MARC PEIGNÉ

Iterated Function Systems and Spectral Decomposition of the Associated Markov Operator

Publications de l'Institut de recherche mathématiques de Rennes, 1993, fascicule 2
« Fascicule de probabilités », , p. 1-28
<http://www.numdam.org/item?id=PSMIR_1993___2_A9_0>

Iterated function systems and spectral decomposition of the associated Markov operator

PEIGNÉ Marc
I.R.M.A.R.
Université de Rennes I
Campus de Beaulieu
35042 RENNES Cedex

Abstract : We consider a discrete-Markov chain on a locally compact metric space (X, d) obtained by randomly iterating maps $T_i, i \in \mathbb{N}$, such that the probability $p_i(x)$ of choosing a map T_i at each step depends on the current position x. Under suitable hypotheses on the T_i's and the p_i's, it is shown that this process converges exponentionally fast in distribution to a unique invariant probability measure ; when the p_i's are constant on X it is possible to control the exponential rate of convergence.We apply this result in order to obtain a strong law of large numbers and a central limit theorem for this chain.

Résumé : On considère une chaîne de Markov à temps discret sur un espace métrique localement compact (X, d) obtenue par itération aléatoire de fonctions $T_i, i \in \mathbb{N}$, la probabilité $p_i(x)$ de choisir la transformation T_i dépendant seulement de la position courante x. Sous des hypothèses assez générales portant sur les applications T_i et p_i, on montre que ce processus converge en loi à vitesse exponentielle vers une unique mesure de probabilité invariante ; lorsque les poids p_i sont constants sur X, il est possible de contrôler le taux de convergence exponentielle. Nous appliquons ce résultat pour obtenir une loi forte des grands nombres et un théorème limite centrale pour cette chaîne.

1 Introduction

Let X be a locally compact space with a countable basis ; X is then metrizable and one can choose a metric d on X compatible with the topology such that (X,d) is a separable complete metric space in which sets of finite diameter are relatively compact. Let $(T_i)_{i\geq 0}$ be a collection of Borel measurable functions from X into X and $(p_i)_{i\geq 0}$ be a non-negative Borel measurable partition of unity on X (that is $\forall i \in \mathbb{N}, \forall x \in X \quad p_i(x) \geq 0 \quad \text{and} \quad \sum_{i=0}^{+\infty} p_i(x) = 1$).
We consider the following discrete-time Markov process on X : for a given $x \in X$ and Borel subset $B \subset X$, the transition probability from x to B is defined by

$$P(x,B) = \sum_{i=0}^{+\infty} 1_B(T_i(x))p_i(x)$$

where 1_B denotes the characteristic function of B.
Closely connected with this transition probability is the Markov operator (also denoted P) defined for complex-valued Borel measurable functions f on X by

$$Pf(x) = \int_X f(y)P(x,dy) = \sum_{i=0}^{+\infty} f(T_i(x))p_i(x).$$

We now state these things in probabilistic notations : let $\Omega = \mathbb{N}^{\mathbb{N}^*} = \{(i_n)_{n\geq 1} : i_n \in \mathbb{N}\}$ and $\mathcal{F}$ be the σ-algebra on Ω with respect to which each coordinate function $X_k, k \geq 1$:

$$\begin{array}{rccc} X_k : & \Omega & \rightarrow & \mathbb{N} \\ & (i_n)_{n\geq 1} & \mapsto & i_k \end{array}$$

is measurable. Let $\mathcal{F}_n, n \geq 1$, be the σ-algebra on Ω generated by the variables $X_1, \cdots, X_n$. For $x \in X$, let $\mathbb{P}_x$ be the probability measure defined by

$$\int f(\omega)\mathbb{P}_x(d\omega) = \sum_{(\omega_1,\ldots,\omega_n)\in\mathbb{N}} f(\omega_1,\ldots,\omega_n)p_{\omega_1}(x)p_{\omega_2}(T_{\omega_1}x)\cdots p_{\omega_n}(T_{\omega_{n-1}}\circ\cdots\circ T_{\omega_1}x),$$

if f denotes a function on Ω depending on the n first coordinates.
For all $x \in X$, define a sequence of X-valued random variables on Ω by $Z_0(x,\omega) = x$ and $Z_n(x,\omega) = T_{X_n(\omega)} \circ \cdots \circ T_{X_1(\omega)}x$ for $n \geq 1$. One can easily see that $(Z_n(x,\cdot))_{n\geq 0}$ is a X-valued Markov chain under $\mathbb{P}_x$ with initial distribution concentrated at x and transition probability P.

In this paper, we only treat the case where the T_i's are locally Lipschitz functions and the p_i's are continuous on X ; these conditions together will guarantee that P maps $\mathcal{C}_b(X, \mathbb{C})$ into itself, where $\mathcal{C}_b(X, \mathbb{C})$ denotes the space of complex-valued bounded and continuous functions on X.

When the p_i's are constant on X, the sequence $(T_{X_{n-1}} \circ \cdots \circ T_{X_1})_{n\geq 0}$ is a random walk on the semi-group $(\mathcal{C}(X,X), \circ)$ of continuous functions from X into X. Several authors have treated such systems : Dubin and Freedman [6], Hutchison[20], Diaconis and Shashahami [8], Elton, Barnsley [3] ... (with some success to make computer pictures). Of course, there are many close connections with the products of random matrices(see by example Furstenberg and Kesten [11], Guivarc'h and Raugi [16] ...) and also with the theory of random recurrent equations (cf Le Page [25], Goldie [12], Letac [26] ...).

Variable p_i's were considered by Doeblin and Fortet (1937 [6]), Ionescu Tulcea and Marinescu [22], Karlin (with motivations in learning models [24]) ..., and, more recently Barnsley, Elton, Demko and Geronimo [2]. Note that there are closed connections with the symbolic dynamic and the Ruelle-Perron-Frobenius operators theory (see [2], [14], [15] $\cdots$ and the example 3 in the section 6 of this paper). One can observe that the sequence $(T_{X_{n-1}} \circ \cdots \circ T_{X_1})_{n\geq 0}$ is not a Markov chain on $(\mathcal{C}(X,X), \circ)$ when the p_i's are variables.

Others articles treat some particular iterated function systems : Random walk on $\mathbb{R}^d$ generated by affine maps (Berger and Mete Soner [4]), random walk on $(\mathbb{R}^+)^d$ with elastic collisions on the axes (Leguesdron [26], Peigné [30] , Fayolles, Malyshev and Menshikov [10]...).

One of the main problems concerning these Markov processes is on the existence and attractiveness of a P- invariant probability measure ν on X and, in case the answer is affirmative, on the rate of convergence of the sequence $(P^n)_{n\geq 0}$ to ν as n goes to infinity.

The Elton and coauthors' argument is based on the following fact : under an assumption on the T_i's called "average-contractiveness" and a "Dini condition" on the $p_i's$ (that is, the moduli of uniform continuity φ_i of the $p_i's$ are such that $t \longrightarrow \frac{\varphi_i(t)}{t}$ is integrable over $]0, \delta]$ for some $\delta > 0$), they prove that the sequence $(P^n f)_{n\geq 0}$ is equicontinuous on X for any continuous complex-valued function with compact support on X. An application of Ascoli's theorem completes the proof. Note that they do require that the probability of strict contraction between any two points is bounded away from zero, in order to prove the attractiveness of the measure ν ; that means, in some sense, that there is a kind of independance in the choice of the T_i's (see the proof of lemma 2.7 in [2]).

Unfortunately, it seems to be impossible to determine the rate of convergence of the sequence $(P^n(x,.))_{n\geq 0}$ to the invariant mesure ν using their method. Our main focus is to obtain such a speed and our argument relies on the fact that, when the p_i's are Holder continuous, P operates on a suitable space $L \subset \mathcal{C}(X)$ constituted by certain Lipschitz functions on X :

$$L = L_{\alpha,\beta} = \{f \in \mathcal{C}(X) : \|f\| = |f| + m(f) < +\infty\}$$

$$\text{with} \quad |f| = \sup_{x\in X} \frac{|f(x)|}{1 + \ d(x,x_0)^\beta} \quad \text{and} \quad m(f) = \sup_{\substack{x,y\in X \\ x\neq y}} \frac{|f(x) - f(y)|}{d(x,y)^\alpha(1 + \ d(x_0,x)^\beta)}$$

where x_0 is a fixed element in X and α and β are strictly positive reals. This space was first introduced by Le Page in ([25]) in order to study the asymptotic behavior of the limit distribution of a random recurrent equation.

We have the following theorem

Theorem 1
Suppose that the following assumptions are satisfied

H0. $\displaystyle \sup_{\substack{x,y,z\in X\\ y\neq z}} \sum_{i=0}^{+\infty} \frac{d(T_i y, T_i z)}{d(y,z)}\, p_i(x) \;<\; +\infty$

H1. $\displaystyle \sup_{x,y\in X} \sum_{i=0}^{+\infty} \frac{d(T_i y, x_0)}{1+d(y,x_0)}\, p_i(x) \;<\; +\infty$

H2. $\displaystyle \sup_{x\in X} \sum_{i=0}^{+\infty} \frac{d(T_i x, x_0)}{1+d(x,x_0)}\, m(p_i) \;<\; +\infty$ *with* $\displaystyle m(p_i) = \sup_{\substack{x,y\in X\\ x\neq y, d(x,y)\leq 1}} \frac{|p_i(x)-p_i(y)|}{d(x,y)}$

H3 (ρ). *There exist* $k_0 \in \mathbb{N}^*$ *and* $\rho \in [0,1[$ *such that*
$\forall x,y,z \in X$

$$\sum_{i_1,\cdots,i_{k_0}\in \mathbb{N}} d(T_{i_{k_0}}\circ\cdots\circ T_{i_1}y, T_{i_{k_0}}\circ\cdots\circ T_{i_1}z)\; p_{i_{k_0}}(T_{i_{k_0-1}}\circ\cdots\circ T_{i_1}x)\cdots p_{i_1}(x) \;<\; \rho\, d(y,z)$$

H4. For all x *and* y *in* X, *there exist sequences of integers* $(i_n)_{n\in\mathbb{N}^*}$ *and* $(j_n)_{n\in\mathbb{N}^*}$ *such that*

$$\lim_{n\to+\infty} d(T_{i_n}\circ\cdots\circ T_{i_1}x, T_{j_n}\circ\cdots\circ T_{j_1}y)\;(1+d(T_{j_n}\circ\cdots\circ T_{j_1}x, x_0)) = 0$$

with $p_{i_n}(T_{i_{n-1}}\circ\cdots\circ T_{i_1}x)\cdots p_{i_1}(x)p_{j_n}(T_{j_{n-1}}\circ\cdots\circ T_{j_1}y)\cdots p_{j_1}(y) > 0 \;\; \forall n\geq 1.$

Then, one can choose α *and* β *in* $\mathbb{R}^{*+}$ *such that*

i) P *operates on the space* L

ii) there exist on L *a positive bounded operator* ν *and a bounded operator* Q *with spectral radius* $\rho(Q) = \lim_{n\to+\infty}\|Q^n\|^{1/n}$ *strictly less than one such that*

$$P = \nu + Q$$

with $\nu^2 = \nu$ *and* $\nu Q = Q\nu = 0$.

In particular, one can identify ν *with an attractive and* P*- invariant probability measure on* X *having a moment of order one (that is* $\int_X d(x_0,x)\nu(dx) < +\infty$*) and one can find* $\delta \in]0,1[, C>0$ *and* $n_0\in\mathbb{N}^*$ *such that*

$$\forall f\in L, \forall x\in X, \forall n\geq n_0 \qquad |P^n f(x) - \nu(f)| \;\leq\; C\,\delta^n\,\|f\|.$$

Remarks and coments

1. The hypothesis $H0$ implies that the transformations $T_i, i \geq 0$, are Lipschitz functions on X.

2. The hypotheses $H0$, $H1$ and $H2$ hold when $(T_i)_{i\geq 0}$ and $(p_i)_{i\geq 0}$ are finite collections of Lipschitz functions on (X, d).

3. If X is compact, the metric d is bounded and the norm $\|.\|_\psi$ is equivalent to the norm $\|.\|_\infty$; the hypothesis $H1$ is always fullfilled and $H2$ becomes equivalent to the fact that the p_i's are Lipschitz functions on (X, d). So, this theorem appears as a generalisation of the survey of iterated functions systems on a compact space.

4. Note that under the hypotheses $H0, H1$ and $H2$, there exist reals r and R such that

$$\forall f \in L \quad \|Pf\| \leq r\|f\| + R|f|;$$

thus, under these hypotheses, P operates on L. Iterating this inequality, one obtains

$$\forall f \in L, \forall n \geq 1 \quad \|P^n f\| \leq r_n\|f\| + R_n|f| \ ;$$

where $(r_n)_{n\geq 0}$ and $(R_n)_{n\geq 0}$ are sequences in $\mathbb{R}^{*+}$. Furthermore, if we suppose that $H3$ is also satisfied, one can choose $(r_n)_{n\geq 0}$ such that $\liminf_{n\to+\infty}(r_n)^{1/n} < 1$.

5. The hypothesis $H3$ is sometime called "uniform average contractivity before k_0 steps" ; using probabilistic notations, it may be stated as follows
There exist $\rho \in [0, 1[$ and $k_0 \in \mathbb{N}^*$ such that

$$\forall x, y, z \in X \quad \mathbb{E}_x[d(Z_{k_0}(y, .), Z_{k_0}(z, .))] \leq \rho\, d(y, z).$$

Iterating this formula, we are led to the next inequality
$\forall x, y, z \in X, \forall n \in \mathbb{N}, \forall l \in \{0, \cdots, k_0 - 1\}$

$$\mathbb{E}_x[d(Z_{nk_0+l}(y, .), Z_{nk_0+l}(z, .))] \leq C\rho^n d(y, z)$$

with $C = \sup_{0\leq l\leq k_0-1} \sup_{\substack{x,y,z\in X\\ y\neq z}} \mathbb{E}_x[\frac{d(Z_l(y, .), Z_l(z, .))}{d(y, z)}] < +\infty$ in virtue of hypothesis $H0$; consequently, we have

$$\limsup_{n\to+\infty}\Big(\sup_{\substack{x,y,z\in X\\ y\neq z}} \mathbb{E}_x[\frac{d(Z_n(y, .), Z_n(z, .))}{d(y, z)}]\Big)^{1/n} \quad < \quad 1.$$

6. Note that condition $H3$ does not require that one of the transformations T_i is a contraction in (X, d) ; by example let us consider the following iterated function system on $(\mathbb{R}^2, \|.\|)$ (where $\|.\|$ is the euclidean norm on $\mathbb{R}^2$) :

$$\forall (x, y) \in \mathbb{R}^2 \quad T_1(x, y) = (\frac{5x}{4}, \frac{y}{4}), \quad T_2(x, y) = (\frac{x}{4}, \frac{5y}{4}) \quad \text{and} \quad p_1 = p_2 = 1/2.$$

We have $m(T_1) = m(T_2) = 5/4$ but hypothesis $H3(\rho)$ holds with $\rho \leq 15/16$ and $k_0 = 2$ since

$$\sum_{i,j \in \{1,2\}} m(T_i T_j) p_i p_j = \frac{1}{4}(2(\frac{5}{4})^2 + 2(\frac{1}{4}\frac{5}{4})) = \frac{15}{16}$$

7. In many papers (see by example [2], [3], [4]), the "uniform average contractivity condition" is stated as follows :
There exist $k_0 \in \mathbb{N}^*$ and $q > 0$ such that

$$\sup_{\substack{x,y,z \in X \\ y \neq z}} \mathbb{E}_x[(\frac{d(Z_{k_0}(y,.), Z_{k_0}(z,.))}{d(y,z)})^q] < 1;$$

this condition seems to be more general than ours, but in fact, introducing the new distance δ on X defined by $\delta(x,y) = d(x,y)^a$ with $a = inf\{1, q\}$, one can see that $H3$ is satisfied on (X, δ).

8. From the remark 5, it follows

$$\forall x, y, z \in X \quad \mathbb{E}_x[\sum_{n=0}^{+\infty} d(Z_n(y,.), Z_n(z,.))] < +\infty$$

and so, for $\mathbb{P}_x-$ almost all $\omega \in \Omega$, the trajectories $(Z_n(y,\omega))_{n\geq 0}$ and $(Z_n(z,\omega))_{n\geq 0}$ are proximal on X, that is

$$\lim_{n \to +\infty} d(Z_n(y,\omega), Z_n(z,\omega)) = 0.$$

We will see that in fact the following stronger result holds

$$\lim_{n \to +\infty} d(Z_n(y,\omega), Z_n(z,\omega))(1 + d(x_0, Z_n(y,\omega)) = 0 \quad \mathbb{P}_x(d\omega) - a.s.$$

Unfortunately, this property is not sufficient in order to compare the trajectories $(Z_n(y,\omega))_{n\geq 0}$ and $(Z_n(z,\omega'))_{n\geq 0}$ when the processes $(Z_n(y,.))_{n\geq 0}$ and $(Z_n(z,.))_{n\geq 0}$ are defined respectively on the spaces $(\Omega, \mathbb{P}_y)$ and $(\Omega, \mathbb{P}_z)$. Hypothesis $H4$ will allow us to make this comparison.

9. Using the preceding remark, one can see that for all $y, z \in X$, there exists sequences of integers $(i_n)_{n\geq 1}$ such that

$$\lim_{n \to +\infty} d(T_{i_n} \circ \cdots \circ T_{i_1} y, T_{i_n} \circ \cdots \circ T_{i_1} z)(1 + d(x_0, T_{i_n} \circ \cdots \circ T_{i_1} y)) = 0.$$

Then, one can easily see that $H4$ is satisfied when $H0, H1$ and $H3$ hold and $\forall x \in X, \forall i \in \mathbb{N} \quad p_i(x) > 0$.

10. We will prove that $H1$ and $H3[\rho]$ lead to the following result

$$\sup_{x \in X} \mathbb{E}_x[\sum_{n=0}^{+\infty} \frac{d(x_0, Z_n(x,.))}{1 + d(x_0, x)}] < +\infty;$$

in particular, for any $x \in X$ and $\mathbb{P}_x$-almost all $\omega \in \Omega$ we shall have

$$\liminf_{n \to +\infty} d(x_0, Z_n(x,\omega)) \ < \ +\infty.$$

Thanks to this property, we shall not need to require the probability of strict contraction between any two points be bounded away from zero (as in [2]) in order to prove the attractiveness of the measure ν ; the hypothesis $H4$ will be sufficient.

The paper is organized as follows. In section 2, under the hypotheses $H0, H1, H2$ and $H3(\rho)$ we prove that one can chose α and β such that P operates on $L_{\alpha,\beta}$ and satisfies the following Doeblin-Fortet inequality $DF(r)$:

$$\forall n \geq 1, \forall f \in L \qquad \|P^n f\| \leq \rho_n \|f\| + R_n |f|,$$

where $(R_n)_{n\geq 0}$ and $(\rho_n)_{n\geq 0}$ are sequences in $\mathbb{R}^{*+}$ such that $\liminf\limits_{n\to+\infty}(\rho_n)^{1/n} = r \leq \rho^\alpha \in [0,1[$. Thus, using a result due to H.Hennion ([19]) which sharpens the Ionescu-Tulcea and Marinescu theorem ([22]) one can describe the spectrum of P on L. In section 3, assuming that the p_i's are strictly positive constants, we take α as closed as 1 as we want and we control the eigenvalues λ of P on L of modulus $> \rho^\alpha$. In section 4, we suppose that the assumption $H4$ is satisfied and we study the eigenfunctions of P on L associated to eigenvalues $\lambda, |\lambda| = 1$. Section 5 is concerned with asymptotic behavior of the Markov chain $(Z_n(x,.))_{n\geq 0}$ on $(\Omega, \mathbb{P}_x)$ and section 6 deals with examples.

2 The spectral decomposition of the operator P

Let $\mathcal{C}(X, \mathbb{C})$ be the space of complex-valued continuous functions on X and let φ and ψ be increasing and continuous functions from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that

$$\varphi(0) = 0, \varphi(1) = \psi(0) = 1 \text{ and } \quad \lim_{t\to+\infty} \psi(t) = \lim_{t\to+\infty} \frac{\psi(t)}{\varphi(t)} = +\infty.$$

We now consider the two following functions Ψ and Φ defined by

$$\forall x, y \in X \qquad \Psi(x) = \psi(d(x_0, x)) \text{ and } \quad \Phi(x,y) = \varphi(d(x,y))$$

where x_0 is a fixed point in X.
We introduce the space $L = L_{\varphi,\psi} = \{f \in \mathcal{C}(X, \mathbb{C}) : \|f\| = |f| + m(f) < +\infty\}$, with

$$|f| = \sup_{x\in X} \frac{|f(x)|}{\Psi(x)} \quad \text{and} \quad m(f) = \sup_{\substack{x,y\in X \\ x\neq y}} \frac{|f(x) - f(y)|}{\Phi(x,y)\Psi(x)}$$

One can easily see that $(L, \|\cdot\|)$ is a complex Banach space and that every bounded subset of $(L, \|\cdot\|)$ is relatively compact in $(L, |\cdot|)$.
The following statement holds

Proposition 2.1 *Assume that*

F1. $\sup_{n\in\mathbb{N}} \quad \sup_{x,y\in X} \mathbb{E}_x[\ \dfrac{\Psi(Z_n(y,.))}{\Psi(y)}\ ] < +\infty$

F2. $\sup_{\substack{x,y\in X\\ x\neq y}} \quad \sum_{i=0}^{+\infty} \dfrac{\Psi(T_i x)|p_i(x)-p_i(y)|}{\Psi(x)\Phi(x,y)} < +\infty$

F3(r). $\liminf_{n\to+\infty}(r_n)^{1/n} = r < 1 \quad$ *with* $\quad r_n = \sup_{\substack{x,y,z\in X\\ y\neq z}} \quad \mathbb{E}_x[\dfrac{\Phi(Z_n(y,.),Z_n(z,.))\Psi(Z_n(y,.))}{\Phi(y,z)\Psi(y)}]$

Then, P operates on the space L, its spectral radius $\rho(P) = \lim_{n\to+\infty} \|P^n\|^{1/n}$ is 1 and we have the following Doeblin-Fortet inequality $DF(r)$:

$$\forall n \geq 1, \forall f \in L \quad \|P^n f\| \leq r_n \|f\| + R_n |f|$$

where the $R_n, n \geq 1$, are positive constants.
Thus, r is greater than the essential spectral radius $\rho_e(P)$ of P on L, that is, for any $r' > r$, there exist subspaces F and H in L satisfying the following conditions

i) $L = F \oplus H, P(F) \subset F, P(H) \subset H.$

ii) $1 \leq dimF < +\infty$ *and the spectrum $\sigma(P_F)$ of the restriction P_F of P on F consists of eigenvalues of modulus $\geq\ r'$.*

iii) H is closed and the spectral radius $\rho(P_H) = \lim_{n\to+\infty} \|P_H^n\|^{1/n}$ is strictly less than r'.

Proof : Fix $f \in L_{\varphi,\psi}, n \geq 1$ and $x, y \in X$; we have

$$|P^n f(x) - P^n f(y)| \quad \leq \quad I_f(n,x,y) + J_f(n,x,y)$$

with

$$\begin{aligned} I_f(n,x,y) \quad &= \quad \sum_{i_1,\cdots,i_n} |f(T_{i_n}\circ\cdots\circ T_{i_1}x) - f(T_{i_n}\circ\cdots\circ T_{i_1}y)| p_{i_n}(T_{i_{n-1}}\circ\cdots\circ T_{i_1}x)\cdots p_{i_1}(x) \\ &\leq \quad r_n\ m(f)\ \Phi(x,y)\ \Psi(y) \end{aligned}$$

and

$$\begin{aligned} J_f(n,x,y) \quad &\leq \quad \sum_{i_1,\cdots,i_n} |f(T_{i_n}\circ\cdots\circ T_{i_1}y)| \\ &\qquad \times |p_{i_n}(T_{i_{n-1}}\circ\cdots\circ T_{i_1}x)\cdots p_{i_1}(x) - p_{i_n}(T_{i_{n-1}}\circ\cdots\circ T_{i_1}y)\cdots p_{i_1}(y)| \\ &\leq \quad |f| \sum_{i_1,\cdots,i_n} |\Psi(T_{i_n}\circ\cdots\circ T_{i_1}y)| \end{aligned}$$

$$
\begin{aligned}
&\times|p_{i_n}(T_{i_{n-1}}\circ\cdots\circ T_{i_1}x)\cdots p_{i_1}(x)-p_{i_n}(T_{i_{n-1}}\circ\cdots\circ T_{i_1}y)\cdots p_{i_1}(y)| \\
\leq\ & |f|\sum_{k=1}^{n} J_f(n,k,x,y)
\end{aligned}
$$

where we denote

$$
\begin{aligned}
J_f(n,k,x,y) \ =\ & \sum_{i_1,\cdots,i_n} \Psi(T_{i_n}\circ\cdots\circ T_{i_1}y)p_{i_n}(T_{i_{n-1}}\circ\cdots\circ T_{i_1}x)\cdots p_{i_{k+1}}(T_{i_k}\circ\cdots\circ T_{i_1}x) \\
&\times|p_{i_k}(T_{i_{k-1}}\circ\cdots\circ T_{i_1}x)-p_{i_k}(T_{i_{k-1}}\circ\cdots\circ T_{i_1}y)| \\
&\times p_{i_{k-1}}(T_{i_{k-2}}\circ\cdots\circ T_{i_1}y)\cdots p_{i_1}(y)
\end{aligned}
$$

Using the hypotheses $F1$, $F2$ and $F3(r)$, we are led to the following overestimation

$$
\begin{aligned}
J_f(n,k,x,y) \ \leq\ & \sum_{i_1,\cdots,i_n} \frac{\Psi(T_{i_n}\circ\cdots\circ T_{i_1}y)}{\Psi(T_{i_k}\circ\cdots\circ T_{i_1}y)} \\
&\times\frac{\Psi(T_{i_k}\circ\cdots\circ T_{i_1}y)|p_{i_k}(T_{i_{k-1}}\circ\cdots\circ T_{i_1}x)-p_{i_k}(T_{i_{k-1}}\circ\cdots\circ T_{i_1}y)|}{\Psi(T_{i_{k-1}}\circ\cdots\circ T_{i_1}y)\Phi(T_{i_{k-1}}\circ\cdots\circ T_{i_1}x,T_{i_{k-1}}\circ\cdots\circ T_{i_1}y)} \\
&\times\Phi(T_{i_{k-1}}\circ\cdots\circ T_{i_1}x,T_{i_{k-1}}\circ\cdots\circ T_{i_1}y)\Psi(T_{i_{k-1}}\circ\cdots\circ T_{i_1}y) \\
&\times p_{i_n}(T_{i_{n-1}}\circ\cdots\circ T_{i_1}x)\cdots p_{i_{k+1}}(T_{i_k}\circ\cdots\circ T_{i_1}x)p_{i_{k-1}}(T_{i_{k-2}}\circ\cdots\circ T_{i_1}y)\cdots p_{i_1}(y) \\
\leq\ & A\ B_{n-k}\ r_{k-1}\ \Phi(x,y)\ \Psi(y)
\end{aligned}
$$

with $B_k=\sup_{x,y\in X}\mathbb{E}_x[\frac{\Psi(Z_k(y,.))}{\Psi(y)}], k\geq 1$, and $A=\sup_{\substack{x,y\in X\\ x\neq y}}\sum_{i=0}^{+\infty}\frac{\Psi(T_ix)|p_i(x)-p_i(y)|}{\Psi(x)\Phi(x,y)}<+\infty.$

Finally, one obtains

$$
m(P^nf)\ \leq\ r_n\ m(f)\ +\ A\ \sum_{k=1}^{n} r_{k-1}B_{n-k}\ |f|;
$$

thus, P operates on the space L and we have the expected inequality $DF(r)$. The end of the proposition follows from the H.Hennion's theorem [19] □

Now, we prove that, under the hypotheses $H0, H1, H2$ and $H3$, for a suitable choice of the functions φ and ψ, the conditions $F1, F2$ and $F3$ of the preceding proposition are fullfilled ; namely, let α and β be two positive constants and let us put

$$
\forall t\in\mathbb{R}^+\quad \varphi(t)=\varphi_\alpha(t)=t^\alpha \quad\text{and}\quad \psi(t)=\psi_\beta(t)=1+t^\beta.
$$

We get $\alpha<\beta$ aso that $\lim_{t\to+\infty}\frac{\psi(t)}{\varphi(t)}=+\infty$ and we note $L_{\alpha,\beta}$ the space L associated to the functions φ_α and ψ_β.
We have the following

Theorem 2.2 *Suppose that hypotheses $H0, H1, H2$ and $H3(\rho)$ are fullfilled.*

Then, one can choose α and β in $]0,1/2[$ such that the conditions $F1, F2$ and $F3(r)$ of the preceding proposition hold on the space $L = L_{\alpha,\beta}$, with $r \leq \rho^\alpha$; consequently, the essential spectral radius of the operator P on $L_{\alpha,\beta}$ is lesser than ρ^α.

In particular, the set G of eigenvalues of P of modulus 1 is finite, the eigenspaces $L_\lambda = \{f \in L : Pf = \lambda f\}, \lambda \in G$, are finite dimensional and there are bounded projections U_λ from L to L_λ such that

$$(*) \qquad \begin{cases} P = \sum_{\lambda \in G} \lambda U_\lambda + Q \\ U_\lambda^2 = U_\lambda, \quad U_\lambda U_\mu = 0 \text{ if } \lambda \neq \mu \\ U_\lambda Q = Q U_\lambda = 0 \end{cases}$$

where Q is a bounded operator on L whose spectral radius $\rho(Q)$ is strictly less than 1.

In order to prove this theorem, we will need the following

Lemma 2.3 *Under the hypotheses $H0$, $H1$ and $H3$, we have*

$$\sup_{\substack{x\in X \\ n \geq 0}} \mathbb{E}_x[d(x_0, Z_n(x_0, .))] \quad < \quad +\infty$$

Proof : It follows from the inequalities

$$\begin{aligned} \mathbb{E}_x[d(x_0, Z_n(x_0, .))] &= \sum_{i_1,\cdots,i_n} d(x_0, T_{i_n} \circ \cdots \circ T_{i_1} x_0) p_{i_n}(T_{i_{n-1}} \circ \cdots \circ T_{i_1} x) \cdots p_{i_1}(x) \\ &\leq \sum_{i_1,\cdots,i_n} \sum_{k=1}^{n} d(T_{i_n} \circ \cdots \circ T_{i_{k+1}} x_0, T_{i_n} \circ \cdots \circ T_{i_k} x_0) \\ &\qquad p_{i_n}(T_{i_{n-1}} \circ \cdots \circ T_{i_1} x) \cdots p_{i_1}(x) \\ &\leq \sum_{k=1}^{n} \sup_{y \in X} \Big(\sum_{i_{k-1},\cdots,i_n} d(T_{i_n} \circ \cdots \circ T_{i_{k+1}} x_0, T_{i_n} \circ \cdots \circ T_{i_k} x_0) \\ &\qquad p_{i_n}(T_{i_{n-1}} \circ \cdots \circ T_{i_k} y) \cdots p_{i_k}(y) \Big) \\ &\leq B \sum_{k=1}^{n} \rho_k \end{aligned}$$

with $\rho_k = \sup_{\substack{x,y,z \in X \\ y \neq z}} \mathbb{E}[\frac{d(Z_k(y,.), Z_k(z,.))}{d(y,z)}]$ and $B = \sup_{x \in X} \mathbb{E}_x[d(Z_1(x_0,.), x_0)]$. Under the hypotheses $H0$ and $H3(\rho)$, the sum $\sum_{k=0}^{+\infty} \rho_k$ is finite since

$$\forall n \in \mathbb{N}, \forall m \in \{0, \cdots, k_0 - 1\} \quad \rho_{nk_0+m} \leq \rho_m \, \rho^n \quad \text{with} \quad \rho_m < +\infty.$$

$\square$

Proof of the theorem 2.2 : From the above lemma, one obtains
$\forall x, y \in X, \forall n \geq 0$

$$\begin{aligned} \mathbb{E}_x[d(x_0, Z_n(y, .,))] &\leq \mathbb{E}[d(x_0, Z_n(x_0, .))] + \mathbb{E}_x[d(Z_n(x_0, .), Z_n(y, .))] \\ &\leq B \sum_{k=1}^{n} \rho_k \; + \; \rho_n \; d(x_0, y) \end{aligned}$$

which implies $\sup_{n \geq 0} \sup_{x,y \in X} \mathbb{E}_x[\frac{d(x_0, Z_n(y, .))}{1 + d(x_0, y)}] < +\infty$ (condition F1).

The condition $F2$ follows immediately from the inequalities
$\forall x, y \in X$

$$\sum_{i=0}^{+\infty} \frac{\Psi(T_i x) \; |p_i(x) - p_i(y)|}{\Psi(x)\Phi(x, y)} \leq 2 \sup_{x,y \in X} \mathbb{E}_x[\frac{\Psi(Z_n(y, .))}{\Psi(y)}] < +\infty \quad \text{when} \quad d(x, y) \geq 1$$

and

$$\sum_{i=0}^{+\infty} \frac{\Psi(T_i x) \; |p_i(x) - p_i(y)|}{\Psi(x)\Phi(x, y)} \leq \sup_{x \in X} \sum_{i=0}^{+\infty} \frac{\Psi(T_i x)}{\Psi(x)} m(p_i) < +\infty \quad \text{when} \quad d(x, y) \leq 1$$

In the same way, if we assume that $2\alpha \leq 1$ and $2\beta \leq 1$, we obtain
$\forall x, y, z \in X, \forall n \geq 0$

$$\begin{aligned} \mathbb{E}_x[\Phi(Z_n(y, .), Z_n(z, .))\Psi(Z_n(y, .))] &\leq \mathbb{E}_x[\Phi(Z_n(y, .), Z_n(z, .))^2]^{1/2} \mathbb{E}_x[\Psi(Z_n(y, .))^2]^{1/2} \\ &\leq \mathbb{E}_x[d(Z_n(y, .), Z_n(z, .))]^{\alpha} \sqrt{2}(1 + \mathbb{E}_x[d(x_0, Z_n(y, .))]^{\beta}) \\ &\leq C(\rho_n)^{\alpha} d(y, z)^{\alpha} (1 + d(x_0, y)^{\beta}) \end{aligned}$$

with $\limsup_{n \to +\infty} (C(\rho_n)^{\alpha})^{1/n} \leq \rho^{\alpha} \; < \; 1.$ $\square$

3 A particular case : Iterated function systems with place independant probability

Throughout this section, we will suppose that the functions $p_i, i \in \mathbb{N}$ are constant on X. Thus, the iterated function system $(T_i, p_i)_{i \in \mathbb{N}}$ may be considered as a random walk on the space $\mathcal{C}(X, X)$ of continuous functions from X into X. This is a very particular case of iterated function system in which hypothesis $H2$ is obviously satisfied ; furthermore, in this simpler case, it suffices to consider the following space :

$$L_\alpha = \{f \in \mathcal{C}(X, \mathbb{C}\ ) : \|f\|_\alpha = |f| + m_\alpha(f) < +\infty\}$$

$$\text{with} \qquad |f| = \sup_{x\in X} \frac{|f(x)|}{1+d(x,x_0)} \quad \text{and} \quad m_\alpha(f) = \sup_{\substack{x,y\in X \\ x\neq y}} \frac{|f(x)-f(y)|}{d(x,y)^\alpha}$$

where x_0 is a fixed element in X and $\alpha \in]0,1[$. When $\alpha < 1$, every bounded subset of $(L_\alpha, \|.\|)$ is relatively compact in $(L_\alpha, |.|)$.
We have the following result

Theorem 3.1 *Let $(T_i, p_i)_{i\in \mathbb{N}}$ be an iterated function system on X such that p_i is constant on X for every $i \in \mathbb{N}$; assume the three following assumptions*

I0. There exists $C \in \mathbb{R}^{*+}$ such that

$$\forall x, y \in X \quad \sum_{i\in \mathbb{N}} d(T_i x, T_i y) p_i \ \leq \ C d(x,y)$$

I1. There exists $\rho \in]0,1[$ and $k_0 \in \mathbb{N}$ such that

$$\forall x, y \in X \qquad \sum_{i_0,\cdots,i_{k_0}} d(T_{i_{k_0}} \circ \cdots \circ T_{i_1} x, T_{i_{k_0}} \circ \cdots \circ T_{i_1} y) p_{i_{k_0}} \cdots p_{i_1} \ \leq \ \rho \, d(x,y)$$

I2. $\displaystyle\sum_{i=0}^{+\infty} d(T_i x_0, x_0) \ p_i \ < \ +\infty$

Then, P operates on the space L_α and there exist on L_α a positive bounded operator ν and a bounded operator Q with spectral radius lesser than ρ^α such that

$$P = \nu + Q$$

with $\nu Q = Q\nu = 0$ and $\nu^2 = \nu$.

In particular, one can identify ν with an attractive and P-invariant probability measure on X such that $\int_X d(x_0,x)\nu(dx) \ < \ +\infty$; furthermore, for any Lipschitz function $f \in L_1$ such that $\sup_{x\in X} \frac{|f(x)|}{d(x_0,x)^{\beta_0}} < +\infty$ for some $\beta_0 \ < \ 1$, we have

$$\limsup_{n\to+\infty} |P^n f - \nu(f)|^{1/n} \ \leq \ \rho(\|f\|_1 + \|f\|_{\beta_0})$$

Proof : Fix $\alpha < 1$; using similar arguments to the ones in the proof of theorem 2.2,one obtains

$$\sup_{\substack{x\in X \\ n\geq 0}} \mathbb{E}_x[\frac{d(x_0, Z_n(x,.))}{1+d(x_0,x)}] < +\infty$$

and

$$\forall f \in L_\alpha, \forall n \geq 0 \qquad m(P^n f) \leq \rho_n\, m(f) \quad \text{with} \quad \lim_{n\to+\infty} (\rho_n)^{1/n} = \rho^\alpha.$$

Thus, P operates on L_α and, following [19], one can see that the essential spectral radius of P on L_α is lesser than ρ^α.

Furthermore, if $f_\lambda, \lambda \in \mathbb{C}$, is an eigenfunction of P on L_α corresponding to the eigenvalue λ, $|\lambda| > \rho^\alpha$, we have

$$m(P^n f) = |\lambda|\, m(f) \leq \rho_n\, m(f),$$

which implies $m(f) = 0$ since $\lim_{n\to+\infty} \rho_n = 0$. Thus, the only eigenfunctions of P on L_α corresponding to the eigenvalue λ, $|\lambda| > \rho^\alpha$, are the constants ; using the definition of the essential spectral radius of P on L_α, we conclude that there exists a positive bounded operator ν on L_α such that the spectral radius of $Q = P - \nu$ is lesser than ρ^α.

Now, fix $f \in L_1$ such that $\sup_{x\in X} \dfrac{|f(x)|}{d(x_0,x)^{\beta_0}} < +\infty$; we have

$$\forall \alpha \geq \beta_0 \qquad m_\alpha(f) \leq \quad m_1(f) + m_\beta(f).$$

Thus, for any $\alpha \in [\beta_0, 1[$, f lies in L_α and we have

$$\begin{aligned} \limsup_{n\to+\infty} |P^n f - \nu(f)|^{1/n} &\leq \limsup_{n\to+\infty} \|P^n f - \nu(f)\|_\alpha^{1/n} \\ &\leq \rho^\alpha \|f\|_\alpha \\ &\leq \rho_\alpha(\|f\|_1 + \|f\|_{\beta_0}). \end{aligned}$$

We obtain the expected result letting $\alpha \to 1$. □

4 General case : Proof of the theorem 1

Assume that the system $(T_i, p_i)_{i\in\mathbb{N}}$ satisfies conditions $H0, H1, H2$ and $H3$; by theorem 2.2, we have the Doeblin-Fortet inequality :

$$\forall f \in L_{\alpha,\beta} \qquad m(P^n f) \leq \rho_n m(f) + R_n |f| \qquad \text{with} \quad \liminf_{n\to+\infty} \rho_n^{1/n} \leq \rho^\alpha,$$

for a suitable choice of α and β in $\mathbb{R}^{*+}$. Unfortunately, when the $p_i, i \in \mathbb{N}$, are not constant on X, we have $R_n \neq 0$ and we cannot apply the method of the preceding section in order to control the eigenvalues λ, $|\lambda| > \rho^\alpha$. Generaly, we are unable to describe all of these eigenvalues ; nevertheless, when the hypothesis $H4$ is fullfilled, one can describe the spectrum of P on the unit circle $C = \{\lambda \in \mathbb{C}\ , |\lambda| = 1\}$. More precisely

Proposition 4.1 *Assume hypotheses $H0, H1$, $H2$, $H3$ and $H4$. Then, the only bounded eigenfunctions of P on L corresponding to eigenvalues λ of modulus one are the constants.*

Admitting for the moment this result which will be proved later, one can establish theorem 1 :

Proof of theorem 1. Assume hypotheses $H0, H1, H2, H3$ and $H4$; by theorem 2.2, P may be decompose as follows on L

$$P = \sum_{\lambda \in G} \lambda U_\lambda + Q \qquad (*)$$

where U_λ is the bounded projection from L to the eigenspace $L_\lambda = \{f \in L : Pf = \lambda f\}$ and where Q is a bounded operator on L with spectral radius strictly lesser than 1. In order to prove our theorem, it is sufficient to show that $U_\lambda = 0$ when $\lambda \neq 1$ and to identify U_1 with a probability measure on X.

Fix $\lambda \in G - \{1\}$ and let $M_{\lambda,n} = \frac{1}{n}\sum_{k=0}^{n-1} \lambda^{-k}P^k$, $n \geq 1$. By the decomposition $(*)$, we obtain

$$\lim_{n \to +\infty} \|M_{\lambda,n} - U_\lambda\| \;=\; 0$$

where $\|M_{\lambda,n} - U_\lambda\|$ denotes the norm of the operator $M_{\lambda,n} - U_\lambda$ on $(L, \|.\|)$.

In particular, for any $f \in L$ and $x \in X$, we have

$$\lim_{n \to +\infty} M_{\lambda,n}f(x) \;=\; U_\lambda f(x).$$

If f is bounded, the same holds good for $U_\lambda f$ since $\|M_{\lambda,n}f\|_\infty \leq \|f\|_\infty$; thus, according to proposition 4.1, one obtains $U_\lambda f = 0$.
If f is not bounded, we consider the truncated function f_c defined by

$$\forall x \in X \quad f_c(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq c \\ c\,\dfrac{f(x)}{|f(x)|} & \text{if } |f(x)| \geq c \end{cases}$$

where $c > 0$ is an arbitrary constant. Note that $f_c \in L$ and $\|f_c\| \leq \|f\|$. Since f_c is bounded, we have $U_\lambda f_c = 0$ and so

$$\forall x \in X, \forall n \geq 1 \quad |M_{\lambda,n}f_c(x)| = |M_{\lambda,n}f_c(x) - U_\lambda f_c(x)| \leq (1 + d(x_0, x)^\beta)\|M_{\lambda,n} - U_\lambda\|\|f\|.$$

In another hand, by the Lebesgue theorem, one obtains

$$\forall x \in X, \forall n \geq 1 \quad \lim_{c \to +\infty} M_{\lambda,n}f_c(x) = M_{\lambda,n}f(x),$$

which implies

$$|M_{\lambda,n}f(x)| \leq (1 + d(x_0, x)^\beta)\|M_{\lambda,n} - U_\lambda\|\|f\|.$$

Letting $n \to +\infty$, one concludes $U_\lambda f = 0$.

When $\lambda = 1$, we show by a similar argument that for any $f \in L$, the function $U_1 f$ is constant on X ; the equality $U_1 1 = 1$ allows us to extend U_1 to the space of bounded continuous functions on X and therefore to identify U_1 with a P-invariant probability measure ν on X ; since $\sup_{\substack{x \in X \\ n \geq 0}} \mathbb{E}_x[\frac{d(x_0, Z_n(x,.))}{1+d(x_0,x)}] < +\infty$, we have $\int_X d(x_0,x)\nu(dx) < +\infty$. This completes the proof of the theorem. □

Proof of proposition 4.1.

a) Survey of bounded P-harmonic functions in L

First, we study the bounded P-harmonic functions in L, that is bounded functions $h \in L$ such that $Ph = h$. We shall need the two following lemmas :

Lemma 4.2 *Let x be in X and let h be a bounded P-harmonic function on X. Then, the random process $(h(Z_n(x,\cdot))_{n\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P}_x)$ is a bounded martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$. This process converges $\mathbb{P}_x$-almost surely and in $\mathbb{L}^q(\Omega, \mathbb{P}_x), q \geq 1$, and we have*

$$\forall n \geq 0 \quad h(x) = \mathbb{E}_x[h(Z_n(x,\cdot))] = \mathbb{E}_x[\lim_{k\to+\infty} h(Z_k(x,\cdot))].$$

Further, for all $q \geq 1$, $i_1, \ldots, i_q \in \mathbb{N}$ and $\mathbb{P}_x$-almost every $\omega \in \Omega$, we have

$$\lim_{n\to+\infty}[(h(Z_n(x,\omega)) - h(T_{i_q}\cdots T_{i_1} Z_n(x,\omega)))^2 \times p_{i_q}(T_{i_{q-1}} \circ \cdots \circ T_{i_1} Z_n(x,\omega)) \cdots p_{i_1}(Z_n(x,\omega))] = 0$$

Proof. Since $(Z_n(x,\cdot))_{n\geq 0}$ is a Markov chain with transition probability P on X and h a bounded P-harmonic function, we have

$$\forall n \geq 0 \quad \mathbb{E}_x[h(Z_{n+1}(x,\cdot))/\mathcal{F}_n] = Ph(Z_n(x,\cdot)) = h(Z_n(x,\cdot)).$$

Then, $(h(Z_n(x,\cdot))_{n\geq 0}$ is a bounded martingale on Ω and the first statement of the lemma is an easy consequence of the theory of martingales [27].
In order to prove the second statement, we use an idea due to A. Raugi in ([32]). Let

$$u_n(x) = \mathbb{E}_x[(h(Z_n(x,\cdot)) - h(Z_{n+q}(x,\cdot)))^2],$$

where q is a fixed integer ; we have

$$\begin{aligned} u_n(x) &= \mathbb{E}_x[h(Z_{n+q}(x,\cdot))^2] + \mathbb{E}_x[h(Z_n(x,\cdot))^2] - 2\mathbb{E}_x[h(Z_{n+q}(x,\cdot))h(Z_n(x,\cdot))] \\ &= \mathbb{E}_x[h(Z_{n+q}(x,\cdot))^2] - \mathbb{E}_x[h(Z_n(x,\cdot))^2] \end{aligned}$$

Hence, by cancellation, for any $N \geq 0$

$$\begin{aligned}
\sum_{n=1}^{N} u_n(x) &= \sum_{n=1}^{N} \mathbb{E}_x[h(Z_{n+q}(x,\cdot))^2] - \sum_{n=1}^{N} \mathbb{E}_x[h(Z_n(x,\cdot))^2] \\
&\leq 2q\|h\|_\infty.
\end{aligned}$$

This yields

$$\begin{aligned}
&\sum_{n=0}^{+\infty} u_n(x) \\
&= \sum_{n=0}^{+\infty} \mathbb{E}_x[\sum_{i_1\cdots i_q \in \mathbb{N}} \Big(h(T_{i_q}\cdots T_{i_1} Z_n(x,\cdot)) - h(Z_n(x,\cdot))\Big)^2 p_{i_q}(T_{i_{q-1}}\cdots T_{i_1} Z_n(x,\cdot))\cdots p_{i_1}(Z_n(x,\cdot))] \\
&= \mathbb{E}_x[\sum_{n\geq 0} \sum_{i_1,\ldots,i_q \in \mathbb{N}} \Big(h(T_{i_q}\cdots T_{i_1} Z_n(x,\cdot)) - h(Z_n(x,\cdot))\Big)^2 p_{i_q}(T_{i_{q-1}}\cdots T_{i_1} Z_n(x,\cdot))\cdots p_{i_1}(Z_n(x,\cdot))] \\
&\leq 2q\|h\|_\infty.
\end{aligned}$$

So, $\sum_{n\geq 0} \sum_{i_1,\ldots,i_q \in \mathbb{N}} \Big(h(T_{i_q}\cdots T_{i_1} Z_n(x,\cdot)) - h(Z_n(x,\cdot))\Big)^2 p_{i_q}(T_{i_{q-1}}\cdots T_{i_1} Z_n(x,\cdot))\cdots p_{i_1}(Z_n(x,\cdot))$ is $\mathbb{P}_x$-a.s finite and the second statement of the lemma follows immediately. □

Lemma 4.3 *Assume that hypotheses $H0, H1$ and $H3$ are satisfied. Then, for all $x \in X$ and $\mathbb{P}_x$-almost all $\omega \in \Omega$ we have*

$$\liminf_{n\to+\infty} d(x_0, Z_n(x,\omega)) < +\infty.$$

Proof. It follows immediately from the lemma 2.3 □

Therefore, we may establish the following result

Assume hypotheses $H0$, $H1$, $H2, H3$ and $H4$; then the bounded P-harmonic functions of P in L are constant on X

Let $h \in L$ be a bounded P-harmonic function. According to lemmas 4.2 and 4.3 for any $x \in X$ there exists $\Omega_x \subset \Omega, \mathbb{P}_x(\Omega_x) = 1$, such that for all $\omega \in \Omega_x$

i) the sequence $(h(Z_n(x,\omega)))_{n\geq 0}$ converges

ii) $\liminf_{n\to+\infty} d(x_0, Z_n(x,\omega)) < +\infty$

iii) $\forall q \geq 1, \forall i_1, \ldots, i_q \in \mathbb{N}$

$$\begin{aligned}
\lim_{n\to+\infty} (h(Z_n(x,\omega)) &- h(T_{i_q}\circ\cdots\circ T_{i_1} Z_n(x,\omega)))^2 \\
&p_{i_q}(T_{i_{q-1}}\cdots T_{i_1} Z_n(x,\omega))\cdots p_{i_1}(Z_n(x,\omega)) = 0
\end{aligned}$$

Fix $x \in X$ and $\omega \in \Omega_x$; by *ii*), there exists a sequence of integers $(\varphi(n,\omega))_{n\geq 0}$ such that $(Z_{\varphi(n,\omega)}(x,\omega))_{n\geq 0}$ converges to a point x_ω.

Since the functions h, T_i, and $p_i, i \geq 0$, are continuous on X, one obtains from *iii*) $\forall q \geq 1, \forall i_1, \dots, i_q \in \mathbb{N}$

$$(h(x_\omega) - h(T_{i_q} \circ \cdots \circ T_{i_1} x_\omega))^2 \, p_{i_q}(T_{i_{q-1}} \cdots T_{i_1} x_\omega) \cdots p_{i_1}(x_\omega) \;=\; 0.$$

In the same way, fixing another point $y \in X$ and $\omega' \in \Omega_y$, we have

$$\forall q \geq 1, \forall j_1, \dots, j_q \in \mathbb{N}$$
$$(h(y_{\omega'}) - h(T_{j_q} \circ \cdots \circ T_{j_1} y_{\omega'}))^2 \, p_{j_q}(T_{j_{q-1}} \cdots T_{j_1} y_{\omega'}) \cdots p_{j_1}(y_{\omega'}) \;=\; 0,$$

where $y_{\omega'}$ is a cluster point of the sequence $(Z_n(y,\omega'))_{n\geq 0}$.
Now, under the hypothesis $H4$ one can choose two sequences $(i_q)_{q\geq 0}$ and $(j_q)_{q\geq 0}$ such that

$$\lim_{q\to+\infty} d(T_{i_q} \circ \cdots \circ T_{i_1} x_\omega, T_{j_q} \circ \cdots \circ T_{j_1} y_{\omega'}) \; (1 + d(x_0, T_{j_q} \circ \cdots \circ T_{j_1} y_{\omega'})) = 0$$

with $p_{i_q}(T_{i_{q-1}} \circ \cdots \circ T_{i_1} x_\omega) \cdots p_{i_1}(x_\omega) p_{j_q}(T_{j_{q-1}} \circ \cdots \circ T_{j_1} y_{\omega'}) \cdots p_{j_1}(y_{\omega'}) > 0 \;\; \forall q \geq 1$.

Then, for every $q \geq 1$, we have

$$\begin{aligned}
|h(x_\omega) - h(y_{\omega'})| &= |h(T_{i_q} \cdots T_{i_1} x_\omega) - h(T_{i_q} \cdots T_{i_1} y_{\omega'})| \\
&\leq \|h\| d(T_{i_q} \cdots T_{i_1} x_\omega, T_{i_q} \cdots T_{i_1} y_{\omega'})^\alpha (1 + d(x_0, T_{i_q} \cdots T_{i_1} y_{\omega'})^\beta) \\
&\leq C \; \rho_q \|h\| d(x_\omega, y_{\omega'})^\alpha (1 + d(x_0, x_{\omega'})^\beta).
\end{aligned}$$

Letting $q \to +\infty$, we obtain $h(x_\omega) = h(y_{\omega'})$.
We conclude using the following equalities

$$\forall x \in X \quad h(x) = \mathbb{E}_x[\lim_{n\to+\infty} h(Z_n(x,\omega))]$$

and

$$\forall \omega \in \Omega_x \quad h(x_\omega) = \lim_{n\to+\infty} h(Z_n(x,\omega)).$$

□

b) Survey of the eigenvalues λ of P on $L, |\lambda| = 1, \lambda \neq 1$

In order to describe the bounded eigenfunctions of P corresponding to eigenvalues $\lambda, |\lambda| = 1, \lambda \neq 1$, we use a standard method ([34]) which bring us round to consider the "space-time" Markov chain on $X \times \mathbb{N}$ with transition operator $\tilde{P}$ defined by

$$\forall (x,n) \in X \times \mathbb{N} \quad \tilde{P}F(x,n) = \sum_{i=0}^{+\infty} F(T_i x, n+1) p_i(x).$$

The key argument which allows us to prove proposition 4.1 is contained in the next lemma, which may be established by an easy modification of the proof of Lemma 4.2 :

Lemma 4.4 *Let (x,k) be in $X \times \mathbb{N}$ and let $g \in L$ be a bounded $\tilde{P}$-harmonic function on $X \times \mathbb{N}$. Then, the random process $(g(Z_n(x,\cdot),k+n))_{n\geq 0}$ defined on $(\Omega,\mathcal{F},\mathbb{P}_x)$ is a bounded martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$. This process converges $\mathbb{P}_x$-almost surely and in $\mathbb{L}^q(\Omega,\mathbb{P}_x), q\geq 1$, and we have*

$$\forall n \geq 0 \quad g(x,k) = \mathbb{E}_x[g(Z_n(x,\cdot),n+k)] = \mathbb{E}_x[\lim_{n\to+\infty} g(Z_n(x,\cdot),n+k)].$$

Further, for all $q \geq 1$, $i_1,\ldots,i_q \in \mathbb{N}$ and $\mathbb{P}_x$-almost every $\omega \in \Omega$, we have

$$\begin{aligned}\lim_{n\to+\infty}(g(Z_n(x,\omega),n+k) &- g(T_{i_q}\cdots T_{i_1}Z_n(x,\omega),n+k+q))^2\\ &\times p_{i_q}(T_{i_{q-1}}\cdots T_{i_1}Z_n(x,\omega))\cdots p_{i_1}(Z_n(x,\omega)) = 0\end{aligned}$$

Let λ be a complex number of modulus one and let us consider the subspace $\mathcal{H}_c(\lambda)$ of L, made of by bounded eigenfunctions of P corresponding to the eigenvalue λ. If $h \in \mathcal{H}_c(\lambda)$, the function g defined on $X \times \mathbb{N}$ by $g(x,k) = \lambda^{-k}h(x)$ is $\tilde{P}$-harmonic. Using lemma 2.4 and an easy modification of the proof in part a), we may establish, for any point $x \in X$, the existence of $\Omega_x \subset \Omega, \mathbb{P}_x(\Omega_x) = 1$ such that $\forall \omega \in \Omega_x, \forall q \geq 1, \forall i_1,\ldots,i_q \in \mathbb{N}$

$$(h(x_\omega) - \lambda^{-q}h(T_{i_q}\circ\cdots\circ T_{i_1}x_\omega))^2\, p_{i_q}(T_{i_{q-1}}\cdots T_{i_1}x_\omega)\cdots p_{i_1}(x_\omega) \;=\; 0.$$

Thus, using a similar argument to the one in the above proof, one can prove that the function h is constant on X that is

$$\mathcal{H}_c(\lambda) = \{0\} \text{ if } \lambda \neq 1 \text{ and } \mathcal{H}_c(1) = \mathbb{R}.$$

5 Application : Asymptotic behavior of the Markov chain $(Z_n(x,.))_{n\geq 0}$ on X

We now apply the theorem 1 in order to study the asymptotic behavior of the Markov chain $(Z_n(x,.))_{n\geq 0}$ on X ; in particular, we will establish a strong law of large numbers (SLLN) and a central limit theorem (CLT) for this chain.

Theorem 5.1 *Suppose that hypotheses $H0, H1, H2, H3$ and $H4$ hold and let us not ν the unique P-invariant probability measure on X.*

For any $x \in X$ and any bounded and Lipschitz function f on X, the sequence $(\frac{1}{n}\sum_{k=0}^{n-1} f(Z_k(x,.)))$, converges almost surely to the constant $\nu(f)$ (SLLN) ; in particular, the Markov chain $(Z_n(x,.))_{n\geq 0}$ is recurrent on each open set $O \subset X$ such that $\nu(O) > 0$.

Furthermore, the distribution of the normalized sums $\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f(Z_k(x,.))$, $n \geq 1$, *tends to the normal law* $\mathcal{N}(\nu(f), \sigma^2(f))$ *with*

$$\sigma^2(f) = \lim_{n\to+\infty} \mathbb{E}_x[\frac{1}{n}\sum_{k=0}^{n-1}(f(Z_k(x,.)) - n\nu(f))^2] ;$$

the limit law $\mathcal{N}(\nu(f), \sigma^2(f))$ *is not degenerated (that is* $\sigma^2(f) \neq 0$*) whenever there does not exist a function* h *such that*

$$\forall i \in \mathbb{N} \qquad f = h \;-\; h \circ T_i \quad \nu - a.s$$

Proof : Fix a bounded and Lipschitz function f on X. Let us first note that it is possible to choose the reals α and β such that conditions $F1, F2$ and $F3$ of proposition 2.1 hold on the spaces $L_{\varphi,\psi}$, $L_{\varphi+\varphi^2,\psi^2}$ and $L_{\varphi+\varphi^4,\psi^4}$. Thus P operates on theses spaces and the spectral radius (on theses different spaces) of the operator $Q = P - \nu$ is strictly less then 1. One can easily see that f lies in these different spaces, and so

$$f = \nu(f) + g - Pg$$

with $g = \sum_{k=0}^{+\infty} P^k(f - \nu(f)) = \sum_{k=0}^{+\infty} Q^k(f) \in L_{\varphi,\psi}$.
Observe that $g^2 \in L_{\varphi+\varphi^2,\psi^2}$, which implies $\forall k \geq 0 \quad g(X_k) \in \mathbb{L}^2(\Omega, \mathbb{P}_x)$; then, for any $x \in X$, the sequence $(\sum_{k=0}^{n-1} X_k(x,.))_{n\geq 0}$ with $X_k(x,.) = g(Z_k(x,.)) - Pg(Z_{k-1}(x,.))$ is a zero-mean martingale on $(\Omega, \mathcal{F}, \mathbb{P}_x)$ whose increments $X_k(x,.), k \geq 0$, have finite variance. Thus, using the strong law of large numbers for martingales ([18]), we have

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} X_k(x,.) = 0 \quad \mathbb{P}_x - a.s.$$

On the other side, we have

$$\mathbb{E}_x[\sum_{n=1}^{+\infty} \frac{Pg(Z_n(x,.))}{n}] \;\leq\; \sum_{n=1}^{+\infty} \frac{1}{n^2} P^{n+1} g^2(x) < +\infty,$$

so that $\lim_{n\to+\infty} \frac{Pg(Z_n(x,.))}{n} = 0 \; \mathbb{P}_x - a.s.$ Finally, we obtain

$$\begin{aligned}\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} g(Z_k(x,.)) - Pg(Z_k(x,.)) &= \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n-1} X_k(x,.) + \frac{g(Z_0(x,.))}{n} - \frac{Pg(Z_{n-1}(x,.))}{n} \\ &= 0 \quad \mathbb{P}_x - a.s.\end{aligned}$$

that is $\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(Z_k(x,.)) = \nu(f) \quad I\!P_x - a.s.$

Thus, we have the strong law of large numbers for the Markov chain $(Z_n(x,.))_{n\geq 0}$; in particular, if O is an open set in X such that $\nu(O) > 0$, choosing an increasing sequence of functions $f_n \in L, n \geq 0$, which converges to the indicator function 1_O of O, one can prove that the chain $(Z_n(x,.))_{n\geq 0}$ is recurrent on O.

In order to establish the last assertion of this theorem, it is sufficient to show that the zero-mean martingale $(\sum_{k=0}^{n-1} X_k(x,.))_{n\geq 0}$ satisfies the following conditions

$$i)\ \frac{1}{\sigma_n^2}\sum_{k=1}^{n} I\!E_x[X_k^2/\mathcal{F}_{k-1}] \quad \xrightarrow{P} \quad 1$$

$$ii)\ \forall\epsilon > 0 \quad \frac{1}{\sigma_n^2}\sum_{k=1}^{n} I\!E_x[X_k^2\ 1_{[X_k^2\geq \epsilon\sigma_n^2]}] \quad \xrightarrow{n\to+\infty} \quad 0$$

where $\sigma_n^2 = \sum_1^n I\!E_x[X_k^2]$ and $\xrightarrow{P}$ means convergence in probability on $(\Omega, I\!P_x)$ ([18]).

We have

$$\frac{1}{n}\sum_{k=1}^{n} I\!E_x[(g(Z_k(x,.)) - Pg(Z_{k-1}(x,.)))^2/\mathcal{F}_{k-1}] = \frac{1}{n}\sum_{k=1}^{n} Pg^2(Z_{k-1}(x,.)) - (Pg)^2(Z_{k-1}(x,.)).$$

Since $g^4 \in L_{\varphi+\varphi^4,\psi^4}$, the variables $g(Z_k(x,.)), k \geq 0$, lie in $\in I\!L^4(\Omega, I\!P_x)$ and one can apply the preceding strong law of large numbers

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n} I\!E_x[(g(Z_k(x,.)) - Pg(Z_{k-1}(x,.)))^2/\mathcal{F}_{k-1}] = \nu(g^2) - \nu((Pg)^2) \quad I\!P_x - a.s.$$

By Lebesgue's theorem, we also obtain

$$\lim_{n\to+\infty}\frac{1}{n}\sigma_n^2 = \nu(g^2) - \nu((Pg)^2).$$

Thus, condition $i)$ is satisfied whenever $\nu(g^2) - \nu((Pg)^2) > 0$.

On the other side, we have

$$\forall\epsilon > 0 \quad I\!E_x[|X_k(x,.)|\ 1_{[X_k^2\geq n\epsilon^2]}] \leq \frac{1}{\sqrt{n}\epsilon} I\!E_x[X_k^2(x,.)]$$

so that

$$\frac{1}{n}\sum_{k=1}^{n} I\!E_x[X_k^2 1_{[X_k^2\geq n\epsilon]}] \leq \frac{1}{n^{\frac{3}{2}}\epsilon}\sum_{k=1}^{n} I\!E_x[X_k^4(x,.)].$$

Since the sequence $(\frac{1}{n}\sum_{k=1}^{n} I\!E_x[X_k^4(x,.)])_{n\geq 1}$ is bounded, condition $ii)$ follows immediately.

At last, we have $\sigma^2(f) = 0$ if and only if

$$\begin{array}{rrcl} & g^2(x) & = & (Pg)^2(x) \quad \nu(dx) - a.s \\ \Longleftrightarrow & Pg(x) & = & g(y)\ \nu(dx) - a.s, P(x,dy) - a.s \\ \Longleftrightarrow & Pg(x) & = & g(T_i x)\ \forall i \in \mathbb{N}, \nu(dx) - a.s. \end{array}$$

Thus, we have

$$\sigma^2(f) = 0 \Longleftrightarrow \forall i \in \mathbb{N}, f \ = \ g \ - \ g \circ T_i \quad \nu - a.s$$

□

Remark : It is also possible to obtain such results using the Fourier transform method, which relies on the spectral study of some "Fourier operator" associated to P ([25], [14]). This method is much more complicated but also powerfull than the one in this paper ; the above theorem is just an illustration of the spectral decomposition of P.

6 Examples

1. Under hypotheses $H1$, $H2$, $H3$ and $H4$, the Markov chain $(Z_n(x,.))_{n\geq 0}$ is recurrent on each open set $O \subset X$ such that $\nu(O) > 0$.
Observe that this result is wrong when O is a general Borel set in X ; let us consider the following simple example on $X = \mathbb{R}$:

$$\forall x \in \mathbb{R} \quad T_1(x) = \frac{x}{2}, \quad T_2(x) = \frac{x+1}{2}$$

$$\text{and} \quad p_1(x) = p_2(x) = \frac{1}{2}.$$

When $x \in \mathbb{Q}$, the sequences $(Z_n(x,.))_{n\geq 0}, \omega \in \Omega$ live in $\mathbb{Q}$ and, when $x \in \mathbb{R} - \mathbb{Q}$, the sequences $(Z_n(x,.))_{n\geq 0}, \omega \in \Omega$ live in $\mathbb{R} - \mathbb{Q}$.
Note that this iterated function system on $[0,1]$ has been studied by many people ; in particular, one can cite J.P. Conze and A. Raugi [5] and L. Hervé [17] with beautiful applications to wavelets.

2. Let us consider the particular case where the T_i's are affine maps on $\mathbb{R}^d$ with linear part A_i, that is :

$$\forall x \in \mathbb{R}^d \quad T_i(x) = A_i x + B_i$$

with $B_i, i \in \mathbb{N}$, fixed vectors in $\mathbb{R}^d$. Theses product of affine maps were studied by many people ; see in particular Grincevicius [13], Le Page [25], Berger and Soner [4]. Let us choose a particular norm $\|.\|$ on $\mathbb{R}^d$; it is not difficult to see that the hypotheses $H0, H1, H2$ and $H3$ may be simplified as follows :

H'0. $\sup_{x\in\mathbb{R}^d} \sum_{i=0}^{+\infty} \|A_i\| \, p_i(x) \; < \; +\infty$

H'1. $\sup_{x\in\mathbb{R}^d} \sum_{i=0}^{+\infty} \|B_i\| \, p_i(x) =< +\infty$

H'2. $\sum_{i=0}^{+\infty} \|A_i\| \; m(p_i) < +\infty$ *and* $\sum_{i=0}^{+\infty} \|B_i\| \; m(p_i) < +\infty$

H'3. *"Uniform average contractivity before k_0 steps"*
There exists $k_0 \geq 1$ such that

$$\sup_{x\in\mathbb{R}^d} \sum_{i_1,\cdots,i_{k_0}} \|A_{i_{k_0}} \cdots A_{i_1}\| p_{i_{k_0}}(T_{i_{k_0-1}} \circ \cdots \circ T_{i_1}x) \cdots p_{i_1}(x) < 1$$

with $\|A_i\| = \sup_{x\in\mathbb{R}^d-\{0\}} \dfrac{\|A_i x\|}{\|x\|}$.

3.Expanding map and associated adjoint operator

Let $X \subset \mathbb{R}$ and $T : X \to X$ be piecewise differentiable with $inf_{x\in X}|T'(x)| > 1$; let $\{T_i, i \in \mathbb{N}\}$ be continuous branches of the inverse of T and make T_i constant outside its natural definition domain in such a way that $T_i : X \to X$ is continuous. Let us note $\mu(dx) = \varphi(x)dx$ the probability measure on X with density φ with respect to Lebesgue measure on $\mathbb{R}$; we have the following result [2]

Let μ be invariant for T (that is $\mu(B) = \mu(T^{-1}(B))$ for all Borel set $B \subset X$) and $\varphi > 0$. Then μ is invariant for the iterated function system $(T_i, p_i)_{i\geq 0}$ with place dependant probabilities p_i defined by

$$p_i(x) = \frac{\varphi(T_i(x))|T_i'(x)|}{\varphi(x)}$$

The operator P associated to the iterated function system $(T_i, p_i)_{i\geq 0}$ is the adjoint of T with respect to the measure μ, that is

$$\forall f, g \in \mathbb{L}^2(X, \mu) \quad \int_X Pf(x)g(x)\mu(dx) = \int_X f(x)g \circ T(x)\mu(dx).$$

In some sense, it plays the role of "T^{-1}" and defines, as we saw, a Markov operator on X ; it is very usefull in many domains, in particular in symbolic dynamic where it allows to establish mixing property, ergodicity $\cdots$ of certain dynamic systems ([14], [15]).

By example, one can consider the continuous fraction transformation

$$\forall x \in]0,1] \quad T(x) = \frac{1}{x} - [\frac{1}{x}],$$

with the usual T-invariant probability measure

$$\mu(dx) = \frac{1}{Log2}\frac{dx}{1+x}.$$

A short calculation gives

$$\forall i \geq 1 \quad T_i(x) = \frac{1}{i+x} \text{ and } \quad p_i(x) = \frac{1+x}{(i+x)(i+1+x)}.$$

Let us in troduce the space of Lipschitz functions on $[0,1]$:

$$L = \{f \in C[0,1] : \|f\| = m(f) + \|f\|_\infty < +\infty\}$$

with $m(f) = \sup\limits_{\substack{x,y \in X \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|}$. From routine calculation [14], we get the following inequality

$$\forall f \in L \quad m(Pf) \leq \frac{5}{8}|f| + 2\|f\|_\infty.$$

(it follows from the inequalities :

$$|p_i(x) - p_i(y)| \leq |x-y|(p_i(x) + \frac{1}{i^2} - \frac{1}{(i+1)^2})$$

$$\sum_{i=1}^{+\infty} |p_i(x) - p_i(y)| \leq 2|x-y|$$

$$p_1(x) + \sum_{i=2}^{+\infty} \frac{p_i(x)}{i^2} \leq \frac{1}{4}(1 - p_1(x)) + p_1(x) = \frac{3}{4}p_1(x) + \frac{1}{4} \leq \frac{5}{8} \quad)$$

Thus, one can easily see that the hypotheses $H0, H1, H2$ and $H4$ are fullfilled, and so, conclusions of theorem 1 hold.

One can also consider the following simple extension of the continuous fraction transformation [15]

$$\forall x, y \in]0,1] \times]0,1] \quad T(x,y) = (\frac{1}{y + [\frac{1}{x}]}, \frac{1}{x} - [\frac{1}{x}])$$

where the T-invariant probability measure μ is $\mu(dxdy) = \frac{1}{Log2}\frac{dxdy}{(xy+1)^2}$.

7 Remark

The ergodic theorem for iterated function systems was first proved by J.Elton in [9]. The key argument of his proof was that, under suitable conditions, for all $x, y \in X$, the probability measure $\mathbb{P}_x$ is absolutely continuous with respect to $\mathbb{P}_y$; this result is interesting in itself and we have the following proposition which sharpens the Elton's result :

Proposition 7.1 *Assume hypotheses $H1, H2, H3$ and $H4$. Suppose also that* $\sup_{i\in\mathbb{N}} m(log(p_i)) < +\infty$.
Then, for any $x, y \in X$, the probability measures $\mathbb{P}_x$ and $\mathbb{P}_y$ are equivalent on Ω.

We will need the two following lemmas

Lemma 7.2 *Assume that $H1$ holds ; then, for all $r \in]\rho; 1[$, all x and y in X and $\frac{1}{2}(\mathbb{P}_x + \mathbb{P}_y)$-almost all $\omega \in \Omega$, we have*

$$\lim_{n\to+\infty} \frac{d(T_{X_n(\omega)} \circ \cdots \circ T_{X_1(\omega)}x, T_{X_n(\omega)} \circ \cdots \circ T_{X_1(\omega)}y)}{r^n} = 0.$$

Proof. Assume H1 ; we have

$$\forall x, y \in X \quad \mathbb{E}_x \left[\frac{d(T_{X_n} \circ \cdots \circ T_{X_1}x, T_{X_n} \circ \cdots \circ T_{X_1}y)}{r^n d(x,y)} \right] \leq (\frac{\rho}{r})^n$$

and so

$$\mathbb{E}_x \left[\sum_{n\geq 0} \frac{d(T_{X_n} \circ \cdots \circ T_{X_1}x, T_{X_n} \circ \cdots \circ T_{X_1}y)}{r^n d(x,y)} \right] < +\infty.$$

Thus

$$\sum_{n\geq 0} \frac{d(T_{X_n} \circ \cdots \circ T_{X_1}x, T_{X_n} \circ \cdots \circ T_{X_1}y)}{r^n} < +\infty \quad \mathbb{P}_x - p.s.$$

The lemma follows immediately. □

Lemma 7.3 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_n)_{n\geq 0}$ a filtration such that the σ-algebra generated by $\cup_n \mathcal{F}_n$ is $\mathcal{F}$. Let $\mathbb{Q}$ be a probability measure on Ω such that $\forall n \geq 0$ $\forall A \in \mathcal{F}_n$ $\mathbb{Q}(A) = \int_A X_n(\omega) P(d\omega)$ where $X_n, n \geq 0$ is a positive random variable on Ω, measurable with respect to $\mathcal{F}_n$. Let $(T_p)_{p\geq 0}$ be a sequence of stopping times relatively to the filtration $(\mathcal{F}_n)_{n\geq 0}$ such that*

i) $\lim_{+\infty} T_p(\omega) = +\infty \quad \mathbb{Q}(d\omega)$*-p.s*

ii) for all $p \geq 0$, the sequence $(X_{n\wedge T_p})_{n\geq 0}$ converges $\mathbb{P}$-a.s and in $\mathbb{L}^1(\mathbb{P})$ to the r.v. X_{T_p}.

Then, the sequence $(X_n)_{n\geq 0}$ converges $\mathbb{P}$-a.s and in $\mathbb{L}^1(\mathbb{P})$ to a random variable X_∞ and we have $\mathbb{Q} = X_\infty \mathbb{P}$, that is

$$\forall A \in \mathcal{F} \quad \mathbb{Q}(A) = \int_A X_\infty(\omega) \mathbb{P}(d\omega).$$

Proof. Using a standart method ([27]), one can show that $(X_n)_{n\geq 0}$ is a positive martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$; therefore, this martingale converges $I\!P$-a.s to a positive random variable X_∞, and we have, using Fatou's lemma $\quad I\!E_{\boldsymbol{P}}[X_\infty] \leq 1$.
Furthermore

$$\begin{aligned} I\!E_{\boldsymbol{P}}[X_{T_p}] &= \sum_{k\geq 0} I\!E_{\boldsymbol{P}}[X_k 1_{[T_p=k]}] + I\!E_{\boldsymbol{P}}[X_\infty 1_{[T_p=+\infty]}] \\ &= \sum_{k\geq 0} \mathbb{Q}\ [T_p = k] + I\!E_{\boldsymbol{P}}[X_\infty 1_{[T_p=+\infty]}] \\ &= \mathbb{Q}\ [T_p < +\infty] + I\!E_{\boldsymbol{P}}[X_\infty 1_{[T_p=+\infty]}]. \end{aligned}$$

Using the hypotheses $i)$ and $ii)$, we have

$$I\!E_{\boldsymbol{P}}[X_{T_p}] = \lim_{k\to+\infty} I\!E_{\boldsymbol{P}}[X_{T_p\wedge k}] = 1 \text{ and } \lim_{p\to+\infty} \mathbb{Q}\ [T_p < +\infty] = 0$$

and so

$$1 = \lim_{p\to+\infty} I\!E_{\boldsymbol{P}}[X_\infty 1_{[T_p=+\infty]}] \leq I\!E_{\boldsymbol{P}}[X_\infty].$$

In another hand, by Fatou's lemma, we obtain

$$\forall n \geq 0, \forall A \in \mathcal{F}_n \quad \int_A X_\infty(\omega) dI\!P(\omega) \ \leq \liminf_{n\to+\infty} \int_A X_n(\omega) dI\!P(\omega) = \mathbb{Q}\ (A).$$

Finally, $\mathbb{Q}$ and $X_\infty I\!P$ are two probability measures on Ω such that for any $n \geq 0$ and any $A \in \mathcal{F}_n, \mathbb{Q}\ (A) \geq \int_A X_\infty(\omega) dI\!P(\omega)$; thus $\mathbb{Q}\ = X_\infty I\!P$. □

Proof of proposition 7.1. Let us observe that the restriction $I\!P_{x,n}$ of $I\!P_x$ on the space of functions from Ω into $I\!R$ which depends only on the n first coordinates is defined by

$$\int f(\omega) I\!P_{x,n}(d\omega) = \sum_{(\omega_1,\cdots,\omega_n)\in I\!N^n} f(\omega_1\cdots\omega_n) p_{\omega_n}(T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1} x) \cdots p_{\omega_1}(x).$$

Thus

$$I\!P_{x,n}(d\omega) = h_{x,y,n}(\omega) I\!P_{y,n}(d\omega)$$

with

$$h_{x,y,n}(\omega) = \prod_{k=1}^n \frac{p_{\omega_k}(T_{\omega_{k-1}} \circ \cdots \circ T_{\omega_1} x)}{p_{\omega_k}(T_{\omega_{k-1}} \circ \cdots \circ T_{\omega_1} y)}.$$

The sequence $(h_{x,y,n}(\cdot))_{n\geq 0}$ is a positive matingale on $(\Omega, \mathcal{F}, I\!P_y)$ and so it converges $I\!P_y$-a.s to a r.v. $h_{x,y}(\cdot)$. Since $\sup_{i\in I\!N} m(lnp_i) < +\infty$, there exists $c > 0$ such that

$$\forall i \in I\!N\ \forall x, y \in X \quad \frac{p_i(x)}{p_i(y)} \leq \exp(c\ d(x,y))\ ;$$

so, we have $\forall n \geq 0 \;\; h_{x,y,n}(\omega) \leq \exp(c(\sum_{k=1}^{n} d(T_{\omega_{k-1}} \circ \cdots T_{\omega_1} x, T_{\omega_{k-1}} \circ \cdots T_{\omega_1} y)))$.
Let us introduce the stopping time T_p relatively to the filtration $(\mathcal{F}_k)_{k\geq 0}$ defined by

$$T_p = \inf\{n \geq p : d(T_{X_{n+1}} \circ \cdots \circ T_{X_1} x, T_{X_{n+1}} \circ \cdots \circ T_{X_1} y) \geq r^{n+1}\}$$

with $\inf \emptyset = +\infty$; following lemma 6.2 we have $\lim_{p \to +\infty} T_p(\omega) = +\infty \;\; \mathbb{P}_x(d\omega)$-p.s.
For any $n \geq p$ and $\mathbb{P}_y$-almost all $\omega \in \Omega$, we have

$$\begin{aligned} h_{x,y,n\wedge T_p(\omega)}(\omega) &\leq \exp(c\sum_{k=1}^{p} d(T_{X_k(\omega)} \circ \cdots T_{X_1(\omega)} x, T_{X_k(\omega)} \circ \cdots T_{X_1(\omega)} y) + c\sum_{k=p+1}^{n\wedge T_p(\omega)} r^k) \\ &\leq \exp(c\sum_{k=1}^{p} d(T_{X_k(\omega)} \circ \cdots T_{X_1(\omega)} x, T_{X_k(\omega)} \circ \cdots T_{X_1(\omega)} y) + c\frac{r^{p+1}}{1-r}) \; ; \end{aligned}$$

thus, using Lebesgue's theorem, one can show that the sequence $(h_{x,y,n\wedge T_p})_{n\geq 0}$ converges $\mathbb{P}_y$-a.s and in $\mathbb{L}^1(\mathbb{P}_y)$ to the random variable h_{x,y,T_p}.

Applying lemma 6.3, one can therefore prove that the sequence $(h_{x,y,n})_{n\geq 0}$ converges $\mathbb{P}_y$-a.s and in $\mathbb{L}^1(\mathbb{P}_y)$ to the random variable $h_{x,y}$ and $\mathbb{P}_x = h_{x,y}\mathbb{P}_y$, where $h_{x,y}$ is defined by

$$\forall \omega \in \Omega \quad h_{x,y}(\omega) = \prod_{n=1}^{\infty} \frac{p_{X_n(\omega)}(T_{X_{n-1}(\omega)} \circ \cdots \circ T_{X_1(\omega)} x)}{p_{X_n(\omega)}(T_{X_{n-1}(\omega)} \circ \cdots \circ T_{X_1(\omega)} y)}.$$

□

Acknowledgements. I would like to thank Albert Raugi for helpfull discussions and for pointing out his work and Max Bauer for suggestions about redaction of this article.

References

[1] ANORLD L., CRAUEL H. *Iterated function systems and multiplicative ergodic theory.* preprint.

[2] BARNSLEY M.F., DEMKO S.G., ELTON J.M., GERONIMO J.S. *Invariant measures for Markov processes arising from iterated function systems with place-dependent probabilities.* Ann. de l'Institut Henri Poincaré, vol. 24, n^o 3, (1988), p. 367-394.

[3] BARNSLEY M.F., ELTON J.M. *A new class of Markov processes for image encoding.* Adv. Appl. Prob. 20, (1988), p. 14-32.

[4] BERGER M.A., METE SONER M. *Random walks generated by affine Mappings.* Journal of theorical probability, vol. 1, n^o 3, (1988).

[5] CONZE J.P., RAUGI A. *Fonctions harmoniques pour un opérateur de transition et applications.* Bulletin soc. Math. France, 118, (1990), p. 273-310.

[6] DOEBLIN W., FORTET R. *Sur des chaînes a liasons complètes.* Bull. Soc. Math. de France , Vol 65, (1937), p. 132-148.

[7] DUBINS L., FREEDMAN D. *Invariant probabilities for certains Markov processes .* Ann. Math. Stat. 37, (196), p. 837-848

[8] DIACONES P., SHAHSHAHANI M. *Product of random matrices and computer image generation.* Random Matrices and their applications. 50, Cont. Math A.M.S., Providence, R.I (1986)

[9] ELTON J.H. *An ergodic theorem for iterated maps.* Ergod. Th. and Dynam Syst., n^o 7, (1987), p. 481-488.

[10] FAYOLLES G., MALYSHEV V.A, MENSHIKOV M.V *Random walks in a quater plane with zero drifts. I Ergodicity and null recurrence.* Ann. Inst. Henri Poincaré, Vol. 28, n^o 2, (1992), p. 179-194.

[11] FURSTENBERG H., KESTEN H. *Products of random matrices.* Ann. Math. Statis., n^o 31, (1960), p. 457-469.

[12] GOLDIE C. Implicit renewal theory and tails of solutions of random equations. The Annals of Applied Probability, Vol 1, n^o 1, (1991), p. 126-166.

[13] GRINCEVICIUS A.K. A central limit theorem for the group of linear transformation of the real axis Soviet Math. Doklady , Vol 15, n^o 6, (1974), p. 1512-1515.

[14] GUIVARC'H Y., HARDY J. *Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov.* Ann. Inst. Henri Poincaré, Vol. 24, n^o 1, (1988), p. 73-98.

[15] GUIVARC'H Y., LE JAN Y. *Asymptotic winding of the geodesic flow on modular surfaces and continuous fractions.* Ann. scient. Ec. Norm. Sup., $4^{ème}$ série, t. 26, 1993, p. 23-50.

[16] GUIVARC'H Y., RAUGI A. *Frontières de Furstenberg, propriétés de contraction et théorèmes de convergence.* Zeït für Wahr., n^o 69, (1985), p. 187-242.

[17] HERVÉ L. *Etude d'opérateurs quasi-compacts positifs. Applications aux opérareurs de transfert.* To appear in Ann. Inst. Henri Poincaré,

[18] HALL P., HEYDE C.C. *Martingale limit theory and its applications.* New York Academic Press(1980)

[19] HENNION H. *Décomposition spectrale des opérateurs de Doeblin-Fortet.* Proceeding of the A.M.S,n^o 69, (1993), p. 627-634

[20] HUTCHINSON J.. *Fractals and self-similarity.* Indiana U. J. of Math., 30, (1981), p. 713-747.

[21] IONESCU TULCEA C. *On a class of operators occuring in the theory of chains of infinite order.* Can. J. Math, Vol 11, 1959, p.112-121

[22] IONESCU C., MARINESCU G.. *Sur certaines chaînes à liaisons complètes.* C.R.A.S., 227, (1948), p. 667-669

[23] JAMISON B. Asymptotic behaviour of successive iterates of continuous functions under a Markov operator. J. of Math. Analysis and Applications, t. 9, (1964), p. 203-214.

[24] KARLIN S. *Some random walks Arising in Learning models.* Pac. J. of Math. , 3, (1953, p. 725-756.

[25] LE PAGE E. *Théorèmes de renouvellement pour les produits de matrices aléatoires. Equations aux différences aléatoires.* Séminaires de Rennes

[26] LEGUESDRON J.P. *Marche aléatoire sur le semi-groupe des contractions de $\mathbb{R}^d$. Cas de la marche aléatoire sur $\mathbb{R}^+$ avec chocs élastiques en zéro.* Ann. Inst. Henri Poincaré, Vol. 25, n^o 4, (1989), p. 483-502.

[27] NEVEU J. *Bases mathématiques du calcul des probabilités.* Masson et Cie, 1964.

[28] NORMAN M.F. *Markov processes and learning models.* Academic Press, 1972.

[29] NUMMELIN E. *General irreductible Markov chains and non-negative operators.* Cambridge U. Press, Cambridge (1985).

[30] PEIGNÉ M. *Marches de Markov sur le semi-groupe des contractions de $\mathbb{R}^d$.* Ann. Inst. Henri Poincaré, Vol. 28, n^o 1, (1992), p. 63-94.

[31] RAUGI A. *Théorie spectrale d'un opérateur de transition sur un espace compact.* Ann. Inst. Henri Poincaré, Vol. 28, n^o 2, (1992), p. 281-309.

[32] RAUGI A. *Fonctions harmoniques et théorèmes limites pour les marches aléatoires sur les groupes.* Bull. Soc. Math. de France , Vol 54, (1977)

[33] ROSENBLATT M. *Equicontinuous Markov operators.* Theor. Prob. and its Appl., vol. 9, (1969), p. 180-197.

[34] REVUZ D. *Markov chains.* North Holland Pub. Comp., 1975