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ON THE REAL SPECTRUM OF A RING OF GLOBAL ANALYTIC FUNCTIONS

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We present here some applications of the theory of real spectra of excellent rings to the ring of global analytic functions on a compact real analytic manifold. Section 1 contains the facts of that theory that shall be used in the sequel. Section 2 describes the good relationship between global semianalytic subsets of the manifold and constructible subsets of the real spectrum of the ring of global analytic functions. This leads to the solution of Hilbert's 17th problem, to the real Nullstellensatz and to the finiteness theorems, all in this global analytic setting. Finally, Section 3 gives a quick outlook on several questions related to connectedness, either of constructible sets or of global semianalytic sets.

§1. Real spectra of excellent rings

Let A be an excellent ring (for excellent rings, regular homomorphisms and related notions, we refer to [M1]). We denote by $X=\mathrm{Spec}_r A$ its real spectrum. We shall use terminology and results from [CR], [L1] and [BCR]. Let us remember some.

A prime cone $\alpha \in X$ corresponds to a unique homomorphism $A \to \kappa(\alpha)$, where $\kappa(\alpha)$ is a real closed field, whose kernel is denoted by $\mathrm{supp}(\alpha)$. The image of an $f \in A$ under that homomorphism is written $f(\alpha)$, which allows us a functionwise use of the elements of A. Thus we define a sub-basis of Harrison's topology as follows: all sets

$$\{f>0\}=\{\alpha \in X: f(\alpha)>0\} \qquad (f \in A)$$

It is also useful to consider the constructible topology, whose standard basis consists of all constructible sets $S \subset X$:

$$S=\bigcup_{1\leq i\leq r}\{f_{i1}>0,\dots,f_{is}>0,\ g_i=0\} \quad (f_{ij},\ g_i \in A),$$

but unless otherwise explicitely stated, X is alwais endowed with Harrison's topology. The dimension is given by:

$$\dim \alpha = \dim(A/\mathrm{supp}(\alpha)),$$

$$\dim S = \sup\{\dim \beta : \beta \in S\}$$

(and $\dim \varnothing = -1$). If $\alpha \in X$ we have the dimension at α:

$$\dim_\alpha S = \sup\{\dim \beta : \beta \in S, \beta \to \alpha\},$$

where $\beta \to \alpha$ means that $\mathrm{supp}(\beta) \subset \mathrm{supp}(\alpha)$ and $A/\mathrm{supp}(\beta) \to A/\mathrm{supp}(\alpha)$ preserves signs. Of course, $\dim_\alpha S = -1$ is equivalent to $\alpha \notin S$.

Finally, to state the main result, set: $\mathrm{cod}_\alpha S = \dim_\alpha S - \dim \alpha$.

Theorem 1.1 (Krull definition of real dimension, [Rz2], [Rz5]).- Let $S \subset X$ be constructible and $\alpha \in X$ such that $\mathrm{cod}_\alpha S \geqslant 1$. Then

$$\mathrm{cod}_\alpha S = \sup \left\{ d \;\middle|\; \begin{array}{c} \text{there is a chain} \\ \alpha_d \to \quad \to \alpha_1 \to \alpha \\ \text{with } \alpha_1,\dots,\alpha_d \in S \end{array} \right.$$

The proof of 1.1 is done by induction on $\mathrm{cod}_\alpha S$. The argument combines the dimension result itself with an abstract curve selection lemma (existence of a suitable homomorphism $A \to \kappa[[t]]$) and the computation of the image of the canonical map $\mathrm{Spec}_r A \to \mathrm{Spec}_r A$ in the case A is local and A its adic completion. The good property here is that the homomorphism $A \to A$ is regular, because A is excellent. As a matter of fact, once 1.1 is available we can deduce:

Theorem 1.2 (real going-down, [Rz6]).- Let $\psi: A \to B$ be a regular homomorphism of noetherian rings, A excellent, and ψ^* the corresponding map $Y = \mathrm{Spec}_r B \to \mathrm{Spec}_r A = X$. Then, for any prime cones $\alpha' \to \alpha$ in X and β in Y with $\psi^*\beta = \alpha$, there is β' in Y such that

$$\beta' \to \beta, \qquad \psi^*\beta' = \alpha'$$

and $\mathrm{ht}(\mathrm{supp}(\beta')) = \mathrm{ht}(\mathrm{supp}(\alpha'))$.

To prove this result one first reduces to the case A is a domain of dimension 1, by means of Theorem 1.1. Then, by a standard base change argument A is substituted by a localization D of its normalization. This D is a discrete valuation ring and the going-down is shown readily for it.

Let us remark here that faithfully flat homomorphism do not have, in general, real going-down ([CR]: $t \to t^2$; cf. [Rz6]).

The previous theorems are useful in dealing with constructibility problems in real spectra. Two most interesting of these are constructibility of closures and constructibility of connected components. We delay till Section 3 a brief discussion of the latter, which appears to be far more involved. With respect to closures we can solve the question as follows:

<u>**Theorem 1.3**</u> ([Rz7]).- Let A be an excellent ring and X its real spectra. If $S \subset X$ is constructible, its closure is constructible too.

The proof is based, once again, in Theorem 1.1, plus the nice properties of the constructible topology. The result was already proved for rings of polynomials over a real closed field ([CR]), for rings of convergent power series over the reals ([L], [FRRz]) and, very recently, for rings of formal power series over a real closed field ([AA]).

It is worth quoting an example in [DG]: let A be the ring of continuous functions $\mathbb{R} \to \mathbb{R}$; then, the closure of $\{Id_{\mathbb{R}} > 0\}$ in $\operatorname{Spec}_r A$ is not constructible.

§2. Rings of global analytic functions

Throughout this section M will stand for a compact real analytic manifold, $\mathcal{O}$ for its sheaf of germs of analytic functions and $\mathcal{O}(M)$ for its ring of global analytic functions. Since M is compact, $\mathcal{O}(M)$ is noetherian ([Fr]). We define in the usual way the zero-set of an ideal I of $\mathcal{O}(M)$:

$$Z(I) = \{x \in M:\ f(x)=0 \ \text{ for all } f\in I\},$$

and the ideal of a set $Z\subset M$:

$$I(Z) = \{f \in \mathcal{O}(M):\ f(x)=0 \ \text{ for all } x\in Z\}.$$

(2.1) Derivations on $\mathcal{O}(M)$.- Fix a point $x\in M$ of, say, dimension n, and let $\mathfrak{m}$ denote the corresponding maximal ideal: $\mathfrak{m} = I(\{x\})$. Starting with C^∞ data and approximating, one easily obtains global analytic functions $x_1,\dots,x_n : M \to \mathbb{R}$ and analytic vector fields $\xi_1,\dots,\xi_n$ on M such that

$$\det(\xi_i.x_j)(x)\neq 0.$$

These $x_1,\dots,x_n$ are a regular system of parameters of the local ring $\mathcal{O}(M)_{\mathfrak{m}}$ which turns out to be regular of dimension n. Furthermore, $\xi_1,\dots,\xi_n$ induce derivations $D_1,\dots,D_n$ of $\mathcal{O}(M)$, and these D_i's form a basis of the module of derivations of $\mathcal{O}(M)_{\mathfrak{m}}$. Using Matsumura's jacobian criteria ([M1], [M2]) it follows:

(2.1.1) $\mathcal{O}(M)$ is excellent, and

(2.1.2) $\mathcal{O}(M) \to \mathcal{O}_x$ is a regular homomorphism.

Finally, we recall the isomorphism $\mathcal{O}_x \simeq \mathbb{R}\{x_1,\dots,x_n\}$, where $\{\dots\}$ means convergent power series, induced by the Taylor expansion at x via te coordinate system $(x_1,\dots,x_n)$. In particular, $\mathfrak{m}_x = \{x_1,\dots,x_n\}\mathcal{O}_x$ is the maximal ideal of $\mathcal{O}_x$.

The facts above enable us to apply all results in §1 to the ring $\mathcal{O}(M)$. On the other hand, convergent power series and, consequently, $\mathcal{O}_x$ behave well from the viewpoint of real spectra (cf. [Rz1], [FRRz]), so that we can combine both things to study $\mathrm{Spec}_r\, \mathcal{O}(M)$.

First of all we obtain:

Theorem 2.2 (Artin-Lang for global analytic functions, [Rz2]).- Let I be an ideal of $\mathcal{O}(M)$ and $f_1,\dots,f_m \in \mathcal{O}(M)$. If $f_1,\dots,f_m$ are positive at a prime cone α with support I, then

$$\{x\in M: f_1(x)>0,\dots,f_m(x)>0\}\cap Z(I)\neq\varnothing.$$

Proof.- Suppose $f_1(\alpha)>0,\dots,f_m(\alpha)>0$, $\mathrm{supp}(\alpha)=I$. We have the homomorphism $\mathcal{O}(M)\to\kappa(\alpha)$ that embeds $\mathbb{R}$ into $\kappa(\alpha)$ and consider

V = convex hull of $\mathbb{R}$ in $\kappa(\alpha) = \{u\in\kappa(\alpha):\ u^2\leqslant r$ for some $r\in\mathbb{R}\}$.

This is a convex valuation ring of $\mathbb{R}$ whose maximal ideal will be denoted by $\mathfrak{m}_V$ (cf. [Br]). Obviously, its residue field $V/\mathfrak{m}_V$ is an arquimedian extension of $\mathbb{R}$ and hence $\mathbb{R}\to V/\mathfrak{m}_V$ is an isomorphism.

Now, we notice that $\mathcal{O}(M)\to\kappa(\alpha)$ factorizes through V. Indeed, if $f\in\mathcal{O}(M)$, as M is compact, there is a real number $r>0$ with $f(x)^2<r$ for all $x\in M$; so $g=\sqrt{(r-f^2)}\in\mathcal{O}(M)$, and $r=f(\alpha)^2+g(\alpha)^2\geqslant 0$. That is, $f(\alpha)\in V$. We have so $\mathcal{O}(M)\to V$, and $\mathfrak{m}_V$ lies over a prime ideal $\mathfrak{m}\supset\mathrm{supp}(\alpha)=I$. But the compositum $\mathbb{R}\to\mathcal{O}(M)/\mathfrak{m}\to V/\mathfrak{m}_V$ is an isomorphism and consequently $\mathfrak{m}$ is maximal: $\mathfrak{m}=\{f\in\mathcal{O}(M):\ f(x)=0\}$, for a unique $x\in M$.

We turn to the homomorphism $\mathcal{O}(M)\to\mathcal{O}_x$. It is regular and $\mathcal{O}(M)$ is excellent, (2.1); hence, we can apply the going-down 1.2 with the data

$\alpha\to\alpha_0$=the unique prime cone of $\mathcal{O}(M)$ with support $\mathfrak{m}$

β_0=the unique prime cone of $\mathcal{O}_x$ with support $\mathfrak{m}_x$,

and find a prime cone $\beta\to\beta_0$ such that

$$I\subset\mathrm{supp}(\beta);\ f_1(\beta)>0,\dots,f_m(\beta)>0.$$

Since I is finitely generated, there is $f\in I$ with $Z(I)=\{x\in M:\ f(x)=0\}$. Finally, we know Artin-Lang holds for the ring $\mathcal{O}_x/\mathrm{supp}(\beta)$ (this is somehow classical, and should be attributed to Risler-Lassalle, cf. [Rz1]): there exists $y \in M$, as close as needed to x, such that $f(y)=0,\ f_1(y)>0,\dots,f_m(y)>0$.

The proof of 2.2 is complete.

Corollaire 2.3 (Hilbert's 17th problem, [Rz3]).- Let $f:M\to\mathbb{R}$ be a non-negative real analytic function. Then f is a sum of squares of meromorphic functions: $f = \sum_{1\leqslant i\leqslant r}(f_i/g_i)^2$,
where $f_i,g_i \in \mathcal{O}(M)$, $\mathrm{int}_M\{g_i=0\}=\varnothing$.

Proof.- We can assume M is connected, so that the identity principle holds true and $\mathcal{O}(M)$ is a domain. By the classical Artin-Schreier theory we only have to check that f is positive at any total ordering of $\mathcal{O}(M)$, i.e. at any prime cone whose support is $I=\{0\}$. This follows by applying 2.2 to $-f$, since the hypothesis is that $-f$ is never positive.

Corollary 2.4 (real Nullstellensatz, [Rz3]).- Let I be an ideal of $\mathcal{O}(M)$. Then:

$$IZ(I) = \text{real-radical}(I),$$

that is, $f\in IZ(I)$ if and only if $f^{2m}+g_1^2+\dots+g_r^2 \in I$ for some $g_i \in \mathcal{O}(M)$ and $m \geqslant 1$.

Proof.- In the standard way one reduces to show $IZ(I)\subset I$ for I real prime. Therefore $I=\mathrm{supp}(\alpha)$ for some prime cone α (Artin-Schreier) and from 2.2 we deduce: if $f\notin I$, then $f(\alpha)^2>0$ and $\{f^2>0\}\cap Z(I)\neq\varnothing$. Thus, $f\notin IZ(I)$.

These results illustrate the way M and $X = \mathrm{Spec}_r\,\mathcal{O}(M)$ are related. To be more formal, let us remark first that

$$x \leftrightarrow \alpha_x = \text{the unique prime cone whose support is the maximal ideal of the point } x,$$

gives a topological embedding $M\to X$. Then we can define a map that reproduces in the global analytic context the well-known "tilda operator" of Coste-Roy (cf. [CR]):

(2.5) $$\left\{\begin{matrix}\text{constructible}\\ \\ \text{sets of } X\end{matrix}\right\} \to \left\{\begin{matrix}\text{semianalytic}\\ \\ \text{sets of } M\end{matrix}\right\} : S \to S\cap M$$

Here, it is important to distinguish the two notions: $T\subset M$ is *semianalytic* if

(2.6.1) $$T = \bigcup_{1\leqslant i\leqslant r}\{x\in M: f_{i1}(x)>0,\dots,f_{is}(x)>0, g_i(x)=0\},\quad f_{ij}, g_i \in \mathcal{O}(M),$$

and $T\subset M$ is *locally semianalytic* if every point $x\in M$ has a nbhd U such that

(2.6.2) $$T\cap U = \bigcup_{1\leqslant i\leqslant r}\{x\in U: f_{i1}(x)>0,\dots,f_{is}(x)>0, g_i(x)=0\},\quad f_{ij}, g_i \in \mathcal{O}(M).$$

The theory of locally semianalytic sets reduces to the one of semianalytic germs and is well developed (cf. [L], and for elementary proofs of the basic facts, [FRRz] and [Rz4]). On the contrary, we do not know any specific result on global semianalytic sets, and we want to show how the real spectrum can give some interesting information about them.

As a consequence of Artin-Lang 2.2, we find that 2.5 is a lattice isomorphism (just reformulate 2.2 as: $S=\emptyset$ iff $S\cap M=\emptyset$), whose inverse is:

(2.7) $$T \to \tilde{T} = \bigcup_{1\leqslant i\leqslant r}\{\alpha\in X: f_{i1}(\alpha)>0,\dots,f_{is}(\alpha)>0, g_i(\alpha)=0\},$$

for a T given as in 2.6.1.

A less immediate property is:

<u>**Lemma 2.8**</u>.- The mapping: $T \to \tilde{T}$ "preserves closures".

<u>**Proof**</u>.- The set $S = cl_X(T) \subset X$ is constructible (1.3 and 2.1.1) and closed, so that

$$S = \bigcup_{1\leqslant i\leqslant r}\{\alpha\in X: f_{i1}(\alpha)\geqslant 0,\dots,f_{is}(\alpha)\geqslant 0\}\quad (f_{ij} \in \mathcal{O}(M)).$$

Consequently, the semianalytic set $S\cap M$ is closed. As $S\cap M\supset T$, we have

$S\cap M\supset cl_M(T)$. Conversely, let $x\in S\cap M$ and U be a nbhd of x in M. We may assume $U=\{x\in M: h(x)>0\}$ for some $h\in \mathcal{O}(M)$. Since $x\in U$, $\alpha_x\in U$; since $\alpha_x\in(S\cap M)=cl_X(T)$ we have $U\cap T\neq\emptyset$ and so $U\cap T\neq\emptyset$. Hence $x\in cl_M(T)$. This shows $cl_M(T)=cl_X(T)\cap M$.

From 2.8 we deduce two global finiteness theorems for semianalytic sets:

Corollary 2.9.- Let $T\subset M$ be semianalytic. Then $cl_M(T)$ (resp. $int_M(M)$) is semianalytic too.

Corollary 2.10.- Let $T\subset M$ be semianalytic. If T is closed (resp. open), it can be described with non-strict (resp. strict) inequalities:

$$T=\bigcup_{1\leqslant i\leqslant r}\{x\in M: f_{i1}(x)\geqslant 0,\dots,f_{is}(x)\geqslant 0\}$$

$$(\text{resp. } T=\bigcup_{1\leqslant i\leqslant r}\{x\in M: f_{i1}(x)>0,\dots,f_{is}(x)>0\}),$$

for some global analytic functions $f_{ij}\in\mathcal{O}(M)$.

§3. Open problems on connectedness

The question whether the connected components of a constructible set S in a real spectrum $X=Spec_rA$ are again constructible is, as was quoted earlier, quite more difficult than its analogue for closures. The most general result we know is:

Theorem 3.1 ([Rz8]).- Let A be an excellent ring whose real spectrum has only finitely many closed points. Let B be a finitely generated A-algebra. Then, the connected components of any constructible set of $Spec_rB$ are constructible.

(3.2) Examples and remarks.- (1) The standard models for an A as in 3.1 are the excellent henselian semilocal rings whose residue fields have a finite number of different orderings. Indeed, in such rings, every prime cone makes convex some maximal ideal (cf. [L2] 3.16 i).

On the other hand, for A henselian semilocal the hypothesis on its residue fields is not only sufficient but also necessary to get the constructibility conclusion of 3.1. The reason behind is that the real spectrum of a field is totally disconnected.

(2) The following particular cases of 3.1 were known before: A a real closed field ([CR]); B=A a ring of convergent power series over the reals ([L], [FRRz]), and B=A a ring of formal power series over a real closed field ([AA]).

(3) As 3.2.1 suggests, the nature of 3.1 is essentially local, at least with respect to the ring of coefficients A. Thus many rings are not covered by that theorem. For instance:

Problem I.- Are the connected components of a constructible set constructible in the (very fair) case

$$A = R[x_1,\dots,x_p]\,[[y_1,\dots,y_q]],$$

R a real closed field and x_i, y_j indeterminates?

We shall not enter here in the proof of 3.1. It is based heavily on the real going-down 1.2 and its streamline are done in the case $A=R[[x_1,\dots,x_n]]$, where R stands for a real cosed field (cf. [AA]). It is also remarkable that an abstract slicing procedure patterned on Coste's *saucissonnage* ([BCR]) is also used.

We finish this short section by considering the case $A = \mathcal{O}(M)$, where M is a real manifold as in §2. Of course, this A is not of the type described in 3.1: the set of closed points of $X = Spec_r A$ is M itself (the proof of 2.1 shows that for any $\alpha \in X$ there is $x \in M$ with $\alpha \to \alpha_x$). Nevertheless, we can prove:

Proposition 3.3.- Let S be a constructible set of X. Then the connected components of S are constructible.

Proof.- Let $T = S \cap M$ be the associated semianalytic set. It is known that the connected components of T are locally semianalytic, and form a locally finite family ([L], or [Rz4]). Since M is compact, they must be finitely many, say r.

Suppose now that the connected components of S are not constructible. Then S cannot be connected: $S = W \cup W'$, where W and W' are non-empty, disjoint, open and closed subsets of S. Then, since *constructible* is equivalent to *clopen in the constructible topology* ([L1]), we deduce that W and W' are constructible. Now, both cannot be connected, say W' is not. Repeating again, we get two constructible, non-empty, closed and disjoint subsets W_1, W_2 of W' with $W' = W_1 \cup W_2$. Inductively, we obtain:

$$S = S_1 \cup \ldots \cup S_r \cup S_{r+1}, \qquad S_i \cap S_j = \varnothing \text{ for all } i \neq j,$$

S_i non-empty, constructible, closed in S for all i.

We then set $T_i = S_i \cap M \neq \varnothing$, so that

$$(*) \qquad T = T_1 \cup \ldots \cup T_r \cup T_{r+1}, \qquad T_i \cap T_j = \varnothing \text{ for all } i \neq j,$$

and, what is essential, the T_i's are closed in T by 2.8. We are done, because T having r connected components, the (*) is not admissible.

After 3.3 and 2.8, taking a step towards a result like 2.9 for connected components, the natural question is:

Problem II.- Does the correspondence: $T \to \tilde{T}$ preserve connected components?.

References

[AA] Alonso, M.E.; Andradas, C.: Real spectra of local complete local rings, preprint 1986.

[BCR] Bochnak, J.; Coste, M.; Roy, M.F.: Géométrie algébrique réelle, book to appear in Springer-Verlag.

[Br] Brumfiel, G.W.: Real valuations rings and ideals, in Lecture Notes in Math. 959 (1982) 55-97, Springer-Verlag.

[CR] Coste, M.; Roy, M.F.: La topologie du spectre réel, in Contemporary Math. 8 (1982) 27-59, AMS.

[D] Delfs, H.: The homotopy axiom in semialgebraic cohomology, J. reine angew. Math. 355 (1985) 108-128.

[DG] Delfs, H.; Gamboa, J.M.: Abstract semialgebraic functions, preprint 1986.

[FRRz] Fernandez, F.; Recio, J.; Ruiz, J.M.: Generalized Thom's lemma in semianalytic geometry, to appear in Bull. Ac. Polonaise Sc.

[Fr] Frisch, J.: Points de platitude d'un morphisme d'espaces analytiques complexes, Invent. Math. 4 (1967) 118-138.

[L1] Lam, T.Y.: An introduction to real algebra, Sexta Escuela Latino-americana de Mat., Oaxtepec (1982).

[L2] Lam, T.Y.: Orderings, valuations and quadratic forms. Conf. Board Math. Sciences, 52 (1983) AMS.

[L] Lojasiewicz, S.: Ensembles semi-analytiques. Lecture notes 1965 at IHES, Bures-sur-Yvette.

[M1] Matsumura, H.: Commutative Algebra, 2d edition (1980) Benjamin.

[M2] Matsumura, H.: Noetherian rings with many derivations, in Contributions to Algebra, Ed's Bass, Cassidy, Kovacic (1977) Academic Press.

[Rz1] Ruiz, J.M.: Central orderings in fields of real meromorphic function germs, Manuscr. Math. 46 (1984) 193-214.

[Rz2] Ruiz, J.M.: Cônes locaux et complétions, C.R.Ac.Sc. Paris, 302 (1986) 67-69.

[Rz3] Ruiz, J.M.: On Hilbert's 17th problem and real Nullstellensatz for global analytic functions, Math. Z. 190 (1985) 447-454.

[Rz4] Ruiz, J.M.: Basic properties of analytic and semianalytic germs, preprint 1985.

[Rz5] Ruiz, J.M.: A dimension theorem for real spectra, preprint 1985.

[Rz6] Ruiz, J.M.: A going-down theorem for real spectra, preprint 1985.

[Rz7] Ruiz, J.M.: Constructibility of closures in real spectra, preprint 1986.

[Rz8] Ruiz, J.M.: Constructibility of connected components in real spectra, preprint 1986.

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