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ON THE INTEGRABILITY OF THE LIMIT OF A SUPERCRITICAL BRANCHING PROCESS

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Let Z_n be a Galton-Watson process with $Z_0=1$ and $1 < m = \mathbb{E}Z_1 < \infty$ and put $W = \lim_{n \rightarrow \infty} Z_n / m^n$. It is well-known that for all $\beta \geq 0$,

$$0 < \mathbb{E}W(\log^+ W)^\beta < \infty \text{ if and only if } \mathbb{E}Z_1(\log^+ Z_1)^{\beta+1} < \infty.$$

Inspired by this, Athreya and Ney (1972, p.63) conjectured that for all $\alpha > 1$ and $\beta > 0$, $0 < \mathbb{E}W^\alpha(\log^+ W)^\beta < \infty$ if and only if $\mathbb{E}Z_1^\alpha(\log^+ Z_1)^{\beta+1} < \infty$. Quite curiously, this is not the case, as is shown in the following

Theorem. (a) If $\alpha > 1$ is not an integer and $\beta \geq 0$, then

$$0 < \mathbb{E}W^\alpha(\log^+ W)^\beta < \infty \text{ if and only if } \mathbb{E}Z_1^\alpha(\log^+ Z_1)^\beta < \infty.$$

(b) If $\alpha \geq 1$ is an integer and $\beta \geq 0$, then we have the following implications:

$$\mathbb{E}Z_1^\alpha(\log^+ Z_1)^{\beta+1} < \infty \Rightarrow 0 < \mathbb{E}W^\alpha(\log^+ W)^\beta < \infty \Rightarrow \mathbb{E}Z_1^\alpha(\log^+ Z_1)^\beta < \infty.$$

We can prove this theorem in a similar way as Asmussen and Hering (1983) for a proof of the Kesten-Stigum theorem. Similar arguments were used in Wen (1986, section 4). We shall need the following generalization of a result of Asmussen and Hering (1983, P.41), whose proof is postponed to the end of the note.

Lemma. Let $S_n = \gamma_1 + \gamma_2 + \dots + \gamma_n$ be the sum of independent and identically distributed random variables $\gamma_i \geq 0$. If $\phi(x) \geq 0$ is a non-decreasing concave function on $[0, \infty)$ with $\phi(0) = 0$, then for all $k \geq 1$,

$$\mathbb{E}S_n^k \phi(S_n) \leq (\mathbb{E}S_n)^k \phi(S_n) + c_k n^{k-1} (\mathbb{E}\gamma_1) \phi(\mathbb{E}S_n) + 2k n^k \mathbb{E}\gamma_1^k \phi(\gamma_1), \quad (1)$$

where $c_k = \frac{1}{2}(k-1)k$.

Proof of Theorem. Using the Lemma above for the sum

$$\frac{Z_{n+1}}{m^{n+1}} = \sum_{i=1}^{Z_n} \frac{X_{n,i}}{m^{n+1}}, \quad (2)$$

where given $\mathbb{F}_n = \sigma(Z_1, \dots, Z_n)$, $\{X_{n,i}\}$ are independent copies of Z_1 , we obtain, for all $k \geq 1$,

$$\mathbb{E}[W_{n+1}^k \phi(W_{n+1}) | \mathbb{F}_n] \leq W_n^k \phi(W_n) + c_k W_n^{k-1} \phi(W_n) \frac{\mathbb{E}Z_1^k}{m^{n+k}} + 2k W_n^k \mathbb{E} \left[\frac{Z_1^k}{m^k} \phi\left(\frac{Z_1}{m^{n+1}}\right) \right],$$

where $W_n = Z_n / m^n$, and c_k and ϕ are as in the lemma. Therefore

$$\begin{aligned} \sup_{n \geq 0} \mathbb{E} W_{n+1}^k \phi(W_{n+1}) &\leq \phi(0) + c_k \sum_{n=0}^{\infty} \frac{\mathbb{E} Z_1^k}{m^{n+k}} \sup_{n \geq 0} \mathbb{E} W_n^{k-1} \phi(W_n) \\ &\quad + 2km^{-k} \mathbb{E} \left[Z_1^k \sum_{n=0}^{\infty} \phi\left(\frac{Z_1}{m^{n+1}}\right) \right] \sup_{n \geq 0} \mathbb{E} W_n^k. \end{aligned} \quad (3)$$

If we choose $\phi(x)=x$, an induction on k shows the well-known result that $\mathbb{E} Z_1^k < \infty$ implies $\mathbb{E} W^k < \infty$. In the general case, let us take $k=[\alpha]$ (the integral part of α) and define

$$\phi(x) = \begin{cases} Ax & \text{if } 0 \leq x \leq x_0 \\ x^{\alpha-[\alpha]} \log^\beta x + B & \text{if } x > x_0, \end{cases} \quad (4)$$

where $x_0 > 1$ is chosen so large that for all $x \geq x_0$, $\frac{d^2}{dx^2} \{x^{\alpha-[\alpha]} \log^\beta x\} < 0$, $A = \frac{d}{dx} \{x^{\alpha-[\alpha]} \log^\beta x\} \big|_{x=x_0} > 0$ and $B = c_1 x_0 - x_0^{\alpha-[\alpha]} \log^\beta x_0 \geq 0$.

Then ϕ satisfies the conditions of the Lemma. We claim that

$$\sum_{n=0}^{\infty} \phi\left(\frac{x}{m^{n+1}}\right) = O(x^{\alpha-[\alpha]} (\log^+ x)^{\beta+1}), \quad (5)$$

which can be improved as

$$\sum_{n=0}^{\infty} \phi\left(\frac{x}{m^{n+1}}\right) = O(x^{\alpha-[\alpha]} (\log^+ x)^\beta) \quad (6)$$

if $\alpha-[\alpha] > 0$. [Here $f(x) = O(g(x))$ means that for some constant $c > 0$ and all large $x > 0$, $|f(x)| \leq cg(x)$.] In fact, for all $x > x_0$, choosing an integer $k \geq 0$ such that $x/m^{k+1} \leq x_0 < x/m^k$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \phi\left(\frac{x}{m^{n+1}}\right) &= c_1 \sum_{n=k+1}^{\infty} \frac{x}{m^{n+1}} + \sum_{n=0}^k \left[\left(\frac{x}{m^{n+1}}\right)^{\alpha-[\alpha]} \log^\beta \frac{x}{m^{n+1}} + c_2 \right] \\ &\leq c x_0 + x^{\alpha-[\alpha]} \sum_{n=0}^k m^{-(n+1)(\alpha-[\alpha])} \log^\beta \frac{x}{m^{n+1}} + c_2 k, \end{aligned}$$

for some constant $c > 0$. Since $k < \log(x/x_0)/\log m$, this gives (5) and (6). By (3), (5) and (6), an induction argument on $k \geq 1$ shows immediately the sufficient parts of both (a) and (b). It remains to prove that for all $\alpha \geq 1$ and $\beta \geq 0$, if $0 < \mathbb{E} W^\alpha (\log^+ W)^\beta < \infty$ then $\mathbb{E} Z_1^\alpha (\log^+ Z_1)^\beta < \infty$. Since the function $f(x) = x^\alpha \log^\beta x + c$ is nonnegative and convex on $(0, \infty)$ if $c > 0$ is sufficiently large, Jensen's inequality gives

$$\begin{aligned} \mathbb{E}(W^\alpha \log^\beta W + c) &= \mathbb{E}[\mathbb{E}(W^\alpha \log^\beta W + c \mid \mathcal{F}_1)] \\ &\geq \mathbb{E}[(\mathbb{E} W \mid \mathcal{F}_1)^\alpha \log^\beta \mathbb{E}(W \mid \mathcal{F}_1) + c] = \mathbb{E}(Z_1/m)^\alpha \log^\beta (Z_1/m) + c. \end{aligned}$$

Therefore $\mathbb{E} W^\alpha (\log^+ W)^\beta < \infty$ implies $\mathbb{E} Z_1^\alpha (\log^+ Z_1)^\beta < \infty$. ■

It remains to prove the Lemma.

Proof of Lemma. Since ϕ is concave with $\phi(0)=0$, ϕ is subadditive. Thus

$$\begin{aligned}
 \mathbb{E} S_n^k \phi(S_n) &= \mathbb{E} \sum_{i_1, \dots, i_k=1}^n \gamma_{i_1} \dots \gamma_{i_k} \phi(S_n) \\
 &\leq \mathbb{E} \sum_{i_1, \dots, i_k=1}^n \gamma_{i_1} \dots \gamma_{i_k} \left[\phi\left(\sum_{j \notin \{i_1, \dots, i_k\}} \gamma_j\right) + \phi\left(\sum_{j \in \{i_1, \dots, i_k\}} \gamma_j\right) \right] \\
 &= \sum_{i_1, \dots, i_k=1}^n \mathbb{E} \gamma_{i_1} \dots \gamma_{i_k} \mathbb{E} \phi\left(\sum_{j \notin \{i_1, \dots, i_k\}} \gamma_j\right) \\
 &\quad + \mathbb{E} \sum_{i_1, \dots, i_k=1}^n \gamma_{i_1} \dots \gamma_{i_k} \left[\phi\left(\sum_{j \in \{i_1, \dots, i_k\}} \gamma_j\right) \right] \\
 &\leq \sum_{i_1, \dots, i_k=1}^n \mathbb{E} \gamma_{i_1} \dots \gamma_{i_k} \mathbb{E} \phi(S_n) \\
 &\quad + \mathbb{E} \sum_{i_1, \dots, i_k=1}^n \gamma_{i_1} \dots \gamma_{i_k} \left[\sum_{j \in \{i_1, \dots, i_k\}} \phi(\gamma_j) \right]. \tag{7}
 \end{aligned}$$

We now estimate the last two sums. For the first, we write

$$\sum_{i_1, \dots, i_k=1}^n \mathbb{E} \gamma_{i_1} \dots \gamma_{i_k} = \sum_{(i_1, \dots, i_k) \in D_k} + \sum_{(i_1, \dots, i_k) \in D_k^c}, \tag{8}$$

where $D_k := \{(i_1, \dots, i_k) : i_m \neq i_n \text{ if } m \neq n\} \subseteq \{1, \dots, n\}^k$, and D_k^c is the complement of D in $\{1, \dots, n\}^k$. We claim that

$$\text{card } D_k^c \leq \frac{1}{2}(k-1)kn^{k-1}. \tag{9}$$

To see this, put $D_{k,1}^c := D_k^c \cap \{i_1=1\}$ and divide it into two parts. The first part is of the form $(1, i_2, \dots, i_k)$ with $i_m=1$ for some $2 \leq m \leq k$, whose cardinality $\leq (k-1)n^{k-2}$. The second part contains the elements $(1, i_2, \dots, i_k)$ with $i_m \neq i_n$ if $m \neq n$; this part has cardinality $\leq \text{card } D_{k-1}^c$. Hence

$$\text{card } D_{k,1}^c \leq (k-1)n^{k-2} + \text{card } D_{k-1}^c,$$

and consequently

$$\text{card } D_k^c \leq (k-1)n^{k-1} + n \text{card } D_{k-1}^c.$$

This gives (9) by induction on k . Since

$$\mathbb{E} \gamma_{i_1} \dots \gamma_{i_k} \leq (\mathbb{E} \gamma_{i_1}^k)^{1/k} \dots (\mathbb{E} \gamma_{i_k}^k)^{1/k} = \mathbb{E} \gamma_1^k,$$

by the generalized Höld's inequality, and $\mathbb{E} \gamma_{i_1} \dots \gamma_{i_k} = \mathbb{E} \gamma_{i_1} \dots \mathbb{E} \gamma_{i_k}$ if

$(i_1, \dots, i_k) \in D_k$, (8) and (9) give

$$\begin{aligned} \sum_{1 \leq i_1, \dots, i_k \leq n} \mathbb{E} \gamma_{i_1} \dots \gamma_{i_k} &\leq \sum_{(i_1, \dots, i_k) \in D_k} \mathbb{E} \gamma_{i_1} \dots \mathbb{E} \gamma_{i_k} + \frac{1}{2}(k-1)k n^{k-1} \mathbb{E} \gamma_1^k \\ &\leq (\mathbb{E} S_n)^k + \frac{1}{2}(k-1)k n^{k-1} \mathbb{E} \gamma_1^k \end{aligned} \quad (10)$$

We now estimate the second sum in (7). Since ϕ is increasing,

$$(\gamma_i^k - \gamma_j^k)[\phi(\gamma_i^k) - \phi(\gamma_j^k)] \geq 0;$$

so

$$\gamma_i^k \phi(\gamma_j^k) + \gamma_j^k \phi(\gamma_i^k) \leq \gamma_i^k \phi(\gamma_i^k) + \gamma_j^k \phi(\gamma_j^k),$$

and consequently

$$\mathbb{E} \gamma_i^k \phi(\gamma_j^k) \leq 2 \mathbb{E} \gamma_1^k \phi(\gamma_1^k).$$

Therefore,

$$\begin{aligned} \mathbb{E} \gamma_{i_1} \dots \gamma_{i_k} \phi(\gamma_j) &= \mathbb{E} \gamma_{i_1} \phi^{1/k}(\gamma_j) \dots \gamma_{i_k} \phi^{1/k}(\gamma_j) \\ &\leq [\mathbb{E} \gamma_{i_1}^k \phi(\gamma_j)]^{1/k} \dots [\mathbb{E} \gamma_{i_k}^k \phi(\gamma_j)]^{1/k} \leq 2 [\mathbb{E} \gamma_1^k \phi(\gamma_1)]. \end{aligned} \quad (11)$$

The conclusion then follows from (7), (10) and (11). ■

Remark. The result and the proof of the Theorem can obviously be generalized.

For example, if $\alpha > 1$ is not an integer and

$$\ell(x) = c (\log^+ x)^{\alpha_1} (\log_2^+ x)^{\alpha_2} \dots (\log_k^+ x)^{\alpha_k}$$

$(\alpha_i \in \mathbb{R})$, where $\log_1^+ x := \log^+ x$, $\log_k^+ x := \log^+ \log_{k-1}^+ x$ if $k > 1$, and the first non-vanishing α is positive, then

$$\mathbb{E} W^\alpha \ell(W) < \infty \text{ if and only if } \mathbb{E} Z_1^\alpha \ell(Z_1) < \infty.$$

Note sur épreuves. -Depuis l'achèvement de cette note, l'auteur s'est rendu compte que N.H.Bingham et R.A.Doney [1974: Asymptotic properties of supercritical branching processes. Adv.Appl.Prob., 6, 711-731] ont obtenu des résultats plus fins par des théorèmes Tauberiens et avec des calculs relativement compliqués; en particulier ils ont montré l'équivalence entre $0 < \mathbb{E} W^\alpha (\log^+ W)^\beta < \infty$ et $\mathbb{E} Z_1^\alpha (\log^+ Z_1)^\beta < \infty$ pour tout $\alpha > 1$ (entier ou non) et $\beta \geq 0$. Cependant, la simplicité de notre nouvelle preuve nous a paru garder un intérêt.

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