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The paper is concerned with two types of estimators of an unknown parameter \( \theta \) of the drift of an observed semimartingale \( X \). A martingale part \( M \) of the semimartingale \( X \) is not a local square integrable martingale in general. As a rule we suppose only that \( M \) has a \( r \)-th moment, \( r \in [1,2] \).

The first part of the paper is devoted to an investigation of strong consistency of the least-square estimators (LS-estimators). Our approach is based on a multidimensional large numbers law for local martingales (see [1], where the results were announced particularly, see also [2] - [3]).

In the second part of the paper another type estimators of \( \theta \) are studied. They are so-called sequential estimators (SQ-estimators), and were systematically investigated in [4] for regression models with local square integrable martingales and quasi-left-continuous local martingales as errors. It was proved there that these estimators have a very important property-a guaranted accuracy. Here we get rid of from these assumptions proved a generalisation of Novikov's [2] inequality and Metivier-Pellaumail's one [5] for general local martingales and using the approach of the paper [4].

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) be a standard stochastic basis on which we consider all stochastic processes whose paths are regular.

Let us denote (see, for references [2]) \( \mathcal{M}_{loc}(\mathbb{R}^d) \) the set of local martingales, which values in \( \mathbb{R}^d, d \geq 1 \);
\( \mathcal{A}^+_{loc}(\mathbb{R}^d) \) the set of predictable processes, whose values are positive definite operators (matrix) from \( \mathbb{R}^d \) into \( \mathbb{R}^d \) such that \( A_t - A_s \geq 0, t \geq s \).

Let \( \lambda_1(A), \lambda_2(A) \) and \( tr(A) \) be the minimal, maximal eigenvalues and the trace of the operator (matrix) \( A \). Let us denote \( A^* \) a transpose matrix of \( A \).

For a random process \( X \) with values in \( \mathbb{R}^d, d \geq 1 \), let \( \{w : X_t \to\} \) be the set of \( \omega \in \Omega \) such that \( \lim_{t \to \infty} X_t(\omega) = X_\infty(\omega) \) exists for the norm \( \| \cdot \| \) of the space \( \mathbb{R}^d \).
If $A, B \in \mathcal{F}$ and $P \{ A \cap (\Omega \setminus B) \} = 0$, then we write $A \leq B$ (a.s.).

Let $M \in \mathcal{M}_{loc} (\mathbb{R}^d)$ and

$$
M_t = M_0^* + \int_0^t \int_{\mathbb{R}^d} x d(\mu - \nu), \quad (1)
$$

be the canonical decomposition of $M$, where $\mathbb{R}^d_0 = \mathbb{R}^d \setminus \{0\}$, $M^c$ be a continuous part of $M$ (and $< M^c >$ be its (matrix) quadratic characteristic), $\mu$ be a random measure of jumps of $M$ and $\nu$ be its compensator (see [2]).

**Theorem 1**: Assume the following conditions : (a.s.)

1) $\lim_{t \to \infty} \lambda_1 (A_t) = \infty$;

2) $\limsup_{t \to \infty} \frac{\lambda_1 (A_t)}{\lambda_2 (A_t)} < \infty$;

3) $\int_0^\infty \lambda_1 (A_s) d < M^c >_s + \int_0^\infty \int_{\mathbb{R}^d_0} \lambda_1 (A_s) \|x\|^r d\nu < \infty$

for some $r \in [1, 2]$.

Then $A_t^{-1} M_t \to 0$ (a.s.) as $t \to \infty$.

Particularly, if $V$ is predictable increasing process such that (a.s.)

$$
\frac{d < M^c >}{d V_t} + \frac{d}{d V_t} \int_0^t ||x||^r d\nu \leq \xi < \infty
$$

and (a.s.)

$$
3') \int_0^\infty \lambda_1^{-r} (A_s) d V_s < \infty,
$$
then 1), 2), 3') \Rightarrow A_t^{-1} M_t \to o \text{ (a.s.) as } t \to \infty.

**Proof:** Denote $\mathcal{B}$ a compensator of an increasing process $B$. Then as in one-dimensional case (see [2]) it is proved that (a.s.).

$$\{\omega : t r < M^\infty + \sum_s \frac{\|M_s\|^2}{1 + \|M_s\|^r} < \infty \} \subset \{ \omega : M_t \to \}.$$  \hfill (2)

Particularly, for some $r \in [1, 2]$ (a.s.)

$$\{\omega : t r < M^\infty + \sum_s \|A M_s\|^r < \infty \} \subset \{ \omega : M_t \to \}$$

The last statement follows from

$$\frac{\|x\|^2}{1 + \|x\|} \leq \|x\|^r \text{ for all } x \in \mathbb{R}^d, r \in [1, 2].$$


$$Y_t = \int_0^t A_{s-1}^d M_s.$$  

Using the same arguments we have that (a.s.)

$$\{A_1 (A_t) \to \infty \} \cap \left\{ \lim_{t \to \infty} \frac{\lambda_2 (A_t)}{\lambda_1 (A_t)} < \infty \right\} \subset \{ A_t^{-1} M_t \to o \}.$$  

To complete the proof note that the condition 3) 3') implies (a.s.)

$$t r < Y^\infty + \sum_s \|\Delta Y_s\|^r < \infty$$

(in the case of 3')) and in view of (2) we get the statement of the theorem 1.

This theorem gives us a possibility to prove the strong consistency of the LS-estimators in regression models with non-square integrable martingale errors.

Consider the following regression model

$$X_t = \int_0^t f s d V_s \theta + m_t, \quad \text{(3)}$$
where $m$ is a pure discontinuous (for simplicity) local martingale from $\mathcal{M}_{loc} (R^d)$, a predictable process $V \in \mathcal{A}_{loc}^+ (R^1)$, $f$ is a predictable $(d \times k)$-matrix , $\theta \in R^k$, $k \geq 1$, is an unknown parameter.

Let $F_t = \int_0^t f_s^* f_s dV_s, F_t > 0, t \geq t_0$.

In this case we can define the estimator of $\theta$:

$$\theta_t = F_t^{-1} \int_0^t f_s^* dX_s = \theta + F_t^{-1} \int_0^t f_s^* dm_s.$$ 

**Theorem 2:** Suppose for the model (3) the following conditions hold (a.s.)

1) \( \lim_{t \to \infty} \lambda_1 (F_t) = \infty; \)

2) \( \limsup_{t \to \infty} \frac{\lambda_2 (F_t)}{\lambda_1 (F_t)} < \infty; \)

3) \( \int_0^\infty \int_{K_0^d} \lambda_1^{-r} (F_s) \| f_s \| r \| x \| d\nu < \infty \)

where $r \in [1, 2], \nu$ is a compensator of a measure $\mu$ of jumps of $M$.

Then $\theta_t \to \theta$ (a.s) as $t \to \infty$.

It is possible to unify the conditions of the theorem 2, if we suppose that (a.s.)

\( \frac{d}{dV_t} \int_0^t \int_{R_0^d} \| x \| r \nu \leq \xi < \infty, \)

and (a.s.)

3) \( \int_0^\infty \lambda_1^{-r} (F_s) \| f_s \| r dV_s < \infty. \)

Then 1) - 2) - 3') $\Rightarrow \theta_t \to \theta$ (a.s) as $t \to \infty$. 

Proof: It is sufficient to note that

\[ \theta_t - \theta = A_t^{-1} M_t, \]

where \( A_t = F_t, M_t = \int_0^t f^* x d (\mu - v). \)

Using the theorem 1 we get immediately the statement of the theorem 2.

Remark: Note that the consistency of LS-estimators for the model (3) with non-random regressors was proved by Novikov [6]. The strong consistency of the LS-estimators for this model with non-random regressors was studied also in [7]-[8].

Now consider another type of estimators of \( \theta \) in the one-dimensional model (3). These are SQ-estimators, which systematically were studied in [4]. But the case of non-square integrable errors was handed there for the quasi-left continuous martingale errors \( m \) only. Here we prove an estimate for pure-discontinuous martingales and apply it to give an upper estimate for the \( r\)-th moment of the difference between the SQ-estimator and \( \theta \). This result gives us (in some sense) a guaranted accuracy of these estimators.

Denote \( \mathcal{B} (R) \)- Borel \( \sigma \)-algebra of the space \( R \). Let

\[ M_t = \int_0^t \int_{R_o} x d (\mu - v) \]

be a purely discontinuous local martingale of the classe \( \mathcal{M}_{loc} (R^1) \) (see decomposition (1)).

Let \( U \) be a \( \mathcal{B} (R^1_+ \otimes \mathcal{T} \otimes \mathcal{B} (R^1_0) \)-measurable function such that for some \( r \in [1, 2] \)

\[ \int_0^t \int_{R_o} |U|^r d v \in \mathcal{A}_{loc}^+ (R^1) \]
and
\[
\int_{R_0} U(t, x, \omega) \nu((t), d x) = 0 \quad (4)
\]

Denote \( Y_t(U) = \int_{R_0}^t U d(\mu - \nu) \) and \( Y_t^*(U) = \sup_{s \leq t} |Y_s(U)| \).

Theorem 3: Suppose the function \( U \) satisfies to the condition (4) and \( \tau \) is a predictable stopping time (s.t.). Then
\[
E |Y^*_\tau(U)|^r \leq A_r \int_{R_0}^\tau |U|^r d \nu, \quad (5)
\]

where \( A_r \leq 3 \left( \frac{r}{r-1} \right) \), \( r \in (1, 2], A_1 = Z \) and \( Y_{\tau^0} \) is left limit of \( Y_t \).

Proof. We shall use Novikov's method [5]. Let us involve the s.t. \( \tau \) and
\[
\tau_a = \inf \{ t \leq \tau : \int_{R_0}^t |U|^r d \nu \geq a \},
\]
\[
\inf \{ \emptyset \} = \tau.
\]

Of course, \( \tau_a \) is a predictable s.t.

Therefore there is a sequence of s.t.'s \( \tau^n_{\alpha} \) \( n \geq 1 \) such that
\[
\tau^n_{\alpha} \uparrow \tau_{\alpha} (a.s.) \text{ as } n \uparrow \infty,
\]
\[
\tau^n_{\alpha} < \tau_{\alpha} \text{ on the set } (\omega : \tau_{\alpha} < \infty).
\]

It follows from here that
\[
\int_{R_0}^\tau |U|^r d \nu < a.
\]

Let us show that \( E Y^*_{\tau^n_{\alpha}}(U) < \infty \), we have (as usually, \( I_c \) is an indicator of \( c \))
Using this fact and the elementary inequality
\[ |x + y|^r - |x|^r | \leq C_r \left( |x|^{r-1} |y| + |y|^r \right) \]

we get that
\[ E \int_{0}^{\tau_a} \int_{R_0} |U|_{>1} \left| Y_s + U \right|^r - |Y_s|^r \, d\nu < \infty, \]
\[ E \left[ \int_{0}^{\tau_a} \int_{R_0} \left| U \right|_{\leq 1} \left| Y_s + U \right|^r - |Y_s|^r \right]^2 d\nu < \infty \] \hspace{1cm} (6)

This first inequality of (6) follows from
\[ E \int_{0}^{\tau_a} \int_{R_0} |U|_{>1} \left| Y_s + U \right|^r - |Y_s|^r \, d\nu \leq \]
\[ \leq \text{const} (r) E \left( 1 + \left| Y_{\tau_a}^n \right|^{-1} \right) \int_0^{\tau_a} \int_{R_0} \left| U \right|^r d\nu \leq \]

\[ \leq a \cdot \text{const} (r) \cdot E \left( 1 + \left| Y_{\tau_a}^n \right|^{-1} \right) < \infty. \]

The second one follows from

\[
E \left[ \int_0^{\tau_a} \int_{R_0} I_{|U| \leq 1} \left| Y_s^- + U \right|^r - \left| Y_s^- \right|^r \, d\nu \right] \leq \]

\[ \leq \text{const} (r) E \left[ \int_0^{\tau_a} \int_{R_0} I_{|U| \leq 1} \left( \left| Y_s^- \right|^{-1} |U| + |U|^r \right)^2 \, d\nu \right] \leq \]

\[ \leq \text{const} (r) \left( 1 + (Y_{\tau_a}^n)^{2(r-1)} \right)^{1/2} \left( \int_0^{\tau_a} \int_{R_0} I_{|U| \leq 1} |U|^r \, d\nu \right)^{1/2} \]

and

\[
E \left( Y_{\tau_a}^n (U) \right)^{-r-1} \leq \left( E Y_{\tau_a}^n (U |I| |U|_{>1}) \right)^{-r-1} + \left( E Y_{\tau_a}^n (U |I| |U|_{\leq 1}) \right)^{-r-1} \]

Now using the Ito's formula (see [2] ; p. 150-151) we get

\[
\left| Y_{\tau_a}^n \right|^r = \int_0^{\tau_a} \int_{R_0} \left( \left| Y_s^- + U \right|^r - \left| Y_s^- \right|^r \right) d(\mu - \nu) + \]

\[ + \int_0^{\tau_a} \int_{R_0} \left( \left| Y_s^- + U \right|^r - \left| Y_s^- \right|^r - r \left| Y_s^- \right|^{-2} Y_s^- U \right) d\nu \]

(7)
It follows from (6) that

$$E (\text{martingale part of } (7)) = 0$$

Applying the elementary inequality

$$|x + y|^r - |x|^r - r x y |x|^r - 2 \leq B_r |y|^r,$$

where $B_r \leq 3$, $r \in [1,2]$ and $B_1 = 2$,

to the second part of (7), we have

$$E |Y_{\tau_a}^n|^r \leq B_r E \int_0^{\tau_a} \int_{R_0} |U|^r d\nu.$$

(8)

Using the Doob's inequality [2], we get

$$E (Y_{\tau_a}^n)^r \leq 3 \left( \frac{r}{r-1} \right)^r E \int_0^{\tau_a} \int_{R_0} |U|^r d\nu.$$  

To tend $n \to \infty$ and $a \to \infty$ we complete the proof.

We note that the inequality (8) for $r = 1$ is true with $B_1 = 2$ and therefore $A_1 = 2$.

Now consider the one-dimensional regression model (3) and suppose that

$$\frac{d}{dV_t} \int_0^t |x|^r d\nu \leq \gamma_t,$$

(9)

where $r \in [1,2]$, $\gamma$ is a predictable process such that

$$K_t = \int_0^t \gamma_s^{1-r} |f_s|^r dV_s \in \mathbb{A}_{bc}^r (R^1).$$

we define the following SQ-estimator

$$\theta_H = H^{-1} \int_0^{\tau_H} \gamma_s^{-1} f_s dX_s + H^{-1} \beta_H \gamma_{\tau_H}^{-1} f_{\tau_H} \Delta X_{\tau_H},$$
where $H > 0$, $\tau_H = \inf \{ t : K_t \geq H \}$, $\beta_H = \mathcal{F}_{\tau_H}^-$-measurable random variable such that $\beta_H \in [0, 1], \gamma^{-1} f_s \, \Delta V_s + \beta_H \gamma^{-1} f_{\tau_H} \, \Delta V_{\tau_H} = H.$ (10)

**Theorem 4**: Let the conditions (9) - (10) are fulfilled. Then $\int_0^\infty \gamma_s^{-1} \, |f_s|^r \, d V_s + \beta_H \gamma_{\tau_H}^{-1} \, |f_{\tau_H}|^r \, \Delta V_{\tau_H} = H.$

**Proof**: The first statement is the direct consequence of (10). Now we have, using the theorem 3, that

$$E |\theta_H - \theta|^r = E |H^{-1} \int_0^{\tau_H^+} \gamma_s^{-1} f_s \, x \, d (\mu - v) +$$

$$+ \beta_H H^{-1} \gamma_{\tau_H}^{-1} f_{\tau_H} \, \Delta M_{\tau_H} |^r \leq A_r 2^{r-1} E H^{-r} \int_0^{\tau_H^+} \gamma_s^{-r} |f_s|^r \, |x|^r \, d v +$$

$$+ 2^{r-1} H^{-r} E \beta_H \gamma_{\tau_H}^{-r} \, |f_{\tau_H}|^r \cdot \int_{R_0}^\infty \gamma^{-r} \, |f_s|^r \, \Delta V_s + E \beta_H \gamma_{\tau_H}^{-r} \, |f_{\tau_H}|^r \, \Delta V_{\tau_H} \leq$$

$$\leq H^{-r} 2^{r-1} \left[ A_r E \int_0^{\tau_H^+} \gamma_s^{-1} \, |f_s|^r \, d V_s + E \beta_H \gamma_{\tau_H}^{-1} \, |f_{\tau_H}|^r \, \Delta V_{\tau_H} \right] \leq$$

$$\leq 2^{r-1} H^{-r} \left[ A_r H + \Delta \right] \leq \text{const} (r) \cdot H^{-r} (H + \Delta)$$

The theorem is proved.
REFERENCES


