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# CONVERGENCE OF SUBMARTINGALES TO AN INCREASING PROCESS UNDER DISCRETIZATION OF FILTRATIONS

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## I. Introduction.

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbf{P})$  be a filtered probability space, where the filtration  $(\mathcal{F}_t) = (\mathcal{F}_t^Y)$  is generated by a càdlàg (right continuous and admitting left limits) process  $Y = (Y_t, t \in [0, T])$ . Note that, in general,  $(\mathcal{F}_t)$  is not right-continuous. Let

$$\pi_n = \{0 = t_0^n < t_1^n < \dots < t_{k_n}^n = T\}, \quad n \in \mathbb{N},$$

be a sequence of refining partitions of an interval  $[0, T]$  such that  $|\pi_n| := \max_i |t_i^n - t_{i-1}^n| \rightarrow 0$ ,  $n \rightarrow \infty$ . Denote  $\mathcal{F}_t^n = \sigma(Y_s^n, s \leq t)$ , where

$$Y_t^n := Y_{t_i^n} \quad \text{for } t \in [t_i^n, t_{i+1}^n), \quad Y_T^n := Y_{t_{k_n}^n}.$$

We suppose that the set of fixed times of discontinuity of  $Y$  is included in the union of  $\pi_n$ .

Given an integrable random variable  $X$  and a sequence of random variables  $X^n$ ,  $n \in \mathbb{N}$ , converging to  $X$  in  $L^1(\mathbf{P})$ , consider the martingale  $M = (M_t = \mathbf{E}(X | \mathcal{F}_t), t \in [0, T])$ , and the sequence of martingales  $M^n = (M_t^n = \mathbf{E}(X^n | \mathcal{F}_t^n), t \in [0, T])$ ,  $n \in \mathbb{N}$ , with respect to the perturbed filtrations  $(\mathcal{F}_t^n)_{t \in [0, T]}$ ,  $n \in \mathbb{N}$ . Since  $\mathcal{F}_t^n \uparrow \mathcal{F}_t$ ,  $n \rightarrow \infty$ , for each  $t \in [0, T]$ , we have that  $M_t^n \rightarrow M_t$  in probability.

In paper [2] it was proved that : 1) In general, the convergence  $M^n \rightarrow M^+$  for the Skorokhod topology can fail; see the example of Sect. 2 in [2].

2) If  $Y$  is a Markov process (not necessarily continuous), then  $M^n \rightarrow M^+$  in probability for the Skorokhod topology, ([2] Theorem 1).

A more general problem is the following : Suppose that  $X$  is a  $\mathcal{F}$  adapted càdlàg process, and consider  $X^n$  the càdlàg version of processes  $\mathbf{E}[X | \mathcal{F}_t^n]$  : i. e.  $X^n$  is the  $\mathcal{F}^n$ -optional projection of  $X$  (see [4], VI-43 and VI-47). The same example in [2] shows that in general we have not the convergence of  $X^n$  to  $X$  in probability for the  $J_1$  topology. This problem was studied in paper [3], under a general assumption of weak convergence of filtrations ([3] Theorems 1, 2 and 3), generalizing the situation here, when process  $Y$  is Markov.

We shall prove in this small note that, without any condition on process  $Y$ , we get the desired convergence when  $X$  is a continuous increasing process.

For shortening notations, the filtrations  $(\mathcal{F}_t^n)$ ,  $(\mathcal{F}_t)$  will be denoted  $\mathcal{F}^n$ ,  $\mathcal{F}$ .

**Theorem.** Let  $X$  is a  $\mathcal{F}$ -adapted, continuous, increasing process. We assume that  $X_T$  is square integrable, and  $X_0 = 0$ . Let us denote  $X^n = \mathbf{E}[X \cdot | \mathcal{F}^n]$ . Then :

a)  $X^n$  is a positive submartingale, with the canonical decomposition  $X^n = M^n + A^n$ , where  $M^n$  is a square integrable  $\mathcal{F}^n$ -martingale of pure jumps, and  $A^n$  is a continuous increasing  $\mathcal{F}^n$ -adapted process, such that  $A_T^n$  is square integrable.

b) We have the convergences for the uniform topology in  $\mathbb{D}$  :  $X^n \xrightarrow{\mathbf{P}} X$ ,  $M^n \xrightarrow{\mathbf{P}} 0$  and  $A^n \xrightarrow{\mathbf{P}} X$ .

*Proof* It will be driven in several steps.

1) We have immediately, for every  $s$  and  $t$ , with  $s \leq t$  :

$$\mathbf{E}[X_t^n - X_s^n | \mathcal{F}_s^n] = \mathbf{E}[X_t - X_s | \mathcal{F}_s^n] \geq 0$$

hence the property of  $\mathcal{F}^n$ -submartingale for  $X^n$ .

Let us consider now for every  $n$ , the jump process

$$M^n = \sum_{i=1}^{k_n} (\mathbf{E}[X_{t_i^n} | \mathcal{F}_{t_i^n}^n] - \mathbf{E}[X_{t_i^n} | \mathcal{F}_{t_{i-1}^n}^n]) 1_{\{ \cdot \geq t_i^n \}}.$$

One can see that  $M^n$  is a square integrable martingale of jumps and that  $X^n - M^n$  is a continuous process because the times of jumps of  $X^n$  belong to the set of elements  $t_i^n$  of partition  $\pi^n$  and that  $X^n$  and  $M^n$  have the same jumps ; finally between 2 successive  $t_i^n$ ,  $A^n$  is increasing, hence the canonical decomposition given in a).

2) Now we show that the sequence  $(M^n)$  is tight and that every limit is a law of continuous martingale.

Let us use the Aldous criterion for tightness ; we are given  $\delta > 0$ ,  $n$ , and  $\sigma, \tau$  stopping times of filtration  $(\mathcal{F}^n)$  such that  $\sigma \leq \tau \leq \sigma + \delta$ , and let  $\varepsilon' > 0$ . As previously, the elements of partition  $\pi^n$  are denoted  $t_i^n$  or  $t_k^n$  for  $i, k \leq k_n$ .

$$\begin{aligned} \mathbf{P}[|M_\tau^n - M_\sigma^n| > \alpha] &\leq \frac{1}{\alpha^2} \mathbf{E}[(M_\tau^n - M_\sigma^n)^2] \\ &\leq \frac{1}{\alpha^2} \mathbf{E}\left[ \sum_{\sigma \leq s \leq \tau} (\Delta M_s^n)^2 \right]. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{E}\left[ \sum_{\sigma \leq s \leq \tau} (\Delta M_s^n)^2 \right] &= \mathbf{E}\left[ \sum_{\sigma \leq t_i^n \leq \tau} (\Delta M_{t_i^n}^n)^2 \right] \\ &= \mathbf{E}\left[ \sum_{\sigma \leq t_i^n \leq \tau} (\mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_i^n}^n] - \mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_{i-1}^n}^n])^2 \right] \\ &\leq \sum_{k=0}^{k_n} \mathbf{E}[1_{[t_k^n, t_{k+1}^n]}(\sigma) \sum_{\{i > k; t_i^n - t_k^n \leq \delta\}} (\mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_i^n}^n] - \mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_{i-1}^n}^n])^2] \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=0}^{k_n} \mathbf{E}[1_{[t_k^n, t_{k+1}^n]}(\sigma) \sum_{\{i>k; t_i^n - t_k^n \leq \delta\}} ((\mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_i^n}^n])^2 - (\mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_{i-1}^n}^n])^2)] \\
&\leq \sum_{k=0}^{k_n} \mathbf{E}[1_{[t_k^n, t_{k+1}^n]}(\sigma) \sum_{\{i>k; t_i^n - t_k^n \leq \delta\}} ((\mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_i^n}^n])^2 - (\mathbf{E}[A_{t_{i-1}^n} | \mathcal{F}_{t_{i-1}^n}^n])^2)] \\
&\leq \sum_{k=0}^{k_n} \mathbf{E}[1_{[t_k^n, t_{k+1}^n]}(\sigma) (A_{t_{k+1}^n + \delta}^2 - (A_{t_k^n}^n)^2)] \\
&\leq \sum_{p=1}^{q(\delta)} \mathbf{E}[1_{[s_p, s_{p+1}]}(\sigma) (A_{s_{p+1} + 2\delta}^2 - (A_{s_p - \delta}^n)^2)]
\end{aligned}$$

(where  $s_p$  with  $0 \leq p \leq q(\delta)$  are the points of a subdivision of interval  $[0, T]$  whose mesh is lower than  $\delta$ ),

$$\leq \sum_{p=1}^{q(\delta)} \mathbf{E}[1_{[s_p, s_{p+1}]}(\sigma) (A_{s_{p+1} + 2\delta}^2 - A_{s_p}^2)] + \varepsilon'$$

(as soon as  $n$  is large enough)

$$\leq \mathbf{E}[A_{\sigma + 2\delta}^2 - A_{\sigma - \delta}^2] + \varepsilon' \leq 2\varepsilon'$$

for  $\delta$  small enough.

Hence we have the following, which proves tightness of  $(M^n)$  :

For every  $\varepsilon > 0$ , for every  $\alpha > 0$ , there exists  $\delta_0$ , such that for every  $\delta \leq \delta_0$

$$\limsup_n \sup_{\{(\sigma, \tau); \sigma \leq \tau \leq \sigma + \delta\}} \mathbf{P}[|M_\tau^n - M_\sigma^n| > \alpha] \leq \varepsilon.$$

Writing now for every  $\mathcal{F}^n$ -stopping time  $\sigma$

$$\mathbf{P}[|\Delta M_\sigma^n| > \alpha] \leq \frac{1}{\alpha^2} \mathbf{E}[(\Delta M_\sigma^n)^2].$$

Then, exactly as above:

$$\begin{aligned}
\mathbf{E}[(\Delta M_\sigma^n)^2] &= \mathbf{E}\left[\sum_{\sigma=t_k^n} (\Delta M_{t_i^n}^n)^2\right] \\
&= \mathbf{E}\left[\sum_{\sigma=t_k^n} (\mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_i^n}^n] - \mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_{i-1}^n}^n])^2\right] \\
&\leq \sum_{k=0}^{k_n} \mathbf{E}[1_{\{t_k^n\}}(\sigma) (\mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_i^n}^n] - \mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_{i-1}^n}^n])^2].
\end{aligned}$$

Using the same tools we get finally, considering  $\sigma = \inf\{t : |\Delta M_t^n| > \varepsilon\}$  : for every  $\varepsilon$ ,

$$\mathbf{P}[\sup_{t \leq T} |\Delta M_t^n| > \varepsilon] \xrightarrow{\mathbf{P}} 0$$

hence the desired result ([5], chapt.VI).

3) Last step. The sequence  $((X, M^n, Y))$  is tight in  $\mathbf{C} \times \mathbb{D}^2$ . By representation Theorem of Skorokhod, we can find, on a suitable space  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{P}})$  a sequence  $((\bar{X}^n, \bar{M}^n, \bar{Y}^n))$  relatively compact for the convergence in probability, and for a subsequence (indexed also by  $n$ ), we have :

$$(\bar{X}^n, \bar{M}^n, \bar{Y}^n) \xrightarrow{\mathbf{P}} (\bar{X}, \bar{M}, \bar{Y})$$

where  $\mathcal{L}((\bar{X}^n, \bar{M}^n, \bar{Y}^n)|) = \mathcal{L}((X, M^n, Y)|\mathbf{P})$  and  $\mathcal{L}((\bar{X}, \bar{Y})|) = \mathcal{L}((X, Y)|\mathbf{P})$ .

Let us consider  $\hat{Y}^n$  the step process of order  $n$  of  $\bar{Y}^n$ , and denote  $\hat{X}^n = \bar{\mathbf{E}}[\bar{X}^n | \mathcal{F}^{\hat{Y}^n}]$ ; we have :

$$\mathcal{L}((\bar{X}^n, \hat{X}^n, \bar{M}^n, \bar{Y}^n, \hat{Y}^n)|) = \mathcal{L}((X, X^n, M^n, Y, Y^n)|\mathbf{P}).$$

Let us denote  $\tilde{A}^n = \hat{X}^n - \bar{M}^n$ ; then  $\mathcal{L}((\tilde{A}^n, \bar{M}^n)|) = \mathcal{L}((A^n, M^n)|\mathbf{P})$ .

We have (for almost all  $t$ ) the convergence  $\hat{X}_t^n \rightarrow \bar{X}_t$  in  $L^1$ .

Actually,

$$\bar{\mathbf{E}}[|\hat{X}_t^n - \bar{X}_t|] \leq \bar{\mathbf{E}}[|\hat{X}_t^n - \bar{X}_t^n|] + \bar{\mathbf{E}}[|\bar{X}_t^n - \bar{X}_t|].$$

We have  $\bar{\mathbf{E}}[|\hat{X}_t^n - \bar{X}_t^n|] = \mathbf{E}[|\mathbf{E}[X | \mathcal{F}_t^n] - X_t|]$  and this expression converges to 0 for  $n \rightarrow \infty$ .

On the other hand,  $\bar{X}_t^n \xrightarrow{\mathbf{P}} \bar{X}_t$  and the sequence  $(\bar{X}_t^n)$  is bounded in  $L^2$ .

Finally we get that, for almost every  $t$ ,  $\tilde{A}_t^n \xrightarrow{\mathbf{P}} \bar{X}_t - \bar{M}_t$ .  $\bar{X} - \bar{M}$  is then a continuous increasing process  $\tilde{A}$ , the convergence is then uniform in  $t$ ; we deduce that  $\bar{M}$  which is a continuous martingale and also difference of two increasing processes is 0. The proof is complete.

**Remark** One can deduce the infinitesimality of jumps of  $M^n$ , from the Aldous work [1]. Since  $(X^n)$  is a sequence of  $\mathcal{F}^n$ -submartingale, that  $(X_t^n)$  is uniformly integrable and  $X$  continuous, we get : for every  $\varepsilon$ ,  $\limsup_n \mathbf{P}[\sup_{s \leq T} (X_s^n - X_s) > \varepsilon] = 0$ . We deduce immediately  $\limsup_n \mathbf{P}[\sup_{s \leq T} \Delta M_s^n > 2\varepsilon] = 0$ , and under the martingality of  $M^n$ ,  $\limsup_n \mathbf{P}[\sup_{s \leq T} |\Delta M_s^n| > \varepsilon] = 0$ . This last deduction holds because of predictable character of jumps of  $M^n$ , it would be wrong if  $M^n$  was, for example, quasileftcontinuous.

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