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On regression models with non-square integrable martingale-like errors

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The paper is concerned with two types of estimators of an unknown parameter θ of the drift of an observed semimartingale X. A martingale part M of the semimartingale X is not a local square integrable martingale in general. As a rule we suppose only that M has a r-th moment, $r \in [1,2]$.

The first part of the paper is devoted to an investigation of strong consistency of the least-square estimators (LS-estimators). Our approach is based on a multidimensional large numbers law for local martingales (see [1] , where the results were announced particularly, see also [2] - [3]).

In the second part of the paper another type estimators of θ are studied. They are so-called sequential estimators (SQ-estimators), and were systematically investigated in [4] for regression models with local square integrable martingales and quasi-left-continuous local martingales as errors. It was proved there that these estimators have a very important property-a guaranted accuracy. Here we get rid of from these assumptions proved a generalisation of Novikov's [2] inequality and Metivier-Pellaumail's one [5] for general local martingales and using the approach of the paper [4].

Let $(\Omega , \mathcal{F} , (\mathcal{F}_t)_{t \geq o} , P)$ be a standard stochastic basis on which we consider all stochastic processes whose paths are regular.

Let us denote (see, for references [2]) $\mathcal{M}_{loc} (R^d)$ the set of local martingales, which values in R^d $d \geq 1$;

$\mathcal{A}^+{}_{loc} (R^d)$ the set of predicatble processes, whose values are positive definite operators (matrix) from R^d into R^d such that $A_t - A_s \geq o , t \geq s$.

Let $\lambda_1 (A)$, $\lambda_2 (A)$ and $tr (A)$ be the minimal, maximal egenvalues and the trace of the operator (matrix) A. Let us denote A^* a transpose matrix of A.

For a random process X with values in R^d , $d \geq 1$, let $\{w : X_t \rightarrow\}$ be the set of $\omega \in \Omega$ such that $\lim_{t \to \infty} X_t (\omega) = X_\infty(\omega)$ exists for the norm $\|.\|$ of the space R^d .

If A , $B \in \mathcal{F}$ and $P\{A \cap \{\Omega \setminus B\}\} = o$, then we write $A \leq B$ (a.s.).

Let $M \in \mathcal{M}_{loc}(R^d)$ and

$$M_t = M_t^c + \int_o^t \int_{R_o^d} x\, d(\mu - \nu)\,, \qquad (1)$$

be the canonical decompostion of M , where $R_o^d = R^d \setminus \{o\}$, M^c be a continuous part of M (and $< M^c >$ be its (matrix) quadratic characteristic), μ be a random measure of jumps of M and ν be its compensator (see [2]).

Theorem 1 : Assume the following conditions : (a.s.)

1) $\lim\limits_{t \to \infty} \lambda_1(A_t) = \infty$;

2) $\limsup\limits_{t \to \infty} \dfrac{\lambda_1(A_t)}{\lambda_2(A_t)} < \infty$;

3) $$\int_o^\infty \lambda_1^{-2}(A_s)\, d< M^c>_s + \int_o^{+\infty} \int_{R_o^d} \lambda_1^{-r}(A_s)\, \|x\|^r\, d\nu < +\infty$$

for some $r \in [1\,,2]$.

Then $A_t^{-1} M_t \to o$ (a.s.) as $t \uparrow \infty$.

Particularly, if V is predictable increasing process such that (a.s.)

$$\frac{d< M^c>_t}{d\,V_t} + \frac{d}{d\,V_t} \int_o^t \int_{R_o^d} ||x||^r\, d\nu \leq \xi < \infty$$

and (a.s.)

3′) $$\int_o^\infty \lambda_1^{-r}(A_s)\, d\,V_s < \infty,$$

then 1) , 2) , 3') $\Rightarrow A_t^{-1} M_t \to o$ (a.s.) as $t \uparrow \infty$.

Proof : Denote $\tilde{B}$ a compensator of an increasing process B. Then as in one-dimensional case (see [2]) it is proved that (a.s.).

$$\{\omega : tr < M^c >_\infty + \widetilde{\sum_s \frac{\|\Delta M_s\|^2}{1 + \|\Delta M_s\|}} < \infty\} \leq \{\omega : M_t \to\}.$$

Particularly, for some $r \in [1, 2]$ (a.s.)

$$\{\omega : tr < M^c >_\infty + \sum_s \|A M_s\|^r < \infty\} \leq \{\omega : M_t \to\} \qquad (2)$$

The last statement follows from

$$\frac{\|x\|^2}{1 + \|x\|} \leq \|x\|^r \text{ for all } x \in R^d, r \in [1, 2].$$

Now define as in [2] - [3] the process

$$Y_t = \int_o^t A_s^{-1} \, d\, M_s.$$

Using the same arguments we have that (a.s.)

$$\{\lambda_1 (A_t) \to \infty\} \cap \left\{ \lim_{t \to \infty} \frac{\lambda_2 (A_t)}{\lambda_1 (A_t)} < \infty \right\} \subset \{A_t^{-1} M_t \to o\}.$$

To complete the proof note that the condition 3) 3') implies (a.s.)

$$tr < Y^c >_\infty + \widetilde{\sum_s \|\Delta Y_s\|^r} < \infty$$

(in the case of 3'))

and in view of (2) we get the statement of the theorem 1.

This theorem gives us a possibility to prove the strong consistency of the LS-estimators in regression models with non-square integrable martingale errors.

Consider the following regression model

$$X_t = \int_o^t f s \, d\, V_s \theta + m_t, \qquad (3)$$

where m is a pure discontinuous (for simplicity) local martingale from $\mathcal{M}_{loc}(R^d)$, a predictable process $V \in \mathcal{A}^+_{loc}(R^1)$, f is a predictable $(d \times k)$ – matrix, $\theta \in R^k$, $k \geq 1$, is an unknown parameter.

Let $F_t = \int_o^t f_s^* f_s d V_s$, $F_t > o$, $t \geq t_o$.

In this case we can define the estimator of θ :

$$\theta_t = F_t^{-1} \int_o^t f_s^* \, d X_s = \theta + F_t^{-1} \int_o^t f_s^* \, d m_s .$$

Theorem 2 : Suppose for the model (3) the following conditions hold (a.s.)

1) $\lim\limits_{t \to \infty} \lambda_1 (F_t) = \infty$;

2) $\limsup\limits_{t \to \infty} \dfrac{\lambda_2 (F_t)}{\lambda_1 (F_t)} < \infty$;

3) $\int_o^\infty \int_{R_o^d} \lambda_1^{-r}(F_s) \, \|f_s\|^r \, \|x\|^r \, d \nu < \infty$

where $r \in [1, 2]$, ν is a compensator of a measure μ of jumps of M.

Then $\theta_t \to \theta$ (a.s.) as $t \uparrow \infty$.

It is possible to unify the conditions of the theorem 2 , if we suppose that (a.s.)

$$\frac{d}{d V_t} \int_o^t \int_{R_o^d} \|x\|^r \, d \nu \leq \xi < \infty,$$

and (a.s.)

3') $\int_o^\infty \lambda_1^{-r} (F_s) \, \|f_s\|^r \, d V_s < \infty$.

Then 1) - 2) - 3') $\Rightarrow \theta_t \to \theta$ (a.s.) as $t \uparrow \infty$.

Proof : It is sufficient to note that

$$\theta_t - \theta = A_t^{-1} M_t ,$$

where $A_t = F_t$, $M_t = \int_o^t \int_{R_o^d} f^* x \, d\,(\mu - v).$

Using the theorem 1 we get immediatly the statement of the theorem 2.

Remark : Note that the consistency of LS-estimators for the model (3) with non-random regressors was proved by Novikov [6]. The strong consistency of the LS-estimators for this model with non-random regressors was studied also in [7]-[8].

Now consider another type of estimators of θ in the one-dimensional model (3). These are SQ-estimators, which systematically were studied in [4]. But the case of non-square integrable errors was handed there for the quasi-left continuous martingale errors m only. Here we prove an estimate for pure-discontinuous martingales and apply it to give an upper estimate for the r–th moment of the difference between the SQ-estimator and θ . This result gives us (in some sense) a guaranted accuracy of these estimators.

Denote $\mathcal{B}\,(R)$– Borel σ-algebra of the space R. Let

$$M_t = \int_o^t \int_{R_o} x \, d\,(\mu - v)$$

be a purely discontinuous local martingale of the classe $\mathcal{M}_{loc}\,(R^1)$ (see decomposition (1)).

Let U be a $\mathcal{B}\,(R_+^1) \otimes \mathcal{F} \otimes \mathcal{B}\,(R_0^1)$–measurable function such that for some $r \in [1\ ,2]$

$$\int_o^t \int_{R_o} |U|^r \, d\,v \in \mathcal{A}_{loc}^+\,(R^1)$$

and

$$\int_{R_o} U(t, x, \omega)\, \nu(\{t\}, dx) = o \qquad (4)$$

Denote $Y_t(U) = \int_o^t \int_{R_o} U\, d(\mu - \nu)$ and $Y_t^*(U) = \sup_{s \le t} | Y_s(U) |$.

Theorem 3 : Suppose the function U satisfies to the condition (4) and τ is a prédictable stopping time (s.t.). Then

$$E\, |Y^*_{\tau-}(U)|^r \le A_r E \int_o^{\tau-} \int_{R_o} |U|^r\, d\nu, \qquad (5)$$

where $A_r \le 3 \left(\frac{r}{r-1}\right)^r$, $r \in (1, 2]$, $A_1 = Z$ and Y_{t-} is left limit of Y_t .

Proof. We shall use Novikov's method [5]. Let us involve the *s.t.* $(a > o)$

$$\tau_a = inf\,(t \le \tau : \int_o^t \int_{R_o} |U|^r\, d\nu \ge a),$$

$inf\ (\emptyset) = \tau$.

Of course, τ_a is a predictable *s.t.*

Therefore there is a sequence of *s.t.'s* $(\tau_a^n)_{n \ge 1}$ such that

$\tau_a^n \uparrow \tau_a$ (*a.s.*) as $n \uparrow \infty$,

$\tau_a^n < \tau_a$ on the set $(\omega : \tau_a < \infty\}$.

It follows from here that

$$\int_o^{\tau_a^n} \int_{R_o} |U|^r\, d\nu < a.$$

Let us show that $E\, Y^*_{\tau_a^n}(U) < \infty$, we have (as usually, I_c is an indicator of c)

$$E\, Y^*_{\tau_a^n}(U) \le E \sup_{t \le \tau_a^n} \left| \int_o^t \int_{R_o} U \,.\, I_{|U|>1}\, d(\mu - \nu) \right| +$$

$$+ E \sup_{t \le \tau_a^n} \left| \int_o^t \int_{R_o} U \,.\, I_{|U| \le 1}\, d(\mu - \nu) \right| \le$$

$$\le const \,.\, E \int_o^{\tau_a^n} \int_{R_o} |U| \,.\, I_{|U|>1}\, d\nu + const \,.\, E \left[\int_o^{\tau_a^n} \int_{R_o} |U|^2 \,.\, I_{|U| \le 1}\, d\nu \right]^{1/2} \le$$

$$\le const \,.\, E \int_o^{\tau_a^n} \int_{R_o} |U|^r \,.\, I_{|U|>1}\, d\nu + const \,.\, E \left[\int_o^{\tau_a^n} \int_{R_o} |U|^r\, d\nu \right]^{1/2} \le$$

$$\le const\, (a + a^{1/2}) < \infty.$$

Using this fact and the elementary inequality

$$\big|\, |x+y|^r - |x|^r \,\big| \le C_r \left(|x|^{r-1} |y| + |y|^r \right)$$

we get that

$$E \int_o^{\tau_a^n} \int_{R_o} I_{|U|>1} \,\big|\, |Y_{s-} + U|^r - |Y_{s-}|^r \,\big|\, d\nu < \infty,$$

$$E \left[\int_o^{\tau_a^n} \int_{R_o} I_{|U| \le 1} \,\big|\, |Y_{s-} + U|^r - |Y_{s-}|^r \,\big|^2\, d\nu \right]^{1/2} < \infty \qquad (6)$$

This first inequality of (6) follows from

$$E \int_o^{\tau_a^n} \int_{R_o} I_{|U|>1} \,\big|\, |Y_{s-} + U|^r - |Y_{s-}|^r \,\big|\, d\nu \le$$

$$\leq const\,(r)\,E\,(1+|Y_{\tau_a^n}|^{r-1})\int_o^{\tau_a^n}\int_{R_o}|U|^r d\,v \leq$$

$$\leq a\;.\;const\,(r)\;.\;E\,(1+|Y^*_{\tau_a^n}|^{r-1})<\infty.$$

The second one follows from

$$E\left[\int_o^{\tau_a^n}\int_{R_o} I_{|U|\leq 1}\left|\,|Y_{s-}+U|^r-|Y_{s-}|^r\,\right|d\;v\right]^{1/2}\leq$$

$$\leq const\,(r)\;E\left[\int_o^{\tau_a^n}\int_{R_o} I_{|U|\leq 1}\,(|Y_{s-}|^{r-1}\;|U|+|U|^r)^2\;d\;v\right]^{1/2}\leq$$

$$\leq const\,(r)\,E\,(1+(Y^*_{\tau_a^n})^{2(r-1)})^{1/2}\left(\int_o^{\tau_a^n}\int_{R_o} I_{|U|\leq 1}\,|U|^r\;d\;v\right)^{1/2}$$

and

$$E\,(Y^*_{\tau_a^n}\;(U\,))^{r-1}\leq (E\;Y^*_{\tau_a}\;(U\,I_{|U|>1}))^{r-1}+(E\;Y^*_{\tau_a^n}(U\,.\,I_{|U|\leq 1}))^{r-1}$$

Now using the Ito's formula (see [2] ; p. 150-151)
we get

$$|Y_{\tau_a^n}|^r=\int_o^{\tau_a^n}\int_{R_o}(|Y_{s-}+U|^r-|Y_{s-}|^r)\,d\,(\mu-\nu)+$$

$$+\int_o^{\tau_a^n}\int_{R_o}\{\,|Y_{s-}+U|^r-|Y_{s-}|^r-r\,|Y_{s-}|^{r-2}\;Y_{s-}U\,\}\,d\;v \tag{7}$$

It follows from (6) that

E (martingale part of (7)) = 0

Applying the elementary inequality

$$|x+y|^r - |x|^r - r\,x\,y\,|x|^{r-2} \le B_r\,|y|^r ,$$

where $B_r \le 3$, $r \in [1,2]$ and $B_1 = 2$,

to the second part of (7), we have

$$E\,|Y_{\tau_a^n}|^r \le B_r E \int_o^{\tau_a^n}\int_{R_o} |U|^r d\,v . \qquad (8)$$

Using the Doob's inequality [2] , we get

$$E\,(Y^*_{\tau_a^n})^r \le 3\left(\frac{r}{r-1}\right)^r E \int_o^{\tau_a^n}\int_{R_o} |U|^r\,d\,v .$$

To tend $n \uparrow \infty$ and $a \uparrow \infty$ we complete the proof.
We note that the inequality (8) for $r=1$ is true with $B_1 = 2$ and therefore $A_1 = 2$.

Now consider the one-dimensional regression model (3) and suppose that

$$\frac{d}{d\,V_t}\int_o^t\int_{R_o} |x|^r d\,v \le \gamma_t , \qquad (9)$$

where $r \in [1,2]$, γ is a predictable process such that

$$K_t = \int_o^t \gamma_s^{1-r}\,|f_s|^r d\,V_s \in \mathcal{A}^+_{loc}\,(R^1).$$

we define the following SQ-estimator

$$\theta_{\mathrm{H}} = H^{-1}\int_o^{\tau_{\mathrm{H}}^-} \gamma_s^{-1}\, f_s\,d\,X_s + H^{-1}\,\beta_{\mathrm{H}}\,\gamma_{\tau_{\mathrm{H}}}^{-1}\,f_{\tau_{\mathrm{H}}}\,\Delta X_{\tau_{\mathrm{H}}},$$

where $H > o$, $\tau_H = inf(t : K_t \geq H)$,
β_H - $\mathcal{F}_{\tau_{H^-}}$ -measurable random variable such that
$\beta_H \in [o, 1]$,

$$\int_o^{\tau_{H^-}} \gamma_s^{-1} \; |f_s|^r \, d\, V_s + \beta_H \; \gamma_{\tau_H}^{-1} \; |f_{\tau_H}|^r \; \Delta \; V_{\tau_H} = H. \tag{10}$$

Theorem 4 : Let the conditions (9) - (10) are fulfield.

Then $\int_o^{\infty} \gamma_s^{1-r} \; |f_s|^r d\, V_s = \infty (a.s) \Rightarrow E\, \theta_H = \theta$ and $E\, |\theta_H - \theta|^r \leq const\,(r).\, H^{-r}\,(H{+}\Delta)$,
where $\Delta = E \sup_t \; \Delta K_t$, $r \in [1,2]$.

Proof : The first statement is the direct consequence of (10). Now we have, using the theorem 3, that

$$E\, |\theta_H - \theta|^r = E\, |H^{-1} \int_o^{\tau_{H^-}} \int_{R_0} \gamma_s^{-1} f_s x \, d\,(\mu - \nu) +$$

$$+ \beta_H H^{-1} \; \gamma_{\tau_H}^{-1} f_{\tau_H} \Delta\, M_{\tau_H} \, |^r \leq A_r \, 2^{\,r-1} \, E\, H^{-r} \int_o^{\tau_{H^-}} \int_{R_o} \gamma_s^{-r} \; |f_s|^r |x|^r d\, \nu \, +$$

$$+ 2^{\,r-1} \, H^{-r} \, E\, \beta_H \; \gamma_{\tau_H}^{-r} \; |f_{\tau_H}|^r . \int_{R_o} |x|^r \nu\,(\{\tau_H\}, d\, x) \leq$$

$$\leq H^{-r} \; 2^{\,r-1} \left[A_r \, E \int_o^{\tau_{H^-}} \gamma_s^{1-r} \; |f_s|^r \; d\, V_s + E\, \beta_H \; \gamma_{\tau_H}^{1-r} \; |f_{\tau_H}|^r \Delta\, V_{\tau_H} \right] \leq$$

$$\leq 2^{\,r-1} \, H^{-r} \; [A_r H + \Delta] \leq const\,(r) \,.\, H^{-r}\,(H + \Delta)$$

The theorem is proved.

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