^\epsilon(T, x) = V(T, x) \), where
\[
V(T, x) = \sup \{ F(\psi) - I(\psi) \mid \psi \in H_x, \psi_T \leq 0 \}.
\]

We will obtain asymptotics of \( v^\epsilon(T, x) \) under the Laplace method usual hypothesis:

\( H1 \) the maximum in \( V(T, x) \) is attained at a unique path \( \psi \);
\( H2 \) \( \psi \) is a non-degenerate maximum.
Let $R = F - I$. For $\psi \in H_2$ and $h \in H_0$, 

$$R'(\psi)h = \int_0^T \left[ \frac{1}{2} \psi_s^2S'(\psi) + \psi_nS(\psi) \right] h_{ts} - \psi_T S(\psi_T) h_T. \quad (2.3)$$

Therefore, (H1) yields:

$$e'(\varphi) + \frac{1}{2} \varphi_s^2 S'(\varphi) + \varphi S(\varphi) = 0 \quad \text{ (Euler equation)}$$

$$\varphi_T \geq 0$$

$$\varphi_T = 0 \quad \text{ (complementary slackness)}$$

$$R'(\varphi)h = -\psi_T S(\varphi_T) h_T \quad \text{ for } h \in H_0. \quad (2.4)$$

The case $\varphi_T < 0$ can be reduced to a problem without constraint and the result is known (see Azencott (1980-81) and Ben Arous (1988)). We will hence study the case $\varphi_T = 0$.

**Remark 2.1.** (2.4) has a geometrical interpretation: since $R$ attains its maximum on $E = \{ \psi \mid \psi_T \leq 0 \}$ on the boundary of $E$, its gradient at $\varphi$ and the outwardly normal of $E$ in $\varphi$ are positively linked.

We will now focus on the meaning of (H2). We have:

$$R''(\varphi)h^2 = \int_0^T \psi''(\varphi) + \frac{1}{2} \varphi_s^2 S''(\varphi) + \varphi_n S'(\varphi) \psi^2 ds - \psi_T S'(0)h_T^2 - (h, h)$$

where $(\cdot, \cdot)$ is the scalar product defined on $H_0$ by $(h, h) = \int_0^T h_s^2 S(\varphi) ds$.

We will say that $\varphi$ is a non-degenerate maximum if there exists $\lambda > 0$ such that 

$$R''(\varphi)h^2 \leq -\lambda (h, h) \quad \text{ for all } h \in H_0. \quad (2.5)$$

Introduce now $A$, the self-adjoint Hilbert-Schmidt operator on $H_0$ defined by 

$$(Ah, h) = \int_0^T \psi''(\varphi) + \frac{1}{2} \varphi_s^2 S''(\varphi) + \varphi_n S'(\varphi) \psi^2 ds - \psi_T S'(0)h_T^2. \quad (2.6)$$

Since $A$ is self-adjoint and compact, there exists a basis $(e_n)_{n \geq 1}$ of eigenvectors of $A$, orthonormal with respect to $(\cdot, \cdot)$. Define $(\lambda_n)_{n \geq 1}$ the corresponding eigenvalues, $\lambda_1$ being the largest one. Condition (2.5) is thus equivalent to 

$$\lambda_1 < 1. \quad (2.7)$$

We now define a gaussian process and its corresponding bridge by 

$$g_1(t) = \int_0^t \sigma(\varphi_s) dW_s, \quad g_1^2(t) = g_1(t) - \int_0^t \sigma(\varphi_s) ds - \int_0^t \sigma^2(\varphi_s) ds \quad t \leq T. \quad (2.8)$$

By extending definition (2.6) to continuous functions $h$ on $[0, T]$, we define 

$$K(h) = \frac{1}{2} (Ah, h) \quad \text{ for all } h \in C^0[0, T]. \quad (2.9)$$
Finally, let \( p = -\varphi_T \sigma^2 (0) \).

**Theorem 2.2.** Assume (H1), (H2) (i.e. (2.5) or (2.7)), \( \varphi_T = 0 \) and \( -\varphi_T > 0 \). Then

\[
v'(T, x) = [A_1 \varepsilon + o(\varepsilon)] \exp[\varepsilon^2 V(T, x)]
\]

where \( A_1 = p^{-1} (2\pi \int_0^T \sigma^2(\varphi_s) ds)^{-\frac{1}{2}} \). 

**Remarks.**

i) If \( \varphi_T < 0 \), by directly using the Laplace method, we get

\[
v'(T, x) = [E \exp K(g_1) + o(1)] \exp[\varepsilon^2 V(T, x)].
\]

ii) If \( \varphi_T = 0 \) and \( -\varphi_T = 0 \), it is easy to get [see section 3]

\[
v'(T, x) = [E \exp K(g_1) + o(1)] \exp[\varepsilon^2 V(T, x)].
\]

The previous constants are finite, as a consequence of

**Lemma 2.4.** There exists \( \beta > 0 \) such that, for all \( \alpha \leq \beta \),

\[
E[\exp (1+\alpha) K(g_1)] < \infty \quad \text{and} \quad E[\exp (1+\alpha) K(g_2^2)] < \infty.
\]

**Proof.** We adapt here a computation of Ben Arous (1988). The equality

\[
(Ah, h) = \sum_{n=1}^{\infty} \lambda_n (h, f_n)^2
\]

can be extended by density to continuous martingales \( h \) such that \( h(0) = 0 \) if we let \( (h, f_n) = \int_0^T f_n(s) S(\varphi_s) \, dh_s \). Moreover,

\[
E[(h, f_n)(g_1, f_m)] = \int_0^T f_n(s) S(\varphi_s) \sigma(\varphi_s) f_m(s) S(\varphi_s) \, ds = (f_n, f_m).
\]

Hence, \( (g_1, f_n) \) are independent, gaussian, centered, reduced. Since \( A \) is a trace class operator \( H_0 \), we have

\[
E[\exp (1+\alpha) K(g_1)] = \prod_n E[\exp \frac{1+\alpha}{2} \lambda_n (g_1, f_n)^2]
\]

\[
= \prod_n [1 - (1+\alpha) \lambda_n]^{-\frac{1}{2}}
\]

\[
= (\det (I - (1+\alpha) A))^{-\frac{1}{2}} < \infty.
\]

A similar computation with the restriction of \( A \) to \( \{ h \in H_0 : h(T) = 0 \} \) yields \( E[\exp (1+\alpha) K(g_2^2)] < \infty \). 

\( \Box \)
2.2. THE NONLINEAR PROBLEM

Let

\[ V^*(T, x) = \sup_{w \in \mathcal{W}} \left\{ \inf_{\psi \in \mathcal{H}} \int_0^T |c(\psi_t) - \frac{1}{2} S(\psi_t)| \, dt \right\} \quad \forall \psi \in H, \, \psi_T \leq 0. \]

Our first hypothesis is

[H3] i) The maximum in \( V^*(T, x) \) is attained at a unique path \( \psi \),

ii) \( V^*(s, \psi_{T-s}) < 0 \) for all \( s \in [0, T] \),

iii) \( V^*(T, x) = V(T, x) \).

i) means that the optimal path \( \psi \) runs always ahead of the front. ii) and iii) are satisfied under condition (N) of Freidlin.

[H3] is equivalent to \( V^*(T, x) < 0 \) and the set of (\( \psi, t \)) realizing the equality in \( V^*(T, x) \) is a singleton of the form \( (\varphi; T) \). This was proved by Ben Arous and Rouault (1993) in the case \( \sigma = 1 \), and it can be extended easily.

[H3] yields \( x > 0 \), \( \varphi_T = 0 \), \( -\dot{\varphi}_T \geq 0 \), and \( e(0) = \frac{2}{\sqrt{\pi}} S(0) \leq 0 \).

We need more than this last inequality to analyze the boundary layer, i.e.

[H4] \( -\dot{\varphi}_T > \sqrt{2e(0)\sigma^2(0)} \).

[H4] now means that \( \varphi \), in reversed time, moves quickly away from the front.

Since [H3] implies \( V(T, x) = H(e) \), we can select, as a non-degeneracy hypothesis for the nonlinear problem, that of the linear problem (H2).

Finally, let

\[
\dot{u}(y) = E \exp \left( -c(0) \int_0^y \left( \hat{u} \dot{u} \hat{u} + y + \sigma(0) W \right) ds \right)
\]

where

\[
\hat{u}(0) = 1 \{ x \leq 0 \}
\]

\[
\dot{u} = \frac{1}{2} \sigma^2(0) \partial^2_{u} \hat{u} + c(0) \dot{u} (1 - (\dot{u})^2)
\]

\[
\text{Theorem 2.5. Under hypothesis (H2), (H3) and (H4),}
\]

\[
w(T, x) = A_2 e + o(x) \exp \left( -V^*(T, x) \right)
\]

where \( A_2 = \int_{-\infty}^\infty g(y) \exp g \, dy \int_0^\infty \sigma^2(\varphi_\nu) \, d\nu \exp g^2 \).\

RemarKs 2.6. i) If \( \sigma = 1 \), we can weaken the hypothesis on \( r \); \( r \) is a one-to-one increasing continuous function from \( [0, 1] \) to \( [0, 1] \), \( C^1 \) on \( [0, 1] \), \( \lim_{u \to 0} w^*(u) = 0 \) and

\[
\exists \theta > 0 \int_0^\theta \frac{r(u)}{u^{1+\theta}} \, du < +\infty. \quad (2.30)
\]

ii) By using transformations \( x \to x - a \) or \( x \to x + a \), we get similar results with initial condition \( \phi(x) = 1 \{ x \leq a \} \) or \( \phi(x) = 1 \{ x \geq a \} \).
We can also treat the case \( f(x) = 0 \) if \( x \geq 0 \), \( f(x) \in ]0, 1[ \) if \( x < 0 \) and \( f \) smooth.

Let \( k = \inf \{ n \geq 1 \mid f^{(n)}(0) \neq 0 \} \). We have
\[
u^*(x) = \left| A_2 e^{x + 1 + o(e^{x + 1})} \right| \exp\left[ -2 \int \nu^*(x) \right]
\]
where \( A_2 = \frac{f^{(1)}(0)}{k^2} \int_{-\infty}^0 \left( 2 \sqrt{\sigma(y)} dy \right)
\int_0^\infty \sigma(y) dy \left. \exp \left( -2 \int \nu^*(x) \right) \right] E \exp K(y^2) \).

iii) Let us prove that \( g(y) > 0 \) for all \( y \), and therefore \( A_2 > 0 \).

Let \( \alpha > \sqrt{2 \sigma(0)} \). We have a.s.
\[
\exists \alpha > 0 \forall s > s_0 \quad -T \frac{\alpha}{y} + y + \sigma(0) W_s \geq 0.
\]
Since \( \hat{u}(s, \cdot) \) is non-increasing (Kolmogorov, Petrovskii and Piscunov (1937)), we get a.s.
\[
\forall s > s_0 \quad \hat{u}(s, -T \frac{\alpha}{y} + y + \sigma(0) W_s) \leq \hat{u}(s, \alpha y).
\]
Since \( \alpha > \sqrt{2 \sigma(0)} \sigma(0) \), we have (see Freidlin (1985)),
\[
\lim_{s \to +\infty} \ln \hat{u}(s, \alpha y) = -\frac{1}{2} \sigma^2(0) - 2 \sigma(0)).
\]
Therefore, \( \int_0^{\infty} \hat{u}(s, -T \frac{\alpha}{y} + y + \sigma(0) W_s) ds < \infty \) a.s. and \( g(y) > 0 \).

2.3. Connection with branching diffusions

Let \( \lambda \) be a non-negative function on \( \mathbb{R} \). Consider the following branching diffusion:

i) a particle starts from \( x \in \mathbb{R} \), and executes a small diffusion
\[dX^*_\tau = \sigma(X^*_\tau) dW_t;\]

ii) its lifetime \( \tau \) is given by
\[P(\tau \in [t, t + dt] \mid X^*_\tau = y, \tau \geq 0) = e^{-\lambda(y) dt + o(dt)}\]

iii) when it dies, it is replaced by a random number of descendants \( N \)

(s) each descendant, starting from where its parent died, repeats i), ii), iii).

All diffusions, lifetimes and number of descendants are independent of one another.

Let \( N_t \) be the number of particles in \( ]-\infty, 0[ \) at time \( t \).
Then, \( P_v(N_t \neq 0) \neq 0 \) is the solution of
\[
\begin{aligned}
\partial_t u - \frac{\sigma^2(\cdot)}{2} \partial_{xx} u + \frac{c(\cdot)}{2} f(u) &= 0 \quad \text{in } [0, \infty) \times \mathbb{R} \\
u^*(0, x) &= 1 \quad \text{for } x \leq 0
\end{aligned}
\]
where \( c(x) = (E N - 1) \lambda(x) \) and \( f(u) = (E N - 1)^{-1} (1 - u - E (1 - u) N) \).

Assume \( P(N = 0) = 0 \) and \( EN > 1 \). Then \( f \) is a KPP type non-linearity, i.e.
\[
f(0) = f(1) = 0, \quad f(u) > 0 \text{ for } u \in [0, 1] \\
f \in C^1 [0, 1], \quad f'(u) < f'(0) \text{ for } u \in [0, 1].
\]

Define \( r \) such that \( f(u) = u(1 - r(u)) \), Then \( r \) is increasing continuous one-to-one from \([0, 1]\) to \([0, 1]\), \( C^1 \) on \([0, 1]\), and \( \lim_{u \to 0} u^r(u) = 0 \).

The additional hypothesis made on \( r \) can be expressed in terms of \( N \): \( r \in C^1 \) on \([0, 1]\) if and only if \( E(N^2) < \infty \) (easy check).

\( v \) and \( F \) are \( C^1 \) on \([0, 1]\), \( C^1 \) on \([0, 1]\) and \( \lim_{u \to 0} u^r(u) = 0 \).

In the homogeneous case \((\sigma \text{ and } c \text{ constant})\), Chernov and Rousset (1988, th.1) obtained asymptotics of \( F_2(N_T^2 \neq 0) \) for \((t, x)\) in the sub-critical area \((i.e. \text{ahead of the front})\) under the weaker condition \( E(N \log N) < \infty \).

We can translate our results into branching diffusions language: since \( F_2(N^2) \) is solution of linear equation \((2.1)\), we get, under hypothesis of th. 2.1 and 2.3, asymptotics for \( F_2(N_T^2) \) and \( F_2(N_T^2 \neq 0) \). For instance, \( E[N_T^2 | N_T^2 \neq 0] \) goes to a finite limit. This means that, when \([ - \infty, 0]\) is visited at time \( T \) \((\text{a rare event})\), the average number of particles in this area is finite.

### 2.4. Summary of the proof

Starting with \((2.2)\) and \((1.4)\), we implement the Laplace method \((\text{section } 3)\) which consists in localizing around \( \varphi \), applying the Girsanov formula, then performing a stochastic Taylor expansion of the diffusion \( \zeta' = \int \sigma(\varphi + \zeta') dW_t \) of the form \( \zeta' = \varphi + \zeta' \sigma \) with remainder \((\varphi \text{ is gaussian})\). We get
\[
v_T^r = E^1 \left[ Z_T^r \leq 0, \|Z'_t\| \leq \rho \right] \exp \left[ \rho \left( e^{-1} g(T) + g(0) \right) + K(\varphi) \right] \\
u_T^r = E^1 \left[ Z_T^r \leq 0, \|Z'_t\| \leq \rho \right] \\
 \exp \left[ \rho \left( e^{-1} g(T) + g(0) \right) + K(\varphi) \right] - e^{-1} F_0(0, T, \varphi + Z^r)
\]

For \( v_T^r \), we now use the following strategy.

1. We prove \( v_T^r = E^\Psi \left[ g_1(T), g_2(T) \right] + o(\varepsilon) \) for some \( \Psi \) [lemma 7.1].
2. We construct a process \( \Delta \) independent of \( g_1(T) \) such that \( g_1 \) and \( g_2 \) can be expressed in terms of \( \Delta \) and \( g_1(T) \) \((\text{lemma } 5.1)\).
3. Therefore \( v_T^r = E^\Psi \left[ \Delta, g_1(T) \right] + o(\varepsilon) \) for some \( \Psi \). We condition on \( \Delta \) and prove that the gaussian integral \( e^{-1} E^\Psi \left[ g_1(T), g_2(T) \right] \) goes to a non zero finite limit \((\text{for fixed } \delta)\). This implies \( v_T^r \sim c \varepsilon \) \((\text{section } 7.1)\).

Concerning the nonlinear problem, we prove in section 4 that we can neglect \( F_0(0, T - \varepsilon^r, \varphi + Z^r) \) for \( \alpha \in [0, 1] \), i.e. \( u_T^r = u_T^r + o(\varepsilon) \) where
\[
u_T^r = E^1 \left[ Z_T^r \leq 0, \|Z'_t\| \leq \rho \right] \\
 \exp \left[ \rho \left( e^{-1} g(T) + g(0) \right) + K(\varphi) \right] - e^{-1} F_0(0, T - \varepsilon^r, \varphi + Z^r)
\]
For $v_2^\epsilon$, the strategy is close to that used for $v_1^\epsilon$.

\begin{enumerate}
\item We condition on $\sigma[Z''_\epsilon] = T - \epsilon^2$, $Z'_\epsilon$ for $d \notin [0, a]$.
\item We prove $u_2^\epsilon = E[\Psi_2(g_1, g_2) + o(\epsilon)]$ for some $\Psi_2$ (lemma 7.4).
\item We construct a process $\Delta_x$ independent of $(g_1(T - \epsilon^2), g_1(T))$ such that $g_1$ and $g_2$ can be expressed in terms of $\Delta_x$ and $(g_1(T - \epsilon^2), g_1(T))$ (lemma 5.3).
\item Therefore $u_2^\epsilon = E[\Psi_2(\Delta_x, g_1(T - \epsilon^2), g_1(T)) + o(\epsilon)]$ for some $\Psi_2$. We condition on $\Delta_x$ and prove that the Gaussian double integral $\epsilon^{-1} E[\delta_2(T - \epsilon^2), g_1(T)]$ goes to a non-zero finite limit (for fixed $\delta$). This implies $u_2^\epsilon \sim c\epsilon$ (section 7.2).
\end{enumerate}

3. The Laplace Method

Let $\rho > 0$. By (H1), (1.2) and large deviation arguments, there exists $\tau > 0$ such that

$$v^\epsilon(t, x) = E[1_{[X^\epsilon_T \leq \rho]}] \leq F(v^\epsilon - \rho) + O(\epsilon^{-2}) + \exp \left[\frac{F(T - \epsilon^2) - G(Z^\epsilon)}{\tau} \right].$$

Then, we apply the Girsanov formula and get (see Azencott (1981) pp. 263-266)

$$v^\epsilon(t, x) = E[1_{[Z^\epsilon_T \leq \rho]}] \exp \left[\frac{1}{\epsilon^2} [F(T - \epsilon^2) - G(Z^\epsilon)] \right]$$

$$Z^\epsilon = \epsilon \int_0^T \sigma_T \varphi T dW_t$$

$$G(Z^\epsilon) = \int_0^\tau \varphi \bar{S} - T \bar{\varphi}^2 + \frac{1}{\epsilon^2} d\bar{\varphi}.$$

$Z^\epsilon$ is known to have the following stochastic Taylor expansion (see Azencott (1981))

$$Z^\epsilon = \epsilon g_1 + \epsilon^2 g_2 + \epsilon^3 T^\epsilon \quad \text{where}$$

$$g_1(t) = \int_0^t \sigma_0(s) dW_s, \quad g_2(t) = \int_0^t \sigma_1(s) g_1(s) dW_s, \quad \text{and} \quad \sigma_1(s) = \sigma^{(2)}(\varphi).$$

The remainder $T^\epsilon$ is such that: $\exists c_1, c_2 > 0 \quad \forall t \geq \epsilon > 0 \forall r \geq c_1 \rho^{-1}$

$$P[\|T^\epsilon\| \leq \rho, \|Z^\epsilon\| \geq r] \leq \exp -c_2 r \rho^{-1}.$$ (3.3)

(3.1) yields, for fixed $\alpha > 0$ and $\rho < c_2 \alpha^{-1}$,

$$\sup_{x \in [0, \rho]} F[1_{[\|Z^\epsilon\| \leq \rho]}] \exp \alpha \epsilon \epsilon \|T^\epsilon\| \leq c\alpha(\alpha, \rho, c_1, c_2).$$
REMARK 3.1. Azencott inequalities can sometimes be improved since their left-hand side members are non-decreasing function of $\rho$.

Thus, the previous inequality becomes: \( \forall \alpha > 0 \quad \exists \rho_0(\alpha) > 0 \)

\[
\sup_{\varepsilon \in \varepsilon \leq \varepsilon_\rho(\alpha)} E \left[ \left| V^\varepsilon \right| \leq \rho \right] \exp \alpha \varepsilon \left| V^\varepsilon \right| < \infty \tag{3.4}
\]

Let \( Z^\varepsilon = h_{g_1} + \varepsilon^2 g_2 \). By composition of Taylor expansions, we get

\[
F(\varphi + Z^\varepsilon) - G(Z^\varepsilon) = F(\varphi) + F'(\varphi) \mathcal{Z}^\varepsilon + \frac{\varepsilon^2}{2} F''(\varphi) g_2^2 + \varepsilon^3 \Lambda_x \tag{3.5}
\]

where \( \mathcal{Z}^\varepsilon = \int_0^T \varphi, \mathcal{S}(\varphi) + \mathcal{S}'(\varphi) \mathcal{Z}^\varepsilon + \frac{\varepsilon^2}{2} \mathcal{S}''(\varphi) g_2^2 \mathcal{Z}^\varepsilon + \frac{1}{2} \varepsilon^3 d\gamma \).

The remainder \( \varepsilon^3 \Lambda_x \) satisfies (see Azencott (1984-81), p. 271 and remark 3.1)

\[
\forall \alpha > 0 \quad \exists \rho_2(\alpha) > 0 \quad \sup_{\varepsilon \in \varepsilon \leq \varepsilon_\rho(\alpha)} E \left[ \left| Z^\varepsilon \right| \leq \rho \right] \exp \alpha \varepsilon \left| Z^\varepsilon \right| < \infty \tag{3.6}
\]

\[
\forall \rho > 0 \quad \exists \rho_0(\rho) > 0 \forall \varepsilon \geq \varepsilon_\rho(\rho) \quad \forall \left| V^\varepsilon \right| \leq \rho, \forall \left| \Lambda_x \right| \leq r \quad \exp \left( - \frac{\rho^2}{c_3(\varepsilon)} \right) \leq \exp \left( - \frac{\rho^2}{c_3(\varepsilon)} \right) \tag{3.7}
\]

We transform (3.5) by using (2.9) and (2.3) (which is still meaningful when \( h \) is a continuous martingale on \( 0, T \)) such that \( h(0) = 0 \). We get

\[
F(\varphi + Z^\varepsilon) - G(Z^\varepsilon) = V(T, x) + \mathcal{H}(\varphi) \mathcal{Z}^\varepsilon + \varepsilon^2 K(g_1) + \varepsilon^3 \Lambda_x \tag{3.8}
\]

We can extend the identity \( \mathcal{H}(\varphi) h = \mathcal{H}(\varphi) h \) by density to continuous martingales \( h \) such that \( h(0) = 0 \). Hence, according to (3.1) and (3.8)

\[
v^\varepsilon(T, x) = \exp \left( - \frac{V(T, x)}{2 \varepsilon^2} \right) E \left[ \left| Z^\varepsilon \right| \leq \rho \right] \exp \left( - \frac{V(T, x)}{2 \varepsilon^2} + K(g_1) + \varepsilon \Lambda_x \right) + O(\exp \left( - \frac{V(T, x)}{\varepsilon^2} \right)).
\]

Let \( v_0^\varepsilon = \exp \left( - \frac{V(T, x)}{2 \varepsilon^2} \right) E \left[ \left| Z^\varepsilon \right| \leq \rho \right] \exp \left( - \frac{V(T, x)}{2 \varepsilon^2} + K(g_1) \right)
\]

LEMMA 3.2. If \( \rho \) is small enough, then

\[
v^\varepsilon(T, x) = v_0^\varepsilon + o(v_0^\varepsilon) + O(\varepsilon) \exp \left( \frac{\varepsilon \Lambda_x}{2 \varepsilon^2} \right).
\]

Proof. Let \( b \in [0, 1] \).

It is enough to prove \( w_1^\varepsilon = o(\varepsilon) \) and \( w_2^\varepsilon = o(\varepsilon) + o(\varepsilon) \) where

\[
w_1^\varepsilon = E \left[ \left| \Lambda_x \right| \geq \varepsilon \right. \left. \left| Z^\varepsilon \right| \leq \rho \right] \exp \left( - \frac{V(T, x)}{2 \varepsilon^2} + K(g_1) \right) \exp (\varepsilon \Lambda_x) - 1
\]

\[
w_2^\varepsilon = E \left[ \left| \Lambda_x \right| \leq \varepsilon \right. \left. \left| Z^\varepsilon \right| \leq \rho \right] \exp \left( - \frac{V(T, x)}{2 \varepsilon^2} + K(g_1) \right) \exp (\varepsilon \Lambda_x) - 1
\]

Lemma 2.2, (3.4), (3.6) and Hölder inequality yield: \( \forall \alpha \in [0, 1] \quad \exists \rho_0(\alpha) > 0 \quad \sup_{\varepsilon \in \varepsilon \leq \varepsilon_\rho(\alpha)} E \left[ \left| V^\varepsilon \right| \leq \rho \right] \exp (1 + \alpha) \left( \left| \Lambda_x \right| \right) < \infty
\]

\[
\tag{3.9}
\]
The other computations carried out for \( v \) are still valid for \( v'(T, x) \).

\section{4. The boundary layer}

This section concerns the nonlinear problem. Let \( \alpha \in [0, 1] \), and \( T(x) = T - \epsilon^x \). Ben Arous and Rouault (1993) and (1995) found that, when \( \sigma = 1 \) and \( r(u) = u \), (H3) and (H4) allow us to neglect \( \espace{F}_0, T(x), \psi + Z^x \) in (3.11). The proof can be extended easily. We will not deal with it in detail. Let \( \espace{F} \eta \) for any function \( f \).

\textbf{LEMMA 4.1.} Let \( \eta > 0 \). Then,

\begin{equation}
\mathcal{F}(\eta) \sqrt{\mathcal{F}(2\eta) + \eta} s \leq y \leq \mathcal{F}(\eta) \eta M^{-1} \implies u'(s, y) \leq 4 \exp -\eta s \cdot 2.\
\end{equation}

\textbf{Proof.} We apply the strong Markov property in (1.4) with \( \tau = \inf \{ u \leq s | [X^y] > \eta \} \).
Lemma 4.2. For $\rho$ small enough, $u_1^\rho = u_2^\rho + o(\epsilon)$ where

$$u_2^\rho = E^1 [Z_T \leq 0, \|Z\| \leq \rho] \exp[pZ_T + K(g_1) - \epsilon^{-2} F_s(T(c), T, \varphi + Z^r)].$$

(4.1)

Proof. For $\delta > 0$, let us define the event

$$G = \{\sigma(s) \sqrt{\|Z\|} + \eta(T - \delta) \leq \varphi_1 + Z^r \leq \frac{\sigma(s)}{\delta}, \text{if } s \in [T - \delta, T(c)]\}.$$

(4.1) yields the existence of $\rho, \delta, \eta$ such that, for $\epsilon$ small enough,

$$P(\Omega \setminus G) \cap \{\|X^r\| \leq \rho\} = O(\exp - \text{cost} \epsilon^{-2}).$$

Thus, it is sufficient to find an upper bound for

$$u^\rho = E^1 [G \cap \{\|X^r\| \leq \rho\}] [Z_T \leq 0, \|Z\| \leq \rho] \left[1 - \exp - \epsilon^{-2} F_s(0, T(c), \varphi + Z^r) \right] \exp[pZ_T + K(g_1)].$$

According to lemma 4.1, on $G$,

$$F_s(T - \delta, T(c), \varphi + Z^r) \leq \sigma(s) \int_{\|u\|}^\delta r(\exp - \eta \text{cost} \epsilon^{-2}) \, ds$$

$$= \epsilon^2 (\sigma(s) \eta)^{-1} \int_1^\delta \exp - \frac{\eta \text{cost} \epsilon^{-2}}{ \text{w}} \, du$$

$$\leq \left\{ \begin{array}{l}
\text{cost} \epsilon^2 (\sigma(s) \eta)^{-1} \exp - \frac{\eta \text{cost} \epsilon^{-2}}{ \text{w}} \quad \text{if } r \text{ is } C^1 \text{ on } [0, 1]
\text{cost} \epsilon^2 (\sigma(s) \eta)^{-1} \exp - \frac{\eta \text{cost} \epsilon^{-2}}{ \text{w}} \quad \text{if } (2.10) \text{ holds,}
\end{array} \right. (4.1)$$

(13b) yields, on $\{\|X^r\| \leq \rho\}$ and for $s \in [0, T - \delta]$,

$$u^\rho(s, \varphi_1, Z^r) \leq \exp - \text{cost} \epsilon^{-2}$$

$$F_s(0, T - \delta, \varphi + Z^r) \leq \text{cost} r(\exp - \text{cost} \epsilon^{-2})$$

$$\left\{ \begin{array}{l}
\text{cost} \epsilon^2 \exp - \text{cost} \epsilon^{-2} \quad \text{if } r \text{ is } C^3 \text{ on } [0, 1]
\text{cost} \epsilon^4 \quad \text{if } (2.13) \text{ holds,}
\end{array} \right. (4.1)$$

Actually, the fact that $r$ is non-decreasing and (2.13) imply:

$$r(\exp - \text{cost} \epsilon^{-2}) = O(\epsilon)$$

for all $\mu > 0$. Therefore, according to (4.1) and (4.2),

$$u^\rho \leq \alpha(\epsilon \|Z^r\| \leq \rho) \exp \left(- \epsilon^{-2} T + K(g_1) \right).$$

We conclude by using (3.30).

\[\square\]
5. Construction of Diffusion Bridges

To understand the asymptotics of $\sigma^2$ and $\omega^2$, Ben Arous and Rouault ([1988]) pp. 272, 274 introduced the Brownian bridge $W^a = (W_s - sT^{-1}W_T)_{0 \leq s \leq T}$, and used independence between $W^a$ and $W_T$. In this section, we construct “bridges” associated to the non gaussian diffusion $(g_1, g_2)$.

For $\alpha, \lambda, \mu > 0$, define $\mathcal{E}(\alpha, \lambda, \mu)$ to be the set of continuous processes $X$ such that

$$\forall r \geq 0 \quad P(||X|| \geq r) \leq \lambda \exp(-\mu r^2).$$

Let $\mathcal{E} = \cup_{\alpha, \lambda, \mu > 0} \mathcal{E}(\alpha, \lambda, \mu)$ and $Y = g_1(T)$.

**Lemma 5.1. Construction of a single bridge $\Delta$**

There exists a process $\Delta = [G_{10}, G_{20}, G_{21}]$, independent of $Y$, whose components are in $\mathcal{E}$, and there exist $G_{11}$ and $G_{22}$ in $C^1[0, T]$ such that $G_{10} = \emptyset$ and

$$g_1 = G_{10} + G_{11}Y, \quad g_2 = G_{20} + G_{21}Y + G_{22}Y^2.$$

**Proof.** Let $a_t = -\int_0^T \sigma^2(s) s^{-1} \int_0^s \sigma(s) ds$ and $\bar{W}_t = W_t - a_t Y$ for $t \in [0, T]$. The gaussian process $\bar{W}$ is independent of $Y$.

Then, define $\int_0^T X_t d\bar{W}_t = \int_0^T X_t dW_t - \int_0^T X_t da_t$ for $X$ a continuous and adapted process of $L^2(\Omega \times [0, T])$. We have,

$$g_1 = G_{10} + G_{11} Y, \quad G_{10} = \int_0^T \sigma_0(s) d\bar{W}_t, \quad G_{11} = \int_0^T \sigma_1(s) ds, \quad G_{20} = \int_0^T \sigma_0(s) G_{10}(s) ds + \int_0^T \sigma_1(s) G_{11}(s) ds.$$

Since $G_{10}$ and $G_{11} Y$ are not adapted to the filtration of $W$, we cannot develop this previous expression of $g_2$ by linearity. So we use the Skorokhod integral which extends its integral and accepts non-adapted integrands. $\int_0^T X_t dW_t$ stands for the Skorokhod integral of $X \in dom \delta$. Define $\delta \bar{W}_t$ in the natural way, we have,

$$g_2 = \int_0^T \sigma_0(s) G_{10}(s) d\bar{W}_t + \int_0^T \sigma_0(s) G_{11}(s) Y d\bar{W}_t$$

$$= \int_0^T \sigma_0(s) G_{10}(s) d\bar{W}_t + Y \int_0^T \sigma_1(s) G_{11}(s) ds + \int_0^T \sigma_1(s) G_{11}(s) \bar{W}_t + Y^2 \int_0^T \sigma_1(s) G_{11}(s) ds.$$

We know that (see for instance Nualart and Pardoux [1998]),

$$\int_0^T Y \int_0^s \sigma_1(s) G_{11}(s) d\bar{W}_t =$$

$$- Y \int_0^T \sigma_1(s) G_{11}(s) ds = 0$$

\[\text{Eq. (5.1)}\]
where $D_s$ stands for the Malliavin derivative. Since $D_s Y = -\sigma_0(s)$, we get

$$
\begin{align*}
\gamma_2 &= G_{22} Y^2 + G_{23} Y + G_{24}, \\
G_{22} &= \int_0^1 \sigma_1(s) G_{11}(s) ds, \\
G_{23} &= \int_0^1 \sigma_1(s) G_{21}(s) ds + \int_0^1 \sigma_0(s) \sigma_1(s) G_{11}(s) ds \\
G_{24} &= \int_0^1 \sigma_1(s) G_{21}(s) ds.
\end{align*}
$$

Let $H_m$ be the $m$th Wiener chaos.

**Lemma 5.2.** Let $X = \{X_1, \ldots, X^n\}$ be a continuous $\mathbb{R}^n$-valued process. Assume

1) $(X, W)$ and $Y$ are independent
2) $\forall s \in [1, \ldots, n] \forall t \in [0, T] X^s_t \in H_0 \oplus H_1$.

Then, $(\int_0^t X \delta W_t, W_t)$ and $Y$ are independent.

Lemma 5.2 is proved below. Lemma 5.2 with $X = \sigma_0$ yields independence between $[G_{10}, W]$ and $Y$. Then, with $X = \{\sigma_0, \sigma_0 G_{11}, \sigma_1 G_{11}\}$, it yields that $[G_{10}, G_{20}, G_{21}]$ and $Y$ are independent.

It remains to prove that $G_{10}, G_{20}$, and $G_{21}$ belong to $\mathcal{E}$. For $G_{10}$ and $G_{20}$, it is a consequence of the stability of $\mathcal{E}$ under sum, product, Riemann and Itô integration (see Azencott [1980], p. 252). For $G_{21}$, we just need to prove $\int \sigma_1(s) G_{21}(s) \delta W_t \in \mathcal{E}$. The space $\mathcal{E}$ is not stable under Skorokhod integration, but identity $G_{21} = G_{11} - Y G_{11}$ and (3.1) allow us to conclude.

**Proof of Lemma 5.2.** Let us prove that

$$
E f\left(\int_0^t X \delta W_t, W_t\right) h(Y) = E f\left(\int_0^t X \delta W_t, W_t\right) E h(Y)
$$

for $t \in [0, T]$ and $f, h$ bounded continuous functions (we only deal with one-dimensional marginals not to overload the notations).

Let $I_l^p : 0 = t_0 < \cdots < t_{p+1} = t$ be a sequence of partitions of $[0, t]$ whose meshes go to 0, and let

$$
X_p = \sum_{l=0}^{p-1} \nabla_{k,p} \left[ t_{k+1} - t_{k+1} \right]^\theta, \quad \nabla_{k,p} = (t_{k+1} - t_k)^{-1} \int_{t_k}^{t_{k+1}} X \delta u.
$$

We have (see Nualart and Pardoux [1988], prop. 4.3 and remark p. 546)

$$
\int_0^t X_p(s) \delta W_t \rightarrow \int_0^t X \delta W_t \quad \text{in } L^2(\Omega).
$$

Therefore, there exists a subsequence of $I_l^p$ such that this convergence holds a.s. It yields

$$
Z_p := \int_0^t X_p(s) \delta W_t \rightarrow \int_0^t X \delta W_t \quad \text{a.s.}
$$
By dominated convergence, it is enough to prove that $Z_p$ is independent of $Y$.

Since the Malliavin derivative and Riemann integral commute,

$$Z_p = \sum_{k=0}^{p-1} X_k (W_{f+1,p} - W_{f,p}) - \int_{f+1}^p \frac{d}{ds} X_k ds$$

$$= \sum_{k=0}^{p-1} X_k (W_{f+1,p} - W_{f,p})$$

$$- \left( \frac{d}{ds} W_{f+1,p} - \frac{d}{ds} W_{f,p} \right) \int_{f+1}^p \frac{d}{ds} X_k ds ds.$$

But $X_k \in H_0 \oplus H_1$ yields that $D_x X_k$ is deterministic. Therefore, $Z_p$ can be expressed in terms of $(X,W)$ which is independent of $Y$. 

Let $V = (V,Y) = (-g_1(T'(c)), -g_0(T))$ where $T'(c) = T - \varepsilon^0$ and $d \in [0,a]$. 

**Lemma 5.3.** Construction of a double bridge $\Delta_x$. There exists a process $\Delta_x = \{G_{ijk} \mid 0 \leq j + k \leq i \leq 2 \}$, independent of $V_x$, such that:

i) $g_i = \sum_{a \leq j \leq k \leq i} G_{ijk} Y_i Y^k$, $i = 1,2$

ii) if $j + k = i$, $G_{ijk}$ is a deterministic $C^1$ function on $[0,T]$.

iii) $\exists \alpha, \lambda, \mu > 0 \forall \varepsilon > 0 \forall 0 \leq j + k \leq i \leq 2 \ G_{ijk} \in \mathcal{E}(\alpha, \lambda, \mu)$.

**Proof.** Define $W_{alpha}$, a gaussian process independent of $V_x$:

$$W_{alpha} = W_T - \Theta(t)Y_T - \sigma(t)Y, \quad 0 \leq t \leq T$$

$$\sigma^2 = s_{\varepsilon} + \frac{1}{2} \sigma_0(s) ds$$

$$\Theta(t) = g_i - \int_0^t \sigma_0(s) ds.$$ 

Then, let $dW_{alpha} = dW_T - Yd\sigma (s) - Yd\Theta (s)$. The proof is now similar to that of lemma 5.1. The only new point is to prove that the processes $G_{ijk}$ belong to the same $\mathcal{E}(\alpha, \lambda, \mu)$ for all $\varepsilon$. If $j+k = i$, it is clear because $G_{ijk}$ is deterministic and uniformly bounded in $\varepsilon$. If $j+k < i$, it comes from Azencott results (see Azencott (1980-81) p. 252). 

Exact expressions of $G_{ijk}$ do not matter, except for

$$G_{2\alpha} = \int_0^T \sigma_0(s) dW_{\alpha}.$$ 

6. The Non-linear Part Contribution
Let \( \alpha \in \{0,1-a\} \) and \( g'(y,z) \) given by

\[
F_{\frac{1}{2}}^{\frac{1}{2}} \left( \left\| Z_n^T \right\|_{T(z)} \leq \frac{1}{2} \right) \exp \left( -\frac{1}{2} F_{\frac{1}{2}}^{\frac{1}{2}} \left( \left\| T(z) \right\|_{T(z)} \leq \frac{1}{2} \right) \right) = c_z, Z_T^T = \frac{1}{2} g'.
\]

(6.1)

In this section, we prove \( g'(y,z) \rightarrow g(y) \), thus understanding the contribution of \( u' \) in the expectation of (2.1.1). The key result is lemma 6.1 that establishes the convergence of \( u'(\cdot \ell, \cdot z) \).

If \( \sigma = 1 \), the equality between the laws of a brownian motion knowing its final position and the related bridge makes the computation more simple (see Ben Arous and Rouault 1993). Our method is based upon the following classical result (see Fitzsimmons, Pitman and Yor (1993)).

Let \( (X_t)_{t \in \mathbb{T}} \) be a real Markov process with transition density \( p_{0,t}(x,y) \), and \( (\mathcal{F}_t)_{t \in \mathbb{T}} \) be its natural filtration. Let \( x \in \mathbb{R}, t < T \) and \( Z \) a r.v. \( \mathcal{F}_t \)-measurable. Then, for almost every \( y \in \mathbb{R} \),

\[
E_Z \left[ X_T = y \right] p_{0,t}(x,y) = E_Z \left[ X_T = y \right].
\]

(6.2)

In order to use (6.2) and make \( u'(\cdot \ell, \cdot z) \) appear, let \( Y_t^z = e^{-z} Z_t, \). We have

\[
\begin{align*}
g'(y,z) &= E_0 \left[ Y_T^z \right] \
y_t^z &= y + \int_0^t \sigma \left( \varphi_{t-s} + e^z Y_s^z \right) dW_s \\
B_t &= \left\{ \left\| Y \right\|_{B_t} \leq e^{-z} \right\} \\
I_t &= \int_0^t e \left( \varphi_{t-s} + e^z Y_s^z \right) \left[ u'(\cdot^z, \cdot \ell, \cdot z) \right] ds.
\end{align*}
\]

We will prove that, if \( z^{-1} \) is not too large, the conditioning on \( Y_{t-s}^z = e^{-z} \) does not alter the asymptotics. Actually, “tying up” a diffusion \( Y \) at time \( t \) on point \( y \in \mathbb{R} \) does not alter \( E_0 \left[ Y_T \right] \) when \( t \) and \( \ell' \) go to \( +\infty \), provided that \( \ell' = o(t) \) and that \( y \) is not too large. Let

\[
g'(y) = E_0 \left[ Y_T \right] = g'(y)
\]

(6.3)

**PROPOSITION 6.1.** Assume \( 2/3 < d < a < 1 \). Then

i) For all \( y \), \( g'(y) \rightarrow g(y) \).

ii) Assume \( \epsilon = O(x^{-\delta-\epsilon}) \) where \( \epsilon \in \{0,1\} \), \( 1+\frac{\delta}{2} - d \). Then, \( g'(y,z) - g'(y) \rightarrow 0 \).

We need, in section 7, the following result on the modulus of continuity of \( g'(y,z) \).

**PROPOSITION 6.2.** Assume \( r \in C^3[0,1] \) (case \( \sigma \) non constant).

Assume \( 2/3 < d < a < 1 \). Let \( \eta > 0 \) small enough, and \( \delta > 0 \). Then

i) If \( |y - y'| \leq \epsilon^\delta \) and \( |z|, |z'| \leq \epsilon^{-\epsilon} \), then \( g'(y) - g'(y') = o(1) \).

ii) If moreover \( |x - x'| \leq \epsilon^{1+\delta} \) and \( |z|, |z'| \leq \epsilon^{1+\delta} \), then \( g'(y,z) - g'(y',z') = o(1) \).

**REMARK 6.3.** We state propositions 6.1 and 6.2.1 to show that the conditioning has no influence and to make the proofs easier to read.
Therefore, the Feynman-Kac formula yields

\[ h^t(u) = E \int_0^t \alpha_k(t, x, x+\sigma(0)W_t) \exp \left( \int_0^t \int_{C^2(0,1)} \frac{\alpha_k(t, x, x+\sigma(0)W_t)}{2} \right) \, ds, \]

where

\[ \alpha_k(t, x, x+\sigma(0)W_t) = \begin{cases} 0 & \text{if } x+\sigma(0)W_t \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \frac{1}{r(t)} \end{cases} \]

\[ \beta_k(t, x) = \begin{cases} 0 & \text{if } x+\sigma(0)W_t \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \frac{1}{r(t)} \end{cases} \]

Therefore, the Feynman-Kac formula yields

\[ h^t(u) = E \int_0^t \alpha_k(t, x, x+\sigma(0)W_t) \exp \left( \int_0^t \int_{C^2(0,1)} \frac{\alpha_k(t, x, x+\sigma(0)W_t)}{2} \right) \, ds, \]

\[ \beta_k(t, x) = \begin{cases} 0 & \text{if } x+\sigma(0)W_t \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \frac{1}{r(t)} \end{cases} \]

Therefore, the Feynman-Kac formula yields

\[ h^t(u) = E \int_0^t \alpha_k(t, x, x+\sigma(0)W_t) \exp \left( \int_0^t \int_{C^2(0,1)} \frac{\alpha_k(t, x, x+\sigma(0)W_t)}{2} \right) \, ds, \]

\[ \beta_k(t, x) = \begin{cases} 0 & \text{if } x+\sigma(0)W_t \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \frac{1}{r(t)} \end{cases} \]

Therefore, the Feynman-Kac formula yields

\[ h^t(u) = E \int_0^t \alpha_k(t, x, x+\sigma(0)W_t) \exp \left( \int_0^t \int_{C^2(0,1)} \frac{\alpha_k(t, x, x+\sigma(0)W_t)}{2} \right) \, ds, \]

\[ \beta_k(t, x) = \begin{cases} 0 & \text{if } x+\sigma(0)W_t \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \frac{1}{r(t)} \end{cases} \]

Therefore, the Feynman-Kac formula yields

\[ h^t(u) = E \int_0^t \alpha_k(t, x, x+\sigma(0)W_t) \exp \left( \int_0^t \int_{C^2(0,1)} \frac{\alpha_k(t, x, x+\sigma(0)W_t)}{2} \right) \, ds, \]

\[ \beta_k(t, x) = \begin{cases} 0 & \text{if } x+\sigma(0)W_t \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \frac{1}{r(t)} \end{cases} \]

Therefore, the Feynman-Kac formula yields

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\[ \beta_k(t, x) = \begin{cases} 0 & \text{if } x+\sigma(0)W_t \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \frac{1}{r(t)} \end{cases} \]

Therefore, the Feynman-Kac formula yields

\[ h^t(u) = E \int_0^t \alpha_k(t, x, x+\sigma(0)W_t) \exp \left( \int_0^t \int_{C^2(0,1)} \frac{\alpha_k(t, x, x+\sigma(0)W_t)}{2} \right) \, ds, \]

\[ \beta_k(t, x) = \begin{cases} 0 & \text{if } x+\sigma(0)W_t \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \frac{1}{r(t)} \end{cases} \]
Therefore, we can assume \( \omega \in \{ \|B^\varepsilon\|_{\infty}^2 \leq \varepsilon^{-1} \} \cap B_0 \) for all \( \varepsilon \).

For fixed \( s \) and \( \varepsilon \leq \varepsilon^{-1}, \)

\[
\|V^\varepsilon_0(\omega) - y - \sigma(0)W_{\varepsilon}(\omega)\| \leq \varepsilon^{1-1}.
\]

Hence, by lemma 6.3, \( e(\varphi_{T_{\sigma}} + \varepsilon Y^\varepsilon_0(\omega)) \rightarrow \varphi_{\varepsilon}Y^\varepsilon_0(\omega) + \varepsilon Y^\varepsilon_0(\omega) \)

goes to \( e(0)r(u\sigma s, \varphi_{T_{\sigma}} + y + \sigma(0)W_{\varepsilon}(\omega)) \) for all \( s \).

(6.6) will be deduced, provided that we have a dominated convergence in \( s \). Let

\[
\Omega_1 = \{ \omega \mid \forall s > 0 \exists \delta(s) \forall s \geq \delta(s) \|V^\varepsilon(s \omega)\| \leq \delta \}.
\]

(8.8) implies \( P(\Omega_1) = 1 \). Hence we assume \( \omega \in \Omega_1 \).

By lemma 4.1, for all \( \eta > 0, \)

\[
\delta(\eta)\phi(2(\eta^2 + \eta)) \leq x \leq \delta(\eta)\phi(2(\eta^2 + \eta)) \quad \implies \quad \delta(\eta) \leq 4 \exp(-\eta).
\]

(6.7) yields the existence of \( \eta > 0 \) such that \( \varphi_{\varepsilon}Y^\varepsilon_0(\omega) \leq \delta(\eta)\phi(2(\eta^2 + \eta)) \).

For \( s \leq 0 \), \( \varepsilon = \varepsilon^{-1} \) and \( \eta \) small enough, \( \|Y^\varepsilon_0(\omega) + \varepsilon Y^\varepsilon_0(\omega) + \varphi_{\varepsilon}Y^\varepsilon_0(\omega) \| \leq \delta(\eta)\phi(2(\eta^2 + \eta)) \).

\[ \Omega_1 \] implies \( \exists \delta(s) \forall s \geq \delta(s) \|V^\varepsilon_0(\omega)\| \leq \delta(s) \).

Hence, according to (6.7), for \( \delta(s) \leq s \leq \delta(s) \) and \( \eta \) small enough,

\[
\delta(\eta)\phi(2(\eta^2 + \eta)) \leq \varphi_{\varepsilon}Y^\varepsilon_0(\omega) \leq \delta(\eta)\phi(2(\eta^2 + \eta)) \quad \implies \quad \delta(\eta) \leq 4 \exp(-\eta).
\]

and \( r \in C^1([0,1]) \) or (2.30) yields dominated convergence in \( s \).


We can define on the same sample space (see remark 6.1.) the diffusions

\[
Y^\varepsilon = y + \int_0^T \sigma(\varphi_{\varepsilon} + \varepsilon Y^\varepsilon_0) dW_t, \quad Y^\varepsilon = y + \int_0^T \sigma(\varphi_{\varepsilon} + \varepsilon Y^\varepsilon_0) dW_t.
\]

Define \( B^\varepsilon_t \) and \( B^\varepsilon_t \) from \( Y^\varepsilon \) as \( B_t \) and \( L_t \) were defined from \( Y^\varepsilon \) (see (6.3)).

Since \( \|B^\varepsilon\| \leq \varepsilon^{-1} \), (8.7) yields

\[
P(B_0 \cap B^\varepsilon_0 \cap B^\varepsilon_t) \leq 2 \exp(-\|\varepsilon^{-1/2} + L\|_1^2) \leq \exp(-econst \varepsilon^{-2-s})
\]

(9.6) \( f(\varepsilon) \geq f(\varepsilon) \)

\[
 = \varepsilon^{-1} \exp(-s) - \exp(-s) \leq \exp(-s)
\]

\[
E[|B^\varepsilon_0 \exp(-L_t) - \exp(-L^\varepsilon_t)|]
\]

\[
= E[|B^\varepsilon_0 \exp(-L_t) - \exp(-B^\varepsilon_t)| + o(1)]
\]

6.6. Let \( \mu \in [0, \inf \{\|B^\varepsilon\|_{\infty}^2 \geq \varepsilon^\gamma \} \cap B_0 \cap B^\varepsilon_t) = o(1) \).

\[
P(\{\|B^\varepsilon - Y^\varepsilon\|_{\infty}^2 \geq \varepsilon^\gamma \} \cap B_0 \cap B^\varepsilon_t) = o(1).
\]
Proof. Let $X_t = \gamma + \sigma(0) W_t$, $X'_t = \gamma' + \sigma(0) W_t$, and $\mu \in [b, 3a/2 - 1]$. We get
\[
P(\|Y^\varepsilon - Y'_\mu\|_0^{a-2} \geq \varepsilon^\delta) \cap B_\varepsilon \cap B'_\varepsilon 
\leq P(\|Y^\varepsilon - X\|_0^{a-2} \geq \varepsilon^\delta) \cap B_\varepsilon 
+ P(\|Y'_\mu - X\|_0^{a-2} \geq \varepsilon^\delta) \cap B'_\varepsilon).
\]
We have $Y^\varepsilon - X_0 = \int_0^\infty A'_t dW_t$, where $A'_t = a(\varphi_{\gamma + \varepsilon^2 Y^\varepsilon} - \sigma(0))$ on $B_\varepsilon$ and if $\varepsilon \leq \varepsilon^{a-2}$. Hence, (8.7) yields
\[
P(\|Y^\varepsilon - X\|_0^{a-2} \geq \varepsilon^\delta) \cap B_\varepsilon \leq \exp(-c \varepsilon^{2+2/3} \varepsilon^\delta).
\]
Let $B_\varepsilon = \{\|Y^\varepsilon - Y'_\mu\|_0^{a-2} \leq \varepsilon^\delta\} \cap B_\varepsilon \cap B'_\varepsilon$. Lemma 6.1 yields
\[
|\varphi^\varepsilon(y) - \varphi^\varepsilon(y')| \leq E_1 D_\varepsilon |Y_\varepsilon - Y'_\mu| + o(1).
\]
On the event $D_\varepsilon$ and if $\varepsilon \leq \varepsilon^{a-2}$, then $\varphi_{\gamma + \varepsilon^2 Y^\varepsilon}$ and $\varphi_{\gamma + \varepsilon^2 Y'_\mu}$ stay in a compact $K$. Let $k_1 = \sup_K |\varphi|$ and $k_2 = \sup_K |\varphi'|$. We get
\[
1_{D_\varepsilon} |Y_\varepsilon - Y'_\mu| \leq k_1 \varepsilon^{2-\varepsilon^{a-2}}
+ k_2 \int_0^{\varepsilon^{a-2}} |\varphi'(\varepsilon^2 s, \varphi_{\gamma + \varepsilon^2 Y^\varepsilon} + \varepsilon^2 Y^\varepsilon) - \varphi'(\varepsilon^2 s, \varphi_{\gamma + \varepsilon^2 Y'_\mu} + \varepsilon^2 Y'_\mu)| ds.
\]
Therefore, $|\varphi^\varepsilon(y) - \varphi^\varepsilon(y')| \leq k_1 l^0(0, \varepsilon^{a-2}) + o(1)$ where $l^0(a_1, a_2) =
\[
E_1 D_\varepsilon \int_0^{\varepsilon^{a-2}} |\varphi'(\varepsilon^2 s, \varphi_{\gamma + \varepsilon^2 Y^\varepsilon} + \varepsilon^2 Y^\varepsilon) - \varphi'(\varepsilon^2 s, \varphi_{\gamma + \varepsilon^2 Y'_\mu} + \varepsilon^2 Y'_\mu)| ds
\]
and $l^0$ depends only on $\gamma_1$ and $\gamma_2$, not on $y, y'$. Proof. According to (H4), there exists $\delta > 0$ such that
\[
l = -\varphi_{\gamma + \varepsilon^2 Y^\varepsilon} \sqrt{\|\varphi_{\gamma + \varepsilon^2 Y^\varepsilon} + \delta} > 0.
\]
Let $V^\varepsilon_1 = \{\varphi_{\gamma + \varepsilon^2 Y^\varepsilon} + Y^\varepsilon < \sqrt{\|\varphi_{\gamma + \varepsilon^2 Y^\varepsilon} + \delta} \text{ for all } s \in [\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_2}]\}
\text{ and } V^\varepsilon_2 = \{\|\varphi_{\gamma + \varepsilon^2 Y^\varepsilon} + \varepsilon^2 Y^\varepsilon\|_2 \leq s \text{ for all } s \in [\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_2}]\}$.
(8.7) yields that, for $\varepsilon$ small enough,
\[
P(\Omega \{V^\varepsilon_1\}) \leq P(3\varepsilon \in [\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_2}] \text{ Y}^\varepsilon > c \varepsilon^{-\delta} \leq \exp(-c \varepsilon^{2+4})
P(\Omega \{V^\varepsilon_2\}) \leq P(3\varepsilon \in [\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_2}] \text{ Y}^\varepsilon < -c \varepsilon^{-\delta} \leq \exp(-c \varepsilon^{2+4}).
\]
Let $C_\varepsilon = \int_{\varepsilon^{-\gamma_1}}^{\varepsilon^{-\gamma_2}} r(u' \varphi_{\gamma + \varepsilon^2 Y^\varepsilon} + \varepsilon^2 Y^\varepsilon) ds$. Therefore, lemma 4.1 implies
\[
EC_\varepsilon = E_1 V^\varepsilon_1 \cap V^\varepsilon_2 C_\varepsilon + o(1) \leq \int_{\varepsilon^{-\gamma_1}}^{\varepsilon^{-\gamma_2}} r(v \exp(-\delta) ds + o(1) = o(1).
\]
The same computation for $Y^r$ ends the proof of lemma 6.5. 

Now, let $\gamma \in [0, \mu/2]$. By iterating lemma 6.5, we get $l^r(\varepsilon^{-\gamma}, \varepsilon^{-\gamma}) = o(1)$. It remains to prove $l^r(0, \varepsilon^{-\gamma}) = o(1)$. Let $\tau \in [\gamma/2, \mu - 3\gamma/2]$.

(8.7) yields

$$P(\|Y^r\|_{L_0}^{-\gamma} \geq \varepsilon^{-\gamma}) \leq \exp -c\|L_0\|^{2\gamma}.$$  (6.8)

Define $E^r(\alpha_1, \alpha_2)$ by replacing $D_c$ by

$$\left(\|Y^r\|_{L_0}^{-\gamma} \leq \varepsilon^{-\gamma}\right) \cap \left(\|Y^r\|_{L_0}^{-\gamma} \leq \varepsilon^{-\gamma}\right) \cap \left(\|Y^r - Y^r\|_{L_0}^{-\gamma} \leq \varepsilon^{-\gamma}\right) \cap B_l \cap B_l^r$$

in the definition of $l^r(\alpha_1, \alpha_2)$. (6.8) yields $l^r(0, \varepsilon^{-\gamma}) = E^r(0, \varepsilon^{-\gamma} + o(1))$.

Appendix 8.1 and $r \in C^1[0, 1]$ imply

$$\sup_{\varepsilon^{-\gamma} \leq x < \varepsilon^{-\gamma}} |g_{\tau}(u^x, \varepsilon^2 x)| = O(\varepsilon^{-\gamma}) + O(\varepsilon^{-\gamma/2}|x|).$$

Therefore,

$$E^r(0, \varepsilon^{-\gamma}) = E^r(0, \varepsilon^{-\gamma}) + E^r(\varepsilon^{-\gamma}, \varepsilon^{-\gamma}) + O(\varepsilon^{-\gamma} + O(\varepsilon^{-\gamma/2}|x|)) = o(1).$$

### 6.4. Proof of Proposition 6.1.31

Together with (6.2), we will use an explicit formula for $q^r$, the transition density of $Y^r$. Let

$$\gamma(x) = \sigma(x^2 x), \quad \alpha_1 = \varepsilon^{-\gamma}_{\tau^{-\gamma}, x},$$

$$G(t, x) = \int_0^x e^{\frac{\gamma}{\alpha_1}} \frac{du}{\gamma(\alpha_1 + u)} \quad L(t, \cdot) \text{ be the inverse function of } G(t, \cdot)$$

$$C(t, x) = \partial_t G(t, L(t, x)) = \frac{1}{2} \gamma'(\alpha_1 + L(t, x))$$

$$= \alpha_1' [\gamma(\alpha_1 + L(t, x)) - \gamma(\alpha_1) + \gamma(\alpha_1 - 1)] - \frac{1}{2} \gamma'(\alpha_1 + L(t, x))$$

$$D(t, x) = -\frac{1}{2} \partial_t C(t, x) = \frac{1}{2} C^2(t, x) - \int_0^\infty \partial_u C(t, u) du$$

$$H(t, x) = \int_0^x e^{\frac{\gamma}{\alpha_1}} C(t, u) du$$

$$= \int_0^\infty \left( \alpha_1' [\gamma(\alpha_1 + v) - \gamma(\alpha_1) + \gamma(\alpha_1 - 1)] - \frac{1}{2} \gamma'(\alpha_1 + v) \right) \frac{dv}{\gamma(\alpha_1 + v)}$$

$$V_\alpha(a, b) = (1 - a)a + ab$$

$$J(s, t, x, y) = \mathbb{E} \left[ \exp(-t) \int_0^t \mathbb{E} \left[ \int_0^1 D(V(s, t, x), V(s, t, y), G(t, y) + \sqrt{t - s} \mathbb{B}_s) \right] du \right]$$

where $\mathbb{B}$ is a standard brownian bridge defined on another probability space $\Omega'$ and $\mathbb{E}[\ldots]$ refers to $\Omega'$ (as well as $\mathbb{P}(\ldots)$ later on).
Lemma 6.8.

\[ q^*_{x,y}(x,y) = \frac{1}{\sqrt{2\pi(1 - \theta)}} \frac{J(s,t,x,y)}{\gamma(\theta + y)} \exp\left( \frac{|\mathcal{G}(t,y) - \mathcal{G}(s,x)|^2}{2(t - s)} + H(t,y) - H(s,x) \right). \]


We will need the following bounds.

Lemma 6.9.

i) \( \exists C > 0 \forall t, x \quad |H(t,x)| \leq C(1 + x^2) \)

ii) \( \exists C > 0 \forall t, x \quad D(t,x) \leq C(1 + |x|) \)

iii) \( \forall \lambda > 0 \exists C_\lambda > 0 \quad |x| \leq \lambda, |x| \leq \lambda \rightarrow |H(t,x)| \leq C_\lambda \lambda^2. \)

Proof. i) We have \( \|y^x\|_\infty = O(e^{\gamma}), \quad \|y^y\|_\infty = O(e^{\gamma}), \) and \( \|y^z\|_\infty < \infty. \)

Hence

\[ |H(t,x)| \leq c_\gamma e^{\gamma} |x| + c_\gamma \int_0^{\infty} |\gamma(x_1 + x) - \gamma(x_1)| dx_1 \leq c_\gamma e^{\gamma} (1 + x^2). \]

ii) Straightforward.

iii) Let \( \alpha_n = \frac{c_\gamma e^{\gamma}}{\|y^x\|_\infty}. \) Easy computations yield the successive results:

\[ e^{z} L_{t_0}(t_0,x_0) = O(e^{\gamma}), \quad e^{z} \alpha_n = \frac{e^{z} t_0 + O(e^{\gamma})}{L(t_0,x_0) = \sigma(0) x_0 + O(e^{\gamma})} \]

\[ C(t_1,x_1) = \alpha_n x_1 + O(e^{\gamma}), \quad \frac{\partial}{\partial x} C(t_1,x_1) = \alpha_n x_1 + O(e^{\gamma}) \]

\[ D(t_1,x_1) = O(e^{\gamma}). \]

According to \( (6.2), (6.3) \) and lemma 6.8, we have

\[ g^x(y,z) = g^y(y) = \frac{\gamma(x_1 + x)}{\gamma(x_1 + x)} \exp(-L_0) |J_0 \exp(G_0 + H_0) - 1| \]

\[ G_0 = \frac{e^{z} e^{\gamma}}{2} (G(e^{z} e^{\gamma} x, e^{z} e^{\gamma} y) - G(0, y))^2 \]

\[ H_0 = H(0, y) - H(e^{z}, e^{z} y) \]

\[ J_0 = J_0 \]

\[ J_0^x = J(e^{z} e^{\gamma} x, e^{z} e^{\gamma} y, e^{z} e^{\gamma} y) \]

As we expected, \( J_0 \exp(G_0 + H_0) \rightarrow 1 \) a.s., and more precisely,

Lemma 6.10. For \( \eta > 0 \) small enough, define \( X_\eta = e^{z} e^{\gamma} + \eta \) \( [e^{z} e^{\gamma} - \eta] \)

i) \( X_\eta \rightarrow 0 \) a.s. and \( \mathbb{E} \exp(X_\eta) \rightarrow 1 \)

ii) \( \mathbb{E} \exp(X_\eta^2) \rightarrow 1 \)

iii) \( \mathbb{E} \exp(X_\eta^2) \rightarrow \mathbb{E} \exp(X_\eta^2) \) a.s. where \( \alpha_n \rightarrow 0. \)
\[ J_i^* \to 1 \text{ a.s.} \quad \text{and} \quad J_f^* \to 1. \]

The proof of this lemma is postponed to the end of the section.

In order to get a dominated convergence, we introduce

\[ B_\varepsilon = B_\varepsilon \cap \{|V_{j=\infty}^y| \leq \varepsilon^{-1}\} \]

as well as \( \mathcal{T}(y) \) and \( \mathcal{T}(y, z) \) defined from \( g^*(y) \) and \( g^*(y, z) \) by replacing \( B_\varepsilon \) by \( B_\varepsilon \).

It can be proved easily that \( \mathcal{T}(y) = g^*(y) \to 0 \).

Let us prove that \( \mathcal{T}(y, z) = g^*(y, z) \to 0 \). According to lemma 6.7.ii,

\[ J_i^* \leq E \exp \int_0^1 \left( 1 + | \epsilon^{-1} + |V_{j=\infty}^y| + \epsilon^{-1/2} |B_\varepsilon| \right) \, dx \text{ a.s.} \]

\[ \leq \exp \int_0^1 \left( | \epsilon^{-1} + \epsilon^{-1/2} |B_\varepsilon| \right) \, dx \text{ a.s.} \]

(notice that a.s. refers to the arguments of \( J_i^*, Y_{j=\infty}^y \) and \( X_2 \)).

Together with lemma 6.8.iii and ii, it yields, for \( \varepsilon \) small enough,

\[ 0 \leq g^*(y, z) - \mathcal{T}(y, z) \leq (J_f^* - 1)^{\alpha} E \left[ |V_{j=\infty}^y| \geq \varepsilon^{-1} \right] J_f^* \exp \left( \alpha \varepsilon \Gamma (1 + X_2^2) \right) \]

\[ \leq \exp \int_0^1 \left( | \epsilon^{-1} + \exp \left( \alpha \varepsilon \Gamma (1 + X_2^2) \right) \right) \, dx \text{ a.s.} \]

where \( \rho_x \to 0 \). Moreover, \( P[|V_{j=\infty}^y| \geq \varepsilon^{-1}] \) \( \leq \exp -c \varepsilon^{-a} \). Hence,

\[ 0 \leq g^*(y, z) - \mathcal{T}(y, z) \leq \exp \int_0^1 \left( | \epsilon^{-1} - \exp \left( -c \varepsilon^{-a} \right) \left| E \left[ \exp (2J_x X_2^2) \right] \right| \right) \]

and we can conclude by using lemma 6.8.ii and \( \alpha > 1 - d \).

It remains to prove that \( \mathcal{T}(y, z) - \mathcal{T}(y) \to 0 \). We introduced \( B_\varepsilon \) because

\[ \left( |V_{j=\infty}^y| \leq \varepsilon^{-1} \right) J_f^* \leq \exp \int_0^1 \left( | \epsilon^{-1} \right) \]

(actually, lemma 6.9.ii below is still valid when \( z = O(\varepsilon^{|-a|}) \)). Therefore

\[ | \mathcal{T}(y, z) - \mathcal{T}(y) | \leq E \left[ |V^y| - \epsilon \Gamma J_x \exp (-L_x) | + | \mathcal{T}(y) - E \left[ \epsilon \Gamma J_x \exp (-L_x) \right] \right]. \]

The second term is smaller that \( E \left[ |V_{j=\infty}^y| \geq \varepsilon^{-1} \right] \Gamma (1 + X_2^2) \) which goes to 0 according to (6.10).

By (6.30), the first term is smaller than \( \exp (G_x + H_x) - 1 | \) which goes to 0 by lemma 6.8.ii and 6.8.iii.

Proof of lemma 6.8.iii. The first part is a consequence of (8.8).

The second part holds since \( \exp \left( X_2^2 \right) \) is uniformly integrable. Actually

\[ E \exp 2X_2^2 = \int_0^\infty P(2X_2^2 \geq \gamma) \, d\gamma = 1 + \int_0^\infty \int_0^\gamma P(X_2 \geq \gamma) \exp r \, dr \, dr \]

\[ \leq 1 + \int_0^\gamma \exp -c \varepsilon^{-1/2} \, dr \leq \exp \text{ for } \varepsilon \text{ small enough}. \]
Lemma 6.7 ii and \( |G(t, x)| \leq m^\ast |v| \) yield

\[ |H(e^{z_x}, Y_x - e^{z_x})| \leq C_1 e^{z_x(1 + e^{z_x} + X_y^2)} \]

\[ \leq C_1 e^{z_x(1 + e^{z_x} + X_y^2)} \]

\[ 2d - \frac{G}{} \]

\[ \left( \right) \]

\[ G \]

\[ \leq C_1 e^{z_x(1 + e^{z_x} + X_y^2)} \]

\[ + \cos |z - z_x| + \cos \theta (Y_x - z_x)^2 + \cos \theta |z - z_x| \]

\[ C_2 \]

\[ \leq \cos |z - z_x| + \cos \theta (Y_x - z_x)^2 + \cos \theta |z - z_x| \]

\[ \leq C_2 |e^{z_x} - e^{z_x}| + o(1) + \varepsilon |e^{z_x} - e^{z_x}|. \]

Lemma 6.7 ii yields that \( J_0 = J_0 \) is smaller than

\[ \mathbb{E} [ |\mathbb{B}^\ast | \leq \varepsilon^{-1} \exp(e^{z_x} - e^{z_x}) \]

\[ \int_{\mathbb{D}} |V_0(e^{z_x}, z, z_x) + V_0(G(e^{z_x}, Y_x - z_x), G(e^{z_x}, z_x, z_x)) + \sqrt{e^{z_x} - e^{z_x}}| \, du. \]

Lemma 6.7 iii yields

\[ \mathbb{E} [ |\mathbb{B}^\ast | \leq \varepsilon^{-1} \exp(e^{z_x} - e^{z_x}) \]

\[ \leq \exp(\varepsilon |e^{z_x} - e^{z_x}|) = o(1). \]

We now prove that \( J_0 \to 1 \) a.s.

There exists \( \lambda > 0 \) such that, a.s. on \( |\mathbb{B}^\ast | \leq \varepsilon^{-1} \)

\[ \varepsilon |V_0(G(e^{z_x}, Y_x - z_x), G(e^{z_x}, z_x, z_x)) + \sqrt{e^{z_x} - e^{z_x}}| \leq \varepsilon + e^2 \lambda \]

\[ \leq \lambda. \]

Since \( e^{2V_0(e^{z_x}, z, z_x)} \leq e^2 \), lemma 6.7 iii yields

\[ J_0 - \mathbb{P} (|\mathbb{B}^\ast | \leq \varepsilon^{-1}) \leq \exp(C_3(\lambda) e^2) - 1, \]

which yields \( J_0 \to 1 \) a.s. We prove \( J_0 \to 1 \) in the same way.

6.5. Proof of Proposition 6.2.ii.

We will need the following results

**Lemma 6.11.** Define \( X_x = e^{5/2 - \varepsilon^{2+\theta}} |Y_x^\ast| \) as in lemma 6.8. Then

i) \( \exp X_x \) is bounded in \( L^1 \), for all \( q \geq 1 \)

ii) \( \|G_x + H_x\| \leq \alpha_q (1 + X_x) \) where \( \alpha_q \to 0 \)

iii) \( \|J_x \| \leq \exp \left( |Y_x^\ast| \right) \leq e^2 \)

iv) \( J_x \geq \frac{1}{2} \) for \( \varepsilon \) small enough

v) \( J_x \leq \exp X_x \) for \( \varepsilon \) small enough.

All these results remain valid if we replace \( X_x, G_x, \ldots \) by \( X_x', G_x', \ldots \).
Proof. i) and ii) The proofs are similar to that of lemmas 6.8.i and 6.8.ii. The fact that \( y \) is not fixed but satisfies only \(|y| \leq \varepsilon^{-\alpha}\) does not alter the result. iii) There exists \( k_0 > 0 \) such that, on \( \{|y| \leq \varepsilon^{-1}\} \), we have

\[
\|v(x,y;z)\|_{L^2} \leq k_0, \quad \varepsilon^{-1}.
\]

According to lemma 6.7 ii and iii, on \( \{|y| \leq \varepsilon^{-1}\} \), we have

\[
\begin{aligned}
J_1' &\leq \mathbb{P}(\|b\|_2 \leq \varepsilon^{-1/2}) \exp C_3(k_0 + 1) \varepsilon^{d_1} \\
&\quad + \mathbb{E}[\|b\|_2 \geq \varepsilon^{-1/2}] \exp C_3 \varepsilon^{d_1} (1 + \text{cost } \varepsilon^{-1} + \varepsilon^{d_1/2} [\|b\|_2^2]) \\
&= O(1 + \exp(\text{cost } \varepsilon^{-1} - \text{cost } \varepsilon^{d_1})) = O(1).
\end{aligned}
\]

iv) We use lemma 6.7.iii to find a lower bound for \( D \).

As \( B_0, L_0, G_0, H_0, J_1, J_1' \) and \( J_2' \) where defined from \( Y^x \) and \( z \), we define \( B'_0, L'_0, \ldots \) from \( Y^z \) and \( z' \) (see proof of prop. 6.2). Then,

\[
g'(y,z) - g'(y',z') = \left(6.11\right)
\]

We define \( k'(y,y',z,z') \) by replacing \( B_0 \) and \( B'_0 \) by \( B_0 \cap B'_0 \) in the right-hand side member of (6.11). Then,

\[
\|g'(y,z) - g'(y',z') - k'(y,y',z,z')\|_{L^2} \leq \left|\mathbb{P}(\Omega \setminus B_0') \mathbb{P}(\Omega \setminus B'_0) \mathbb{E}1_{B_0} \mathbb{E}1_{B'_0} \exp 2(G_x + H_x)\right|^2
\]

(8.7) yields \( \mathbb{P}(\Omega \setminus B_0') = \mathbb{P}(\Omega \setminus B'_0) = o(1) \). Lemma 6.9.i, ii, iv and v, yield that \( E1_{B_0} \mathbb{E}1_{B'_0} \exp 2(G_x + H_x) \) and \( E1_{B'_0} \mathbb{E}1_{B'_0} \exp 2(G'_x + H'_x) \) are bounded. Therefore, \( g'(y,z) - g'(y',z') = k'(y,y',z,z') + o(1) \).

We now fix \( \mu \in [0, \inf(0,3\alpha/2 - 1)] \) and define \( g'(y,y',z,z') \) by replacing \( B_0 \) and \( B'_0 \) by \( B_0^\mu \) and \( B'_0^\mu \) in (6.11), where

\[
B_0^\mu = \left[\{y + \varepsilon^{-\mu}y = 0\} \cap \{\|y\|_{L^2} \leq \varepsilon^{-1}\} \cap \{\|y\|_{L^2} \leq \varepsilon^{-1}\} \cap B_0 \right] \cap \left[\{y' + \varepsilon^{-\mu}y' = 0\} \cap \{\|y'\|_{L^2} \leq \varepsilon^{-1}\} \cap B_0 \right].
\]
Lemma 6.4 yields $P(B_e \cap B_0^c \cap (\Omega \setminus B_0^f)) = o(1)$, and the same computation as before thus implies $k^v(y, y', z, z') = g^v(y, y', z, z') + o(1)$.

It is therefore enough to prove that $g^v(y, y', z, z') = o(1)$. Since $a_1, \ldots, a_n = a_1' \ldots a_n'$, we have lemma 6.9.ii and iv yield $[g^v(y, y', z, z')] \leq \text{cost } [g^v + g^v + g^v]$ where

$g^v = E_1 \frac{H}{H} - \frac{H}{H} \exp(G_v + H_z)$, $g^v = E_1 \frac{H}{H} \exp(G_v - \exp(G_v) \exp H_z$,

$g^v = E_1 \frac{H}{H} \exp(G_v - \exp(G_v) \exp H_z$, $g^v = E_1 \frac{H}{H} - J^v \exp(G_v + H_z^\prime)$.

We conclude by proving that, for $i = 1$ to $4$, $g^v = o(1)$.

i=1: We change slightly the proof of $E_1 \frac{H}{H} \exp(\mathcal{G}) - \frac{H}{H} \exp(\mathcal{G} + \mathcal{H}_z) = o(1)$ (proof of prop. 6.2.3) by using lemma 6.9.iii and iv.

ii=2: $|a - b|^3 - (a' + b')^3 | \leq 4 \sup |a, |a'|, |b|, |b'| |(a - a') + (b - b')| \exp(\mathcal{G} + \mathcal{H}_z)\exp H_z$ yields

$G(e^{a-1} z, z^{-1}) - G(0, y)^2 - G(e^{a-1} z, z^{-1}) - G(0, y)^2 = O(e^{3/2})$.

In the same way, on $H_z^2$,

$G(e^{a-1} z, z^{-1}) - G(e^{a-1} z, z^{-1}) - G(e^{a-1} z, z^{-1}) - G(e^{a-1} z, z^{-1}) = O(e^{3/2})$.

Therefore $\frac{1}{2} \exp(\mathcal{G} + \mathcal{H}_z) \leq e^2$ where $\mu > 0$. Hence

$g^v = E_1 \frac{H}{H} \exp(\mathcal{G} - \mathcal{G}_v) - \frac{H}{H} \exp(\mathcal{G} + \mathcal{H}_z) \leq \text{cost } e^{\mu} - 1$.

i=3: Since

$|H(t, x) - H(t, x')| \leq ||C||_\infty |G(t, x) - \mathcal{G}(t, x')| \leq m^{-1} ||C||_\infty |x - x'|$,

we have $\frac{1}{2} \exp(\mathcal{G} + \mathcal{H}_z) = O(e^2)$ and thus $g^v = O(\exp(e^2) - 1)$.

ii=1: Inequality $|D(t, x) - D(t, x')| \leq \text{cost } e^2 |x - x'|$ yields

$|J^v - J^v| \leq J^v \exp(\text{cost } e^2) \int_{-1}^{1} A^v_z(du) = 1$ where $A^v_z$ equals

$|V_0(G(z, 0, y) \exp(\text{cost } e^2) - V_0(G(z, 0, y) \exp(\text{cost } e^2))| \leq |V_0(G(z, 0, y) \exp(\text{cost } e^2))| \leq k(z) |J^v|$.

Let $k(z) = \text{exp(\text{cost } e^2(\text{cost } e^2)) - 1}$. Then

$1 \frac{1}{H} A^v_z \leq m^{-1} |x + e^2(\text{cost } e^2)) | \Rightarrow 1 \frac{1}{H} |J^v - J^v| \leq k(z) |J^v|$. Similarly, $|J^v - J^v| \leq k(z) |J^v|$, and

$g^v \leq \text{cost } k(z) \exp(G_v + H_z^\prime) = o(1)$.
7. END OF THE PROOF

7.1. THE LINEAR PROBLEM

By lemma 3.1, we have to study

\[ v_1^T = E_1 \left[ Z_T^T \leq 0, \| z^T \| \leq \rho \right] \exp \left[ \gamma^T Z_T \right] + K \{ g_i \} \]

Let \( \gamma > 0 \) small enough. Lemma 5.1 yields the existence of \( Q = (Q_0, Q_1, K_0, K_1) \) independent of \( Y = -g(T) \) and \( Q_2, K_2 \in \mathbb{R} \) such that

\[ Z_T = \varepsilon^T Q' \]

where \( Q' = (Q_0 + Q_1 Y + Q_2 y^2, K_0 = K(g)) \)

\[ P(\Omega, D_T) = O(\exp -\varepsilon^{-1}) \]

where \( D_A(A_1, \ldots, A_n) = \{ |A| \leq \varepsilon^{-1}, i = 1, \ldots, n \} \).

Lemma 7.1. We have \( v_1^T = v_0^T + o(1) \) where

\[ v_0^T = E_1 \{ D_A (Y, Q), -\varepsilon^T \leq Z_T^T \leq 0 \} \exp \left[ \varepsilon^T Z_T \right] + K \{ g_i \} \]

Proof. Let \( T = D_A(Y, Q, \| Y \|^2) \) and

\[ v_0^T = E_1 \left[ Z_T \leq 0 \right] \exp \left[ \varepsilon^T Z_T + K \{ g_i \} \right] \]

\[ v_0^T = E_1 \left[ Z_T \leq 0 \right] \exp \left[ \varepsilon^T Z_T + K \{ g_i \} \right] \]

Azencott ([1980-81], p. 270) proved that there exists \( C > 0 \) such that

\[ \rho^T \geq C \Rightarrow P(\| Y \|^2 \geq \rho^T, \| Y \|^2 \geq \tau) \leq \exp -C^{-1} \tau^2 / \tau \]

(7.1)

and \( P(\| Y \|^2 \geq \rho) = O(\exp -\varepsilon^{-1}) \) yield \( P(\Omega, D_T) = O(\exp -\varepsilon^{-1}) \).

Therefore, (3.10) implies \( v_0^T = v_0^T + O(\exp -\varepsilon^{-1}) \). Moreover

\[ |v_0^T - v_0^T| \leq E_1 \left[ Z_T \leq 0 \right] \exp \left[ \varepsilon^T Z_T + K \{ g_i \} \right] \]

\[ \leq E_1 \left[ Z_T \leq 0 \right] \exp \left[ \varepsilon^T Z_T + K \{ g_i \} \right] \]

\[ \leq \exp -E_1 \left[ D_A (Y, Q), \| Q' \|^2 \leq \varepsilon^T \right] \exp \left[ \varepsilon^T Z_T + K \{ g_i \} \right] \]

\[ \leq \exp -E_1 \left[ D_A (Y, Q), \| Q' \|^2 \leq \varepsilon^T \right] \exp \left[ \varepsilon^T Z_T + K \{ g_i \} \right] \]

Let \( \xi = \int_0^T \sigma^T g(s) \, ds \) and \( K_0 = K_2 - \frac{1}{2 \pi} \). This last expression equals

\[ E_1 D_A Q H_0 \exp K_0 \]

where

\[ H_0 (Q) = \frac{1}{2 \pi} \int_{-\infty}^{\infty} \xi \left[ \| Q' \|^2 \right] \exp \left[ K_0 \| Y \|^2 + K \| y \|^2 \right] \, dy \]
We need the following lemma whose proof is straightforward.

**Lemma 7.2.** On $D_0$, the function $f_0^p$ is one-to-one from $[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]$ to an interval $I_0^p$ which contains $[-1, 1]$. Its inverse function $\Phi_0^p$ satisfies

$$\forall u \in I_0^p, \quad |\Phi_0^p(u) + u| \leq 3 \varepsilon^{-\gamma} \quad \text{and} \quad \left| \frac{d\Phi_0^p}{du}(u) + 1 \right| \leq 4 \varepsilon^{-\gamma}.$$  

Therefore, on $D_0$, we have

$$H_1(Q) = -\frac{1}{\varepsilon^{2\gamma}} \int_{Q} \left( |u| \leq \varepsilon^{-\gamma} \right) \exp[K_1\Phi_0^p(u) + K_2\Phi_0^p(u)^2]du.$$  

Therefore, $1_{D_0}H_1(Q) = O(\varepsilon^{-\gamma})$ and $\nu^1_1 = \nu^2_1 + o(\varepsilon)$. Finally, the presence of $\exp -\gamma T_2$ and (3.10) yield $\nu^2_1 = \nu^5_1 + O(\exp -\varepsilon^{-\alpha})$.

Since $Q$ and $Y$ are independent, we have $\nu^5_1 = E[1_{D_0}H_1^*(Q)\exp K(y)]$ where

$$\sqrt{2\pi H_1^*(Q)} = \sqrt{\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \left( |u| \leq \varepsilon^{-\gamma} \right) |u|^{1/2}K_1Y + K_2Y^2|du}$$

$$= \frac{1}{\varepsilon^{2\gamma}} \int_{Q} \left( |u| \leq \varepsilon^{-\gamma} \right) \exp[K_1\Phi_0^p(u) + K_2\Phi_0^p(u)^2]du.$$  

Finally, joint dominated convergence in $\omega$ and $v$ ends the proof of th. 2.1:

$$\frac{\nu^5_1}{\varepsilon} \rightarrow \frac{1}{\sqrt{2\pi H_1^*(Q)}} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \exp[p(v)dv] E \exp K(y).$$

### 7.2. The Nonlinear Problem

In this section, the proofs of the lemmas are postponed to the end.

By lemmas 3.2 and 4.2, we have to study

$$\nu^5_1 = E[1_{\{x^2 < 0, \|Z\| \leq \rho\}} \exp[p(\gamma)T_2 + K(g_1 - \varepsilon^{-\gamma}T_2(T, V, \gamma) + Z)].$$

In order to apply prop. 6.1 and 6.2, we first condition with respect to $\sigma(x^2, \gamma) = T'(\gamma)$ where $T'(\gamma) = T - \varepsilon^{-\gamma}$ and $\epsilon[2/3, a]$. Let $\alpha \not\in \epsilon[2/3, a]$ and

$$\nu^5_1 = E[1_{\{x^2 < 0, \|Z\| \leq \varepsilon^{1+\gamma}\}} \exp[p(\gamma)T_2 + K(g_1)\gamma^{-1}(T, \gamma) + \varepsilon^{-\gamma}T_2(V, y)].$$

The conditioning yields

**Lemma 7.3.** $\nu^5_1 = \nu^6_1 + o(\varepsilon)$.

According to the strategy described in section 2.1, we introduce the following functional of $g_1$ and $g_2$

$$\nu^6_1 = E[I_D(V, Q), \quad \gamma^{-1} \leq T_2 \leq 0, \quad \|T_2\| \leq \varepsilon^{1+\gamma} \gamma^{-1} \exp[p(\gamma)T_2 + K(g_1)\gamma^{-1}(T, \gamma) + \varepsilon^{-\gamma}T_2(V, y)].$$

where the r.v. \( Q^x \) is defined as follows.  
By lemma 5.3, there exists \( Q_x = \{ Q_j^x, R_j^x, K_j^x, 0 \leq j+k \leq 1 \} \) independent of \( V_x = \{ Y, V_x \} = \{ -g_x(T), -g_x(T') \} \), and there exists \( \{ Q_j^x, R_j^x, K_j^x, j+k=2 \} \in \mathbb{R}^6 \) uniformly bounded in \( \varepsilon \) such that:

\[
Z_{\varepsilon} = \varepsilon M_0^x(Y, V_x) \text{ where } M_0^x(y, z) = -y + \varepsilon \sum_{u \leq j+k \leq 2} Q_j^x y^2
\]

\[
Z_{\varepsilon} = \varepsilon N_0^x(Y, V_x) \text{ where } N_0^x(y, z) = -z + \varepsilon \sum_{u \leq j+k \leq 2} R_j^x y^2
\]

\[
k_u = \sum_{u \leq j+k \leq 2} K_j^x y^4 \text{ where } K_u = K(\varepsilon u)
\]

We now have to study \( u \) since

**Lemma 7.4.** \( u = u + o(\varepsilon) \).

Since \( Q_x \) and \( V_x \) are independent, \( u = E_1 \ell D \), where

\[
H_1^x(Q_x) = \mu_x \int C_{\varepsilon} \left\{ -\varepsilon^{\varepsilon} \leq M_0^x(y, z) \leq 0, |N_0^x(y, z)| \leq \varepsilon^{\varepsilon} \right\}
\]

\[
\exp(-\varepsilon^{\varepsilon} M_0^x(y, z) + \sum_{u \leq j+k \leq 2} K_j^x y^2) g'(\varepsilon^{\varepsilon} M_0^x(y, z), N_0^x(y, z))
\]

\[
\exp(-\frac{\varepsilon^2}{2\mu_x}) \left\{ \frac{-y + \varepsilon y^2}{2\mu_x} \right\} dy dz
\]

\( C_{\varepsilon} = \{ (y, z) \ | \ |y|, |z| \leq r \} \), \( \xi = \int \xi^2(u) du, \mu_x = (2\pi)^{-1} |\varepsilon^2| \xi^2 \).

In order to compute this gaussian double integral, we use

**Lemma 7.5.** On \( D \), \( \{ M_2, N_2 \} \) is a \( C^\infty \)-diffeomorphism from a neighborhood of \( C_{\varepsilon} \) to a neighborhood of \( C(1) \). Let \( \lambda_u = \{ \Phi_u^2, \Psi_u^2 \} \) be its inverse function. Then, for all \( (u, v) \in C(1) \),

\[
\text{Jac} \lambda_u(1 + O(\varepsilon)) = 1 + O(\varepsilon) \text{ and } |\Psi_u^2(1, v) + v| + |\Phi_u^2(1, v) + v| = O(\varepsilon^2).
\]

Let \( \lambda_u = \sqrt{\xi^2_{\varepsilon}} \). A change of variables yields (see lemma 7.5)

\[
H_1^x(Q_x) = \mu_x \int \left\{ \left\{ -\varepsilon^{\varepsilon} \leq u \leq u, \ |v| \leq \varepsilon^{\varepsilon} \right\}
\]

\[
\exp(-\frac{u^2}{2\lambda_u^2}) + \sum_{u \leq j+k \leq 2} K_j^x \Phi_u^2(1, u) \Psi_u^2(1, u) g'(\frac{u}{\varepsilon^2}, v)
\]

\[
\exp(-\frac{\varepsilon^2}{2\mu_x}) \left\{ \frac{-u + \varepsilon u^2}{2\lambda_u^2} \right\} du dv
\]

\[
= \mu_x \lambda_u(1 + O(\varepsilon)) \int \left\{ \left\{ -\varepsilon^{\varepsilon} \leq u \leq u, \ |v| \leq \varepsilon^{\varepsilon} \right\}
\]

\[
\exp(-\frac{u^2}{2\lambda_u^2}) + \sum_{u \leq j+k \leq 2} K_j^x \Phi_u^2(xu, \lambda_u) \Psi_u^2(xu, \lambda_u) g'(u, \lambda_u v)
\]

\[
\exp(-\frac{-u + \varepsilon u^2}{2\lambda_u^2}) \left\{ \frac{-u + \varepsilon u^2}{2\lambda_u^2} \right\} du dv.
\]
The following lemma will allow us to end the proof of theorem 2.3.

**Lemma 7.6.** Let \( D_{j}^{p} = D_{n} \| g \| \cap \{ V - V_{4} \leq \varepsilon_{j}^{2/3} \} \). Then,

i) \( P(\Omega, D_{j}^{p}) = O(\exp(-\varepsilon_{j}^{-1/3})) \) and \( 1_{D_{j}^{p}} | K_{n}^{m} - K(\theta) | = O(\varepsilon_{j}^{-1}) \) a.s.

ii) There exists \( c_{j} > 0 \) such that \( (\exp K_{n}^{m})_{2} \leq c_{j} \) is uniformly integrable.

iii) \( \varepsilon_{j}^{-1} D_{j} H_{j}^{m}(Q_{k}) \to (2\pi)^{-1} \int_{-\infty}^{0} g(u) \exp pu \, du \) a.s.

iv) There exists \( M_{0} > 0 \) such that for all \( \varepsilon, \varepsilon_{j}^{-1} D_{j} H_{j}^{m}(Q_{k}) \leq M_{0} \) a.s. Therefore,

\[
\varepsilon_{j}^{-1} u_{j}^{*} = \varepsilon_{j}^{-1} E_{1,j} Q_{k} H_{j}^{m}(Q_{k}) \exp K_{n}^{m} = \varepsilon_{j}^{-1} E_{1,j} \exp\{ -\varepsilon_{j}^{-1} D_{j} H_{j}^{m}(Q_{k}) \} \exp K_{n}^{m} + O(\exp(-\varepsilon_{j}^{-1/3}))
\]

\[= (2\pi)^{-1} \int_{-\infty}^{0} g(u) \exp pu \, du \exp K(\theta) + o(1).\]

**Proof of Lemma 7.3.** (3.10) and (8.7) yield \( u_{j}^{*} = r_{j}^{*} + o(\varepsilon) \) where

\[ r_{j}^{*} = E_{1,j} \exp\{ -\varepsilon_{j}^{-1} D_{j} + K(\theta) \} = -\varepsilon_{j}^{-1} E_{j}(T(u), T, T, \varepsilon^{2}\Omega) \]

\[ A_{j} = \{ \Omega_{j}^{+} \leq 0, \| \Omega \| \leq \varepsilon_{j}^{-1} \}.\]

Before conditioning with respect to \( G_{j} = \sigma(\Omega_{j}^{+}, s \leq T(u); \Omega_{j}^{+}) \), we introduce a \( G_{j} \)-measurable r.v. close to \( \exp K(\theta) \). Integration by part in (2.6) yields

\[ K(\theta) = \int_{0}^{T} \int_{0}^{T} \psi(s)\theta(s)dg_{j} + \int_{0}^{T} \int_{0}^{T} \psi(s)g_{j}(s)dg_{j}(s) \]

with continuous \( \psi_{j} \) and \( \psi_{j}^{*} \). Therefore, \( K(\theta) = \alpha_{j} + \alpha_{j}^{*} \) where \( \alpha_{j} \) (resp. \( \alpha_{j}^{*} \)) corresponds to the integrals on \( [0, T(u)] \) (resp. \( [T(u), T] \)). Hence

\[ \varepsilon_{j}^{-1} D_{j} + K(\theta) = \varepsilon_{j}^{-1} D_{j}^{+} + \alpha_{j} + U_{j} \]

where \( U_{j} = -\varepsilon_{j}^{-1} D_{j}^{+} + \alpha_{j}^{*}. \]

(7.1) yields that, for \( \lambda \in ]-\infty, \alpha[ \),

\[ P(\| \Omega_{j}^{+} \| \leq \varepsilon_{j}^{-1}, |U_{j}| \geq \varepsilon_{j}^{-1} ) \leq \exp -\lambda |U_{j}| \exp -\varepsilon_{j}^{-1/3} \].

(7.2)

Besides, for \( \eta \) > 0 small enough, \( P(|U_{j}| \geq 2\varepsilon_{j}^{-1} \) is smaller than

\[ P(\Omega_{j}^{+} \leq \varepsilon_{j}^{-1}, |U_{j}| \geq \varepsilon_{j}^{-1} ) \leq \exp -\lambda |U_{j}| \exp -\varepsilon_{j}^{-1/3} \]

\[ + P(\| g(s) \| \geq \varepsilon_{j}^{2}, |U_{j}| \geq \varepsilon_{j}^{-1} ) \]

\[ \leq \varepsilon_{j}^{2} \exp -\lambda |U_{j}| + \varepsilon_{j}^{2} \exp -\varepsilon_{j}^{-1/3} \]

\[ \leq \exp -\varepsilon_{j}^{2} \exp -\varepsilon_{j}^{-1/3} \]

(7.3)

\[ [\lambda < \alpha \leq 1/3 \text{ and } d > 2/3], (7.2) \text{ and } (7.3) \text{ with } \lambda \in [0, \alpha) \text{ yield}

\[ P(\| \Omega_{j}^{+} \| \leq \varepsilon_{j}^{-1}, |U_{j}| \geq 3c_{j}^{-1} ) = O(\exp -\varepsilon_{j}^{-1/3}) \].

(7.4)
(7.3) with \( \lambda = \eta > \eta \) such that
\[
P(\|Z^r\| \leq \varepsilon^{-\eta}, |V_r| \geq 3\varepsilon^{-\eta}) = O(\exp^{-\varepsilon^{-\eta}}). \tag{7.5}
\]
(7.5) imply that, for all \( q \in \mathbb{R} \),
\[
\sup_{\varepsilon \leq \varepsilon_0} E_1 \{ \exp |Z^r| \leq \varepsilon^{-\eta} \} \exp q |Z^r| < \infty. \tag{7.6}
\]
Let \( r_2' = E_1 \exp \{ \exp -\varepsilon Z_T^r + \kappa \} = \exp \{ \exp \{ \exp (\varepsilon - \varepsilon Z_T^r + \kappa) \} \} \). Then,
\[
|Z_T^r - r_2'| \leq E_1 \exp \{ \exp -\varepsilon Z_T^r + K(g_j) \} \exp \{ q \varepsilon^2 \} - 1).
\]
Therefore, (3.10), (7.1) and (7.6) imply \( r_2' - r_2 = o(\varepsilon) \).

With the same techniques, if we replace in \( r_2' \) the event \( A_0 \) by \( \{ Z_T^r \leq 0, \|Z^r\| \leq \varepsilon^{-\eta}, \|Z_T^r\| \leq \varepsilon^{-\eta} \} \), then the difference is \( o(\varepsilon) \).

Finally, we consider \( Z_T^r \); the Markov property yields that \( r_2' \) equals
\[
E_1 \{ Z_T^r \leq 0, \|Z^r\| \leq \varepsilon^{-\eta}, \|Z_T^r\| \leq \varepsilon^{-\eta} \} \exp \{ \exp -\varepsilon Z_T^r + \kappa \} \].
\]
We conclude by proving \( r_2' = \omega_2 + o(\varepsilon) \) (same techniques). \( \Box \)

Proof of lemma 7.4. Let
\[
S_2 = E_0 \exp \{ \|Z^r\| \leq \varepsilon^{-\eta} \} \exp K(g_j).
\]
\[
S_2' = E_1 \exp \{ \|Z^r\| \leq \varepsilon^{-\eta}, \|Z_T^r\| \leq \varepsilon^{-\eta} \} \exp \{ \exp -\varepsilon Z_T^r + \varepsilon^{-1} Z_T^r \} \exp K(g_j).
\]
\[
S_2'' = E_1 \exp \{ \|Z^r\| \leq \varepsilon^{-\eta}, \|Z_T^r\| \leq \varepsilon^{-\eta} \} \exp \{ \exp -\varepsilon Z_T^r + \varepsilon^{-1} Z_T^r \} \exp K(g_j).
\]
(3.10) and (7.1) yield \( S_2 = S_2 + O(\exp -\varepsilon^{-\eta}) \). We have
\[
\begin{align*}
1D & \{ \|Z^r\| \leq \varepsilon^{-\eta}, \|Z_T^r\| \leq \varepsilon^{-\eta} \} \exp \{ \exp -\varepsilon Z_T^r \} \leq \exp \{ \exp -\varepsilon Z_T^r \} \\
1D & \{ \|Z^r\| \leq \varepsilon^{-\eta}, \|Z_T^r\| \leq \varepsilon^{-\eta} \} \exp \{ \exp -\varepsilon Z_T^r \} \exp K(g_j). \\
\end{align*}
\]
We divide this last expectation into two parts by introducing \( 1D_0 \) and \( 1D_0 \) (see section 7.1). The first part is \( o(\varepsilon) \) (see proof of lemma 7.2), and the second one is \( O(\exp -\varepsilon^{-\eta}) \). Hence \( S_2 = s_2 + o(\varepsilon) \).

Write \( s_2' = s_2 + s_2' \) where \( s_2' \) (resp. \( s_2' \)) corresponds to \( \{ \|Z^r\| \leq \varepsilon^{-\eta} \} \) (resp. \( \{ \|Z^r\| \leq 2\varepsilon^{-\eta} \} \)). Then \( s_2' = O(\exp -\varepsilon^{-\eta}) \). Moreover, (8.7) yields
\[
P(\|Z^r\| \leq 2\varepsilon^{-\eta}) \leq \exp \{ \exp -\varepsilon^{-\eta} \}. \tag{7.7}
\]
Since \( \{ D_0 \|Z^r\| \leq \varepsilon^{-\eta} \} \subseteq \{ \|Z^r\| \leq 2\varepsilon^{-\eta} \} \), inequality (7.7) yields
\[
s_2' = E_{1D_0} \{ \|Z^r\| \leq \varepsilon^{-\eta}, \|Z_T^r\| \leq \varepsilon^{-\eta}, \|Z_T^r\| \leq 0, \|Z_T^r\| \leq 2\varepsilon^{-\eta} \} \exp \{ \exp -\varepsilon Z_T^r \} \exp K(g_j) + o(\varepsilon).
\]
Define $s'_k$ by replacing $g'(\varepsilon z^T, \varepsilon^* z^T)$ by $g'_{/} \varepsilon^* z^T + K(y_z)$ in the previous expectation. According to proposition 6.2.2,

$$|s'_k - s'_k| \leq o(1) \{ \mathbb{E} 1 \{ X_T \leq 0, \| Z \| \leq \varepsilon' \} \exp[-\varepsilon z^T + K(y_z)] \}.$$ 

This last expectation is smaller than $\varepsilon'$ if $\varepsilon' \leq \varepsilon$. Therefore, theorem 2.1 yields $s'_k = s'_k + o(\varepsilon)$. Finally, easy computations imply $s'_k = s'_k + o(\varepsilon).$ 

Proof of lemma 7.5. On $D_0 Q_0$, we have $y \in (c(\varepsilon))$. According to the local inversion theorem, there exists $V^0_\varepsilon$, an open neighborhood of $C(c(\varepsilon))$, such that $\frac{d}{d} \Phi V^0_\varepsilon \to L^{2}_{2}(V^0_\varepsilon)$. We can choose $V^0_\varepsilon$ simply connected. Thus $L^{2}_{2}(V^0_\varepsilon)$ is also simply connected. Consider a closed path whose support is the boundary of $C(c(\varepsilon))$. Its image by $L^{2}_{2}(V^0_\varepsilon)$ is a closed path whose winding number with respect to $(0,0)$ is not zero and whose intersection with $C(1)$ is empty (easy check). Hence $L^{2}_{2}(V^0_\varepsilon)$ contains $C(1)$. The rest of the proof is straightforward. 

Proof of lemma 7.6. The first part is a consequence of (8.7). According to (5.2), we have

$$G^0_{2n}(t) = g^0(t) = \left( \frac{\xi_0}{\xi_0'} \right) Y - \frac{\xi_0}{\xi_0'} (Y - Y_0) \quad \text{if } t \leq T'(c),$$

and $\|G^0_{2n} - g^0(t)\| = O(\varepsilon^2)\|Y_0\| + \|Y - Y_0\| = O(\varepsilon^2)$ on $D_{0}'$.

(2.11) yields $K_{2n} = K(g^0(t)) = O(\|G^0_{2n}\| + \|g^0(t)\|)$. Therefore,

$$1_{\mathcal{D}_2} K_{2n} - K(g^0(t)) = O(\varepsilon^2 (\|G^0_{2n}\| + \|g^{0}\|))$$

and we conclude by using $\|G^0_{2n}\| + \|g^0\| = O(\|y_z\|)$. 

Proof of lemma 7.6. Let $\beta$ given by lemma 2.2. 

A conditioning on $\{K_{2n}, K_{2n}'(k_{2n}), k_{2n}'\}$ yields

$$E \exp(1+ \beta) K(y_z) = E \Lambda e^{\exp(1+ \beta)} k_{2n} < \infty$$

$$\Lambda = \mu_{s} \int \exp(1+ \beta)(\sum_{i \leq j \leq 2 \leq 2} K'_{2n} y^{j} z^{j}) \exp(-\frac{z^2}{2\xi_0^2} + \frac{(y - z)^2}{2\xi_0'^2}) dy dz.$$ 

Since the deterministic $K'_{2n}, K'_1, k_{2n}'$ are bounded uniformly in $\varepsilon$,

$$\sum_{j \leq 2} k'_{2n} y^{j} z^{j} \geq -c (z^2 + \|y_z\| + y^2) \geq -c \varepsilon (z^2 + \|y_z\|),$$

It yields
\[ \lambda_x \geq \mu_x \int \exp\left(1 + \beta\right) \left(K_{\alpha}(y + K_{\alpha}z) \exp\left(-ct \left(x - z\right)^2 - (y - z)^2 \left(\frac{\xi_y}{\xi_y(y)} \right)^{1 + \gamma}\right) dz. \]

If \( X \) is a one-dimensional gaussian centered r.v., then \( E \exp \lambda X \geq 1 \) for all \( \lambda \in \mathbb{R} \). Hence, there exists \( m_0 > 0 \) such that \( \inf \lambda_x > m_0 \) a.s., and \( \sup E \exp(1 + \beta) K_{\alpha} < \infty \).

**Proof of Lemma 7.6.iii** and **7.6.iv**. Let \( \nu \in [1, d/2 + 1 + \alpha/2 - d] \). Since \( |\nu| \leq \lambda^{1/2} - \nu \) we have \( \lambda_x = O(\nu^{-\alpha}) \) and prop. 6.1 yields \( g'(u, \lambda_x) \) tends to \( g(u) \). Assume \( \lambda^{1/2} - \nu \leq u \leq 1 \leq \lambda^{1/2} - \nu \). Lemma 7.5 yields
\[
\begin{align*}
1 & \sum_{i \leq j \leq 2} K_{\alpha} \Phi^2_{\alpha}(c(u, \lambda_x) - \Phi^2_{\alpha}(c(u, \lambda_x)) \leq \varepsilon^2 \\leq \frac{1}{2} \frac{\lambda^{1/2}}{2^{1/2}(\nu)} \\
\frac{\Phi^2_{\alpha}(c(u, \lambda_x) - \Phi^2_{\alpha}(c(u, \lambda_x))}{2^{1/2}(\nu)} & \rightarrow 0
\end{align*}
\]
and we can conclude easily.

**8. Appendix**

**Appendix 8.1**. There exists \( C > 0 \) such that, for all \( \varepsilon, t, x \)

\[ \left| \partial_t \phi \right| \left( c^2 t, c^2 x \right) \leq C \left( \varepsilon^{-1/2} + t^{1/2} |x| + t \right) \]  \hspace{1cm} (8.1)
\[ \left| \partial_x \phi \right| \left( c^2 t, c^2 x \right) \leq C \left( \varepsilon^{-1/2} + t^{1/2} |x| + t \right) \]  \hspace{1cm} (8.2)

**Proof**. A similar result is known if \( \sigma \) and \( c \) are constant (see Uchiyama (1978)), and we use the same line of proof. Let \( f(u) = 1_{0 < u < 1} (1 - r(u)) \).

Let \( p(x, y) \) be the transition density of diffusion \( dX_t = \sigma(X_t) dW_t \).

Then, the following problems 1 and 2 both have a unique solution, and they coincide.

**Problem 1**
\[ \partial_t u + \frac{1}{2} \sigma^2 \partial_{xx} u + c(x) f(u) \quad \text{if} \quad t > 0 \]
\[ \lim_{t \to 0} u(t, x) = 1_{[x \leq 0]} \quad \text{uniformly on every compact of} \ \mathbb{R}^+. \]
\[ u \in C^1 \left( [0, +\infty[ \times \mathbb{R}, [0, 1] \right) \]

**Problem 2**
\[ u(t, x) = \int_0^t \int_{-\infty}^{+\infty} p(x, y) dy + \int_0^t ds \int_{-\infty}^{+\infty} p(x, y) f(u(s, y)) dy \quad \text{if} \quad t > 0 \]
Consider a Brownian martingale $\{X_t\}_{t \geq 0}$ since $\int \sigma(x,y) \, dx$ and $\int \sigma(x,y) \, dy$ imply $\int \sigma(x,y) e^{x^2+y^2} \, dx \, dy$ (last equality is a consequence of integration by parts formula).

Let $f_t(x) = e(x^2) f(u^2(x,y))$ and $w^t(t,x) = \int_0^t \sigma f_t(x) \, ds$.

Then, since $A^t$ and $P_{s\rightarrow t}$ commute,

\[
A^t w^t(t,x) = \int_0^t A^t P_{s\rightarrow t} f_t(x) \, ds = \int_0^t P_{s\rightarrow t} A^t f_t(x) \, ds
\]

\[
= -\frac{1}{2} \int_0^t ds \int_0^s \partial_y \sigma P_{s\rightarrow t} f_t(x,y) e(x^2+y^2) \, dy,
\]

(last equality is a consequence of integration by parts formula).

Appendix 8.2. Consider a Brownian martingale $Z_t = x + \int_0^t A_s \, dW_s$, where $A$ satisfies $\|A\|_\infty \leq M$ a.s. ($M \in \mathbb{R}$). Then

\[
\forall \varepsilon > 0 \quad \mathbb{P}(Z_t \geq \varepsilon \mid Z_0 = x) \leq \exp \left(\frac{(x - \varepsilon)^2}{2M^2}\right) + \exp \left(\frac{(x + \varepsilon)^2}{2M^2}\right)
\]

\[
\lim_{\varepsilon \to 0} \sup_{t \geq 0} \frac{Z_t}{\mathbb{P}(Z_t \geq \varepsilon)} = \frac{1 + M^2}{2} \quad \text{a.s.}
\]

Proof. \([8.7]\) is a classical consequence of Doob inequality. Concerning \([8.8]\), we copy the proof of the law of iterated logarithm for the Brownian motion (see for instance Revuz and Yor (1991)).
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